Numerical treatment of Periodic and Oscillatory Problems Using Leapfrog Method

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Abstract *— In this paper, the Leapfrog method is used to study the periodic and oscillatory problems. Results obtained using Leapfrog and Single-term Haar wavelet series (STHW) [10] methods are compared with the exact solutions of the periodic and oscillatory problems. The results obtained using Leapfrog is found to be very closer to the exact solutions of these problems. Error graphs for the obtained results and exact solutions are presented in a graphical form to highlight the efficiency of this method. This Leapfrog can be easily implemented in a digital computer and the solution can be obtained for any length of time.*

Keywords — *Periodic problems, Oscillatory problems, Ordinary differential equations, Leapfrog Method, Single-term Haar wavelet series.*

I. INTRODUCTION

Oscillatory IVPs frequently arise in areas such as classical mechanics, celestial mechanics, quantum mechanics, and biological sciences. Several numerical methods based on the use of polynomial basis functions have been developed for solving this class of important problems (see Lambert [7,8], Hairer *et al* in [4], Hairer [5], and Sommeijer [17]). Other methods based on exponential fitting techniques which take advantage of the special properties of the solution that may be known in advance have been proposed (see Simos [16], Vanden et al [18], Franco [3], Fang et al [2], Nguyen et al [9], and Jator et al [6]). The motivation governing the exponentially-fitted methods is inherent in the fact that if the frequency or a reasonable estimate of it is known in advance, these methods will be more advantageous than the polynomial based methods.[3]

The goal of this article is to construct a numerical method for addressing periodic and oscillatory problems by an application of the Leapfrog method which was studied by Sekar and team of his researchers [11-15]. Recently, Sekar *et al*. [10] discussed the periodic and oscillatory problems using STHW. In this paper, the same periodic and oscillatory problems was considered (discussed by Sekar *et al*. [10]) but present a different approach using the Leapfrog method with more accuracy for periodic and oscillatory problems.

II. LEAPFROG METHOD

The most familiar and elementary method for approximating solutions of an initial value problem is Euler's Method. Euler's Method approximates the derivative in the form of $y' = f(t,y),\ y(t_0) = y_0,\ y \in R^d$ by a finite difference quotient $y'(t) \approx (y(t+h) - y(t))/h$. We shall usually discretize the independent variable in equal increments:

$$t_{n+1} = t_n + h,\ n = 0,1,\ldots,t_0 .$$

Henceforth we focus on the scalar case, $N = 1$. Rearranging the difference quotient gives us the corresponding approximate values of the dependent variable:

$$y_{n+1} = y_n + hf(t_n, y_n),\ n = 0,1,\ldots,t_0$$

To obtain the leapfrog method, we discretize t_n as in $t_{n+1} = t_n + h,\ n = 0,1,\ldots,t_0$, but we double the time interval, h, and write the midpoint approximation $y(t+h) - y(t) \approx hy'\left(t + \frac{h}{2}\right)$ in the form

$$y'(t+h) \approx (y(t+2h) - y(t))/h$$

and then discretize it as follows:

$$y_{n+1} = y_{n-1} + 2hf(t_n, y_n),\ n = 0,1,\ldots,t_0$$

The leapfrog method is a linear $m = 2$-step method, with $a_0 = 0, a_1 = 1, b_{-1} = -1, b_0 = 2$ and $b_1 = 0$. It uses slopes evaluated at odd values of n to advance the values at points at even values of n, and vice versa, reminiscent of the children's game of the same name. For the same reason, there are multiple solutions of the leapfrog method with the same initial value $y = y_0$. This situation suggests a potential instability present in multistep methods, which must be addressed when we analyze them—two values, y_0 and y_1, are required to initialize solutions of $y_{n+1} = y_{n-1} + 2hf(t_n, y_n),\ n = 0,1,\ldots,t_0$ uniquely, but the analytical problem $y' = f(t,y),\ y(t_0) = y_0,\ y \in R^d$ only provides one. Also for this reason, one-step methods are used to initialize multistep methods.

III. PERIODIC AND OSCILLATORY PROBLEMS

3.1. Inhomogeneous equation.

Consider the following problem

$$y'' = -100y + 99\sin x$$

with initial condition $y(0) = 1$ and $y'(0) = 11$

Whose analytical solution is $y(x) = \cos 10x + \sin 10x + \sin x.$

Equation has been solved numerically using the STHW and Leapfrog method and the obtained results (with step size time = 0.1) along with the exact solutions are presented in Table 1 along with absolute errors calculated between them. A graphical representation is shown for the inhomogeneous equation in Fig. 1, using three-dimensional effects. This result reveals the superiority of the Leapfrog method with less complexity in implementation and at the same time the error reduction is 1000 times less than the STHW method.

3.2. Duffing's equation.

Consider the nonlinear undamped Duffing equation

$$y'' + y + y^3 = B\cos(\omega x)$$

where B = 0.002 and $\omega = 1.01$.

The analytical solution of the above equation is given by

$$y(x) = \sum_{i=0}^{3} A_{2i+1}\cos[(2i+1)\omega x]$$

where $A_1 = 0.200179477536$, $A_3 = 0.246946143\times10^{-3}$, $A_5 = 0.304016\times10^{-6}$ and $A_7 = 0.374\times10^{-9}$.

Equation (5) has been solved numerically with boundary conditions of the form

$$y(0) = A_1 + A_3 + A_5 + A_7,\quad y'(0) = 0$$

The results obtained (with step size time = 1) using the Leapfrog and STHW methods along with exact solutions and absolute errors between them are calculated and are presented in Table 2. A graphical representation is given for Duffing's equation in Fig. 2, using three-dimensional effect. It is inferred that, the Leapfrog method gives better solution for the non-linear undamped Duffing's equation when compared to STHW method.

3.3. An orbit problem.

Consider the following 'almost' periodic orbit problem studied by Stiefel and Bettis [13]

$$z'' + z = 0.001e^{ix},\quad z(0) = 1,\quad z'(0) = 0.9995i,\quad z \in C,$$

whose analytical solution is given by $z(x) = u(x) + iv(x)$, $u, v \in R$

$$u(x) = \cos x + 0.0005x\sin x,\ v(x) = \sin x - 0.0005x\cos x.$$

The true solution in equation represents the motion on a perturbation of a circular orbit in the complex plane. Re-writing the equation in the following equivalent form

$$u'' + u = 0.001\cos x,\quad u(0) = 1,\quad u'(0) = 0,$$

$$v'' + v = 0.001\sin x,\quad v(0) = 0,\quad v'(0) = 0.9995,$$

Equation has been solved numerically using the STHW method and Leapfrog method. The obtained results (with step size time = 0.1) along with exact solutions and the absolute errors between them are calculated and are presented in Table 3. A graphical representation is presented for the orbit problem in Fig. 3-5, using three-dimensional effect. From Table 3-5 and the error graphs 3-5 reveals that Leapfrog method works well (with out any error) when compared to STHW method, which yields a little error.

3.4. Two-body problem.

Consider the system of coupled differential equations, which is well known as two-body problem

$$y'' = -\frac{y}{\left(y^2 + z^2\right)^{3/2}},$$

$$z'' = -\frac{z}{\left(y^2 + z^2\right)^{3/2}},\ y(0) = 1,\ y'(0) = 0,\ z(0) = 0,\ z'(0) = 1$$

whose analytical solution is given by

$$y(x) = \cos(x),\ z(x) = \sin(x)$$

The above system of equation has been solved numerically using the STHW method and Leapfrog method. The obtained results (with step size time = 0.1) along with exact solutions and absolute errors between them are calculated and are presented in Table 6-7. A graphical representation is given for the two-body problem in Fig. 6-7, using three-dimensional effect.

IV. NUMERICAL TREATMENT

Time t	Discrete solution for inhomogeneous equation				
	Exact Solutions	STHW Solutions	STHW Error	Leapfrog Solutions	Leapfrog Error
0	1.0000000	1.0000000	0	1.0000000	0
0.1	1.4816067	1.4816067	0	1.4816067	0
0.2	0.6918199	0.6918200	1E-07	0.6918199	1E-09
0.3	-0.5533524	-0.5533525	1E-07	-0.5533524	2E-09
0.4	-1.0210278	-1.0210279	1E-07	-1.0210278	3E-09
0.5	-0.1958365	-0.1958368	3E-07	-0.1958365	4E-09
0.6	1.2453975	1.2453978	3E-07	1.2453975	5E-09
0.7	2.0551066	2.0551069	3E-07	2.0551066	6E-09
0.8	1.5612134	1.5612137	3E-07	1.5612134	7E-09
0.9	0.2843139	0.2843144	5E-07	0.2843139	8E-09
1	-0.5416219	-0.5416224	5E-07	-0.5416219	9E-09
1.1	-0.1043556	-0.1043561	5E-07	-0.1043556	1E-08
1.2	1.2393225	1.2393232	7E-07	1.2393225	1.1E-08
1.3	2.2911729	2.2911736	7E-07	2.2911729	1.2E-08
1.4	2.1127925	2.1127932	7E-07	2.1127925	1.3E-08
1.5	0.8880915	0.8880922	7E-07	0.8880915	1.4E-08
1.6	-0.2459909	-0.2459919	1E-06	-0.2459909	1.5E-08
1.7	-0.2448940	-0.2448950	1E-06	-0.2448940	1.6E-08
1.8	0.8831813	0.8831823	1E-06	0.8831813	1.7E-08
1.9	2.0848846	2.0848857	1.1E-06	2.0848846	1.8E-08
2	2.2303235	2.2303247	1.2E-06	2.2303235	1.9E-08

Table 1 Results for the inhomogeneous equation at various values of "*x*".

Time t	Discrete solution for Duffing equation				
	Exact Solutions	STHW Solutions	STHW Error	Leapfrog Solutions	Leapfrog Error
0	0.2004294	0.2004294	0	0.2004294	0
1	0.3066526	0.3066526	0	0.3066526	0
2	0.2199634	0.2199634	0	0.2199634	0
3	0.0207932	0.0207932	0	0.0207932	0
4	-0.1036664	-0.1036664	0	-0.1036664	0
5	-0.0375666	-0.0375667	1E-07	-0.0375666	1E-09
6	0.1578427	0.1578428	1E-07	0.1578427	2E-09
7	0.2990128	0.2990129	1E-07	0.2990128	3E-09
8	0.2543050	0.2543051	1E-07	0.2543050	4E-09
9	0.0651066	0.0651069	3E-07	0.0651066	5E-09
10	-0.0919355	-0.0919358	3E-07	-0.0919355	6E-09
11	-0.0682613	-0.0682613	6E-08	-0.0682613	7E-09
12	0.1123593	0.1123593	8E-08	0.1123593	8E-09
13	0.2815429	0.2815434	5E-07	0.2815429	9E-09

14	0.2809778	0.2809783	5E-07	0.2809778	1E-08
15	0.1111868	0.1111875	7E-07	0.1111868	1.1E-08
16	-0.0689371	-0.0689377	6E-07	-0.0689371	1.2E-08
17	-0.0905881	-0.0905889	8E-07	-0.0905881	1.3E-08
18	0.0662643	0.0662653	1E-06	0.0662643	1.4E-08
19	0.2550834	0.2550844	1E-06	0.2550834	1E-09
20	0.2986884	0.2986894	1E-06	0.2986884	2E-09

Table 2 Results for the Duffing's equation at various values of "*x*".

Time t	Discrete solution for orbit problem				
	Exact Solutions	STHW Solutions	STHW Error	Leapfrog Solutions	Leapfrog Error
0	1.0000000	1.0000000	1E-07	1.0000000	1E-09
0.1	0.9950091	0.9950091	2E-07	0.9950091	2E-09
0.2	0.9800864	0.9800864	3E-07	0.9800864	3E-09
0.3	0.9553807	0.9553807	4E-07	0.9553807	4E-09
0.4	0.9211388	0.9211388	5E-07	0.9211388	5E-09
0.5	0.8777024	0.8777024	6E-07	0.8777024	6E-09
0.6	0.8255050	0.8255050	7E-07	0.8255050	7E-09
0.7	0.7650676	0.7650676	8E-07	0.7650676	8E-09
0.8	0.6969935	0.6969935	9E-07	0.6969935	9E-09
0.9	0.6219623	0.6219623	1E-06	0.6219623	1E-08
1	0.5407229	0.5407229	1.1E-06	0.5407229	1.1E-08
1.1	0.4540861	0.4540861	1.2E-06	0.4540861	1.2E-08
1.2	0.3629168	0.3629168	1.3E-06	0.3629168	1.3E-08
1.3	0.2681249	0.2681249	1.4E-06	0.2681249	1.4E-08
1.4	0.1706567	0.1706567	1.5E-06	0.1706567	1.5E-08
1.5	0.0714850	0.0714850	1.6E-06	0.0714850	1.6E-08
1.6	-0.0284001	-0.0284001	1.7E-06	-0.0284001	1.7E-08
1.7	-0.1280018	-0.1280018	1.8E-06	-0.1280018	1.8E-08
1.8	-0.2263259	-0.2263259	1.9E-06	-0.2263259	1.9E-08
1.9	-0.3223908	-0.3223908	2E-06	-0.3223908	2E-08
2	-0.4152377	-0.4152377	2.1E-06	-0.4152377	2.1E-08

Table 3 Results for an orbit problem at various values of "*u*".

Fig.2 Error graph for inhomogeneous equation

Time t	Discrete solution for inhomogeneous equation				
	Exact Solutions	STHW Solutions	STHW Error	Leapfrog Solutions	Leapfrog Error
0	0.0000000	0.0000000	1E-07	0.0000000	1E-09
0.1	0.0997836	0.0997836	2E-07	0.0997836	1E-09
0.2	0.1985713	0.1985713	3E-07	0.1985713	3E-09
0.3	0.2953769	0.2953769	4E-07	0.2953769	3E-09
0.4	0.3892341	0.3892341	5E-07	0.3892341	5E-09
0.5	0.4792061	0.4792061	6E-07	0.4792061	5E-09
0.6	0.5643948	0.5643948	7E-07	0.5643948	7E-09
0.7	0.6439500	0.6439500	8E-07	0.6439500	7E-09
0.8	0.7170774	0.7170774	9E-07	0.7170774	9E-09
0.9	0.7830472	0.7830472	1E-06	0.7830472	9E-09
1	0.8412008	0.8412008	1.1E-06	0.8412008	1E-09
1.1	0.8909579	0.8909579	1.2E-06	0.8909579	1E-09
1.2	0.9318217	0.9318217	1.3E-06	0.9318217	3E-09
1.3	0.9633843	0.9633843	1.4E-06	0.9633843	3E-09
1.4	0.9853307	0.9853307	1.5E-06	0.9853307	5E-09

1.5	0.9974419	0.9974419	1.6E-06	0.9974419	5E-09
1.6	0.9995969	0.9995969	1.7E-06	0.9995969	7E-09
1.7	0.9917743	0.9917743	1.8E-06	0.9917743	7E-09
1.8	0.9740520	0.9740520	1.9E-06	0.9740520	9E-09
1.9	0.9466071	0.9466071	2E-06	0.9466071	9E-09
2	0.9097134	0.9097134	2.1E-06	0.9097134	1E-09

Table 4 Results for an orbit problem at various values of "*z*".

Time t	Discrete solution for orbit problem				
	Exact Solutions	STHW Solutions	STHW Error	Leapfrog Solutions	Leapfrog Error
0	1.0000000	1.0000000	0	1.0000000	0
0.1	1.0947928	1.0947928	1E-07	1.0947928	2.00E-09
0.2	1.1786577	1.1786577	1E-07	1.1786577	3.00E-09
0.3	1.2507576	1.2507576	1E-07	1.2507576	4.00E-09
0.4	1.3103730	1.3103730	1E-07	1.3103730	5.00E-09
0.5	1.3569085	1.3569085	3E-07	1.3569085	6.00E-09
0.6	1.3898999	1.3898999	3E-07	1.3898999	7.00E-09
0.7	1.4090176	1.4090176	3E-07	1.4090176	8.00E-09
0.8	1.4140710	1.4140710	3E-07	1.4140710	9.00E-09
0.9	1.4050096	1.4050096	3E-07	1.4050096	1.00E-08
1	1.3819239	1.3819239	5E-07	1.3819239	1.10E-08
1.1	1.3450441	1.3450441	5E-07	1.3450441	1.20E-08
1.2	1.2947385	1.2947385	5E-07	1.2947385	1.30E-08
1.3	1.2315093	1.2315093	5E-07	1.2315093	1.40E-08
1.4	1.1559875	1.1559875	5E-07	1.1559875	1.50E-08
1.5	1.0689270	1.0689270	7E-07	1.0689270	1.60E-08
1.6	0.9711968	0.9711968	7E-07	0.9711968	1.70E-08
1.7	0.8637724	0.8637724	7E-07	0.8637724	1.80E-08
1.8	0.7477261	0.7477261	7E-07	0.7477261	1.90E-08
1.9	0.6242161	0.6242161	1E-06	0.6242161	2.00E-08
2	0.4944756	0.4944756	1E-06	0.4944756	2.10E-08

Table 5 Results for an orbit problem at various values of "*z* ".

Fig.2 Error graph for Duffing equation

Time t	Discrete solution for inhomogeneous equation				
	Exact Solutions	STHW Solutions	STHW Error	Leapfrog Solutions	Leapfrog Error
0	1.0000000	1.0000000	0	1.0000000	0
0.1	0.9950041	0.9950041	1E-07	0.9950041	2.00E-09
0.2	0.9800665	0.9800665	1E-07	0.9800665	3.00E-09
0.3	0.9553365	0.9553365	1E-07	0.9553365	4.00E-09
0.4	0.9210609	0.9210609	1E-07	0.9210609	5.00E-09
0.5	0.8775825	0.8775825	3E-07	0.8775825	6.00E-09
0.6	0.8253356	0.8253356	3E-07	0.8253356	7.00E-09
0.7	0.7648421	0.7648421	3E-07	0.7648421	8.00E-09
0.8	0.6967066	0.6967066	3E-07	0.6967066	9.00E-09
0.9	0.6216098	0.6216098	3E-07	0.6216098	1.00E-08
1	0.5403022	0.5403022	5E-07	0.5403022	1.10E-08
1.1	0.4535959	0.4535959	6E-07	0.4535959	1.20E-08
1.2	0.3623575	0.3623575	5E-07	0.3623575	1.30E-08
1.3	0.2674986	0.2674986	5E-07	0.2674986	1.40E-08
1.4	0.1699669	0.1699669	5E-07	0.1699669	1.50E-08
1.5	0.0707369	0.0707369	7E-07	0.0707369	1.60E-08

1.6	-0.0291997	-0.0291997	7E-07	-0.0291997	1.70E-08
1.7	-0.1288447	-0.1288447	7E-07	-0.1288447	1.80E-08
1.8	-0.2272024	-0.2272024	7E-07	-0.2272024	1.90E-08
1.9	-0.3232898	-0.3232898	7E-07	-0.3232898	2.00E-08
2	-0.4161470	-0.4161470	1E-06	-0.4161470	2.10E-08

Table 6 Results for two-body problem at various values of "*y*".

Time t	Discrete solution for orbit problem				
	Exact Solutions	STHW Solutions	STHW Error	Leapfrog Solutions	Leapfrog Error
0	0.0000000	0.0000000	0	0.0000000	0
0.1	0.0998334	0.0998334	0	0.0998334	0
0.2	0.1986693	0.1986693	0	0.1986693	0
0.3	0.2955202	0.2955202	0	0.2955202	0
0.4	0.3894183	0.3894183	0	0.3894183	0
0.5	0.4794255	0.4794255	0	0.4794255	0
0.6	0.5646424	0.5646424	1E-07	0.5646424	2.00E-09
0.7	0.6442177	0.6442177	1E-07	0.6442177	3.00E-09
0.8	0.7173561	0.7173561	1E-07	0.7173561	4.00E-09
0.9	0.7833269	0.7833269	1E-07	0.7833269	5.00E-09
1	0.8414710	0.8414710	1E-07	0.8414710	6.00E-09
1.1	0.8912073	0.8912073	1E-07	0.8912073	7.00E-09
1.2	0.9320391	0.9320391	1E-07	0.9320391	8.00E-09
1.3	0.9635582	0.9635582	1E-07	0.9635582	9.00E-09
1.4	0.9854497	0.9854497	3E-07	0.9854497	1.00E-08
1.5	0.9974949	0.9974949	3E-07	0.9974949	1.10E-08
1.6	0.9995735	0.9995735	3E-07	0.9995735	1.20E-08
1.7	0.9916647	0.9916647	3E-07	0.9916647	1.30E-08
1.8	0.9738475	0.9738475	3E-07	0.9738475	1.40E-08
1.9	0.9462999	0.9462999	3E-07	0.9462999	1.50E-08
2	0.9092973	0.9092973	3E-07	0.9092973	2.00E-09

Table 7 Results for two-body problem at various values of "*z*".

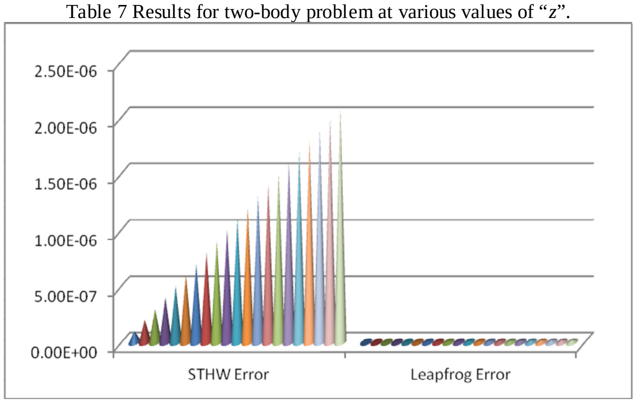

Fig. 3 Error graph for "*u*" at various time intervals (orbit problem)

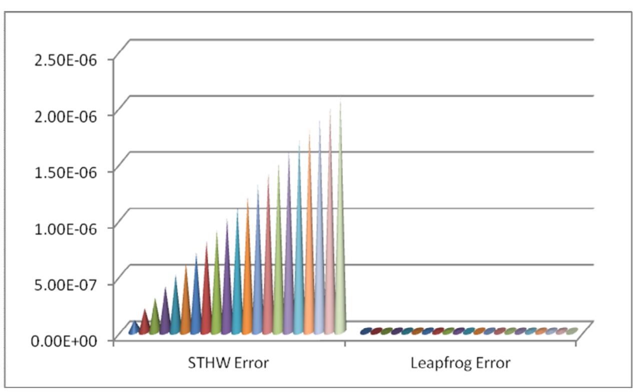

Fig. 4 Error graph for "*v*" at various time intervals (orbit problem)

Fig. 5 Error graph for "*z*" at various time intervals (orbit problem)

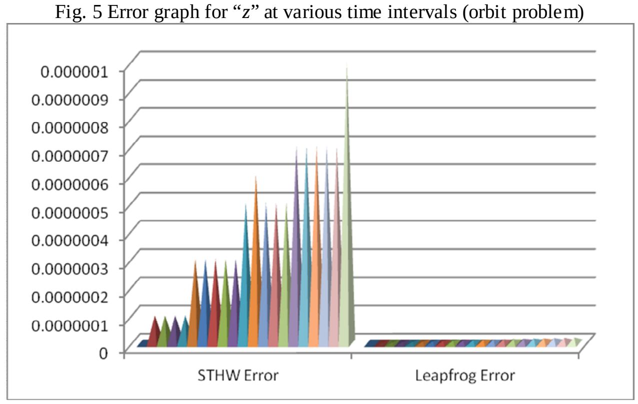

Fig. 6 Error graph for "*y*" at various time intervals

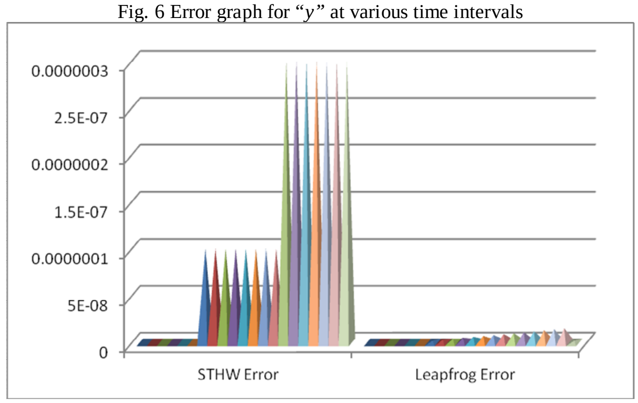

Fig. 7 Error graph for "*z*" at various time intervals

V. CONCLUSIONS

The obtained results of the periodic and oscillatory problems using Leapfrog is very closer to these exact solutions of the problem when compared to the STHW method. From the Table 1-7, one can observe that for most of the time intervals, the absolute error is less in Leapfrog when compared to the STHW method which yields a little error, along with the exact solutions. From Fig. 1-7, one can predict that the error is very less in Leapfrog when compared to the STHW method and especially Leapfrog method works well for the orbit problem and the two body problem. Hence, the Leapfrog method is more suitable for studying the periodic and oscillatory problems and especially it is recommended for the problems of orbit and two-body problems.

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