Numerical treatment for the Nonlinear Fuzzy Differential Equations using Leapfrog Method

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Abstract* — In this paper, we study the numerical method for Nonlinear Fuzzy Differential Equations by an application of the Leapfrog method for fuzzy differential equations. And, we present an example with initial condition [28] having two different solutions to illustrate the efficiency of the proposed Leapfrog method. Solution graphs are presented to highlight the efficiency of the Leapfrog method.

Keywords — *Single term Haar wavelet series, Leapfrog method, Fuzzy Differential Equations, Nonlinear Fuzzy Differential Equations.*

I. INTRODUCTION

The study of fuzzy differential equations is rapidly expanding as a new branch of fuzzy mathematics. Both theory and applications have been actively discussed over the last few years. According to Vorobiev and Seikkala [34], the term 'fuzzy differential equation' was first coined in 1978. Since then, it has been a subject of interest among scientists and engineers. In the literature, the study of fuzzy differential equations has several interpretations. The first one is based on the notion of Hukuhara derivative [17,31]. Under this interpretation, the existence and uniqueness of the solution of fuzzy differential equations have been extensively studied (see [13,20,33,36,37]).

In 1987, the concept of Hukuhara derivative was further explored by Kaleva [20] and Seikkala [32]. Subsequently, the theory of fuzzy differential equations has been developed and fuzzy initial value problems have been studied. However, this approach produces many solutions that have an increasing length of support as the independent variable increases (see [7,12,31,34]). Moreover, different formulations of the same fuzzy differential equation might lead to different solutions. According to Diamond [12] the approach based on the Hukuhara derivative does not produce the variety of behaviours as in the case of ordinary differential equations. This shortcoming has been alleviated by Hullermeier [19], who studied a fuzzy differential equation as a family of differential inclusions. According to Bede et al. [6], the main shortcoming of Hullermeier's approach is that it does not include a ''fuzzification'' of the differential operator. The same authors also claim that the solution of a fuzzy differential equation is not necessarily a fuzzy interval-valued function. In [4] it is shown that in some situations, the approach based on Hullermeier's interpretation also yields different solutions.

The third interpretation was suggested by Buckley and Feuring [9], who applied the extension principle to the crisp solution of ordinary differential equations in order to obtain a solution in the fuzzy setting. In this case, different formulations of the same ordinary differential equation lead to the same solution ensuring its uniqueness. In 2005, Bede and Gal [5] introduced a new concept of fuzzy derivatives called the generalised differentiability of fuzzy interval-valued functions. In this setting, the solution of a fuzzy differential equation may have a decreasing length of support as the independent variable increases. However, it depends on the selection of the fuzzy derivatives. Moreover, different formulations of the same fuzzy differential equation will lead to different solutions as well. Therefore, the uniqueness is not ensured. This generalisation was further studied by Chalco-Cano and Roman-Flores [10,11] who established relationships with other interpretations (see also [27]). Indeed, the uniqueness of the solution of a fuzzy differential equation has been challenging. However, according to Remark 4 in [6], the existence of several solutions is not a deficiency of the methods: we can choose the solution which better reflects the characteristics of the problem. In [23], the authors used the generalised differentiability to study the existence of solutions of a class of first-order linear fuzzy differential equations with periodic boundary conditions.

Recently, Gasilov et al. [15] proposed a new method to solve a system of linear differential equations with real coefficients and with an initial condition described by a vector of fuzzy intervals. The proposed method is based on properties of linear transformations. However, the authors considered a fuzzy set of real vector functions rather than a fuzzy vector function. In order to solve fuzzy differential equations with fuzzy coefficients, fuzzy initial values and fuzzy forcing functions, Akin et al. [3] proposed a new algorithm based on an analysis of the crisp solution.

Results in [11,9,10,32] have motivated several authors to propose numerical methods for solving fuzzy differential equations. One of the earliest contributions was a fuzzy version of Euler's method by Ma et al. [26]. A new version of Euler's method based on generalised differentiability has been studied in [29]. However, a serious shortcoming is that the authors did not take into account the dependency problem, which arises when multiple occurrences of the same fuzzy interval are treated independently in fuzzy interval arithmetic. As discussed in [11,14,18,24,25,28,35,38], this can lead to a repetition of some numerical computations, producing approximations that are less accurate. In [2], the authors developed a 4-th order Runge–Kutta method for solving fuzzy differential equations. However, their proposed method suffers from the same problem as in [26,29]. We can see the same problem in numerical methods proposed by Abbasbandy and Allahviranloo [1], Khastan and Ivaz [22], Palligkinis et al. [30], and the latest by Ghazanfari and Shakerami [16]. By considering the dependency problem in fuzzy interval arithmetic, we propose a new fuzzification of Euler's method for a more general class of problems.

In this study, we develop numerical methods for nonlinear fuzzy differential equations by an application of the Leapfrog which was studied by Sekar and team of his researchers [39-44]. Recently, M. Z. Ahmad *et al*. [43] and M. Rostami *et al*. [44] discussed the nonlinear fuzzy systems by extension principle and second order Runge-Kutta method. In this paper, the same nonlinear fuzzy differential equations is considered (discussed by M. Z. Ahmad *et al*. [43] and M. Rostami *et al*. [44]) but a different approach using the Leapfrog method with more accuracy is presented.

II. Leapfrog Method

In mathematics Leapfrog integration is a simple method for numerically integrating differential equations of the form $\ddot{x} = F(x)$, or equivalently of the form $\dot{v} = F(x), \dot{x} \equiv v$, particularly in the case of a dynamical system of classical mechanics. Such problems often take the form $\ddot{x} = -\nabla V(x)$, with energy function $E(x,v) = \frac{1}{2}|v|^2 + V(x)$, where V is the potential energy of the system. The method is known by different names in different disciplines. In particular, it is similar to the Velocity Verlet method, which is a variant of Verlet integration. Leapfrog integration is equivalent to updating positions $x(t)$ and velocities $v(t) = \dot{x}(t)$ at interleaved time points, staggered in such a way that they 'Leapfrog' over each other. For example, the position is updated at integer time steps and the velocity is updated at integer-plus-a-half time steps.

Leapfrog integration is a second order method, in contrast to Euler integration, which is only first order, yet requires the same number of function evaluations per step. Unlike Euler integration, it is stable for oscillatory motion, as long as the time-step Δt is constant, and $\Delta t \le 2/w$. In Leapfrog integration, the equations for updating position and velocity are

$$x_i = x_{i-1} + v_{i-1/2}\Delta t, a_i = F(x_i), v_{i+1/2} = v_{i-1/2} + a_i \Delta t,$$

where x_i is position at step $i, v_{i+1/2}$, is the velocity, or first derivative of x, at step $i + 1/2, a_i = F(x_i)$ is the acceleration, or second derivative of x, at step i and Δt is the size of each time step. These equations can be expressed in a form which gives velocity at integer steps as well. However, even in this synchronized form, the time-step Δt must be constant to maintain stability.

$$x_{i+1} = x_i + v_i \Delta t + \frac{1}{2} a_i \Delta t^2,$$

$$v_{i+1} = v_i + \frac{1}{2}(a_i + a_{i+1})\Delta t.$$

One use of this equation is in gravity simulations, since in that case the acceleration depends only on the positions of the gravitating masses, although higher order integrators (such as Runge–Kutta methods) are more frequently used. There are two primary strengths to Leapfrog integration when applied to mechanics problems. The first is the time-reversibility of the Leapfrog method. One can integrate forward n steps, and then reverse the direction of integration and integrate backwards n steps to arrive at the same starting position. The second strength of Leapfrog integration is its symplectic nature, which implies that it conserves the (slightly modified) energy of dynamical systems. This is especially useful when computing orbital dynamics, as other integration schemes, such as the Runge-Kutta method, do not conserve energy and allow the system to drift substantially over time.

III. Nonlinear Fuzzy Differential Equations

An arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$ for all r ∈ [0, 1], which satisfy the following requirements [O. Kaleva (1990)]:

(i) $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0, 1],

(ii) $\overline{u}(r)$ is a bounded right continuous non-increasing function over [0, 1],

(iii) $\underline{u}(r) \le \overline{u}(r)$ ◎ r ◎ [0, 1],

Let E be the set of all upper semi-continuous normal convex fuzzy numbers with bounded α-level intervals.

Lemma

Let $[\underline{v}(\alpha), \overline{v}(\alpha)]$, α ◎ (0, 1] be a given family of non-empty intervals. If

(i) $[\underline{v}(\alpha), \overline{v}(\alpha)] \supset [\underline{v}(\beta), \overline{v}(\beta)]$ for 0 < α ≤ β,

and

(ii) $\left[\lim_{k\to\infty} \underline{v}(\alpha_k), \lim_{k\to\infty} \overline{v}(\alpha_k)\right] = [\underline{v}(\alpha), \overline{v}(\alpha)]$,

whenever (α_k) is a non-decreasing sequence converging to α ◎ (0, 1], then the family $[\underline{v}(\alpha), \overline{v}(\alpha)]$, α

◎ (0, 1], represent the α-level sets of a fuzzy number v in E.

Conversely if $\left[\underline{v}(\alpha), \bar{v}(\alpha)\right]$, α ◎ (0, 1], are α-level sets of a fuzzy number v ◎ E, then the conditions (i) and (ii) hold true.

Definition

Let I be a real interval. A mapping v : I → E is called a fuzzy process and we denoted the α-level set by $[v(t)]_\alpha = \left[\underline{v}(t,\alpha), \bar{v}(t,\alpha)\right]$. The Seikkala derivative $v'(t)$ of v is defined by $[v'(t)]_\alpha = \left[\underline{v}'(t,\alpha), \bar{v}'(t,\alpha)\right]$, provided that is a equation defines a fuzzy number $v'(t)$ ◎ E.

Definition

Suppose u and v are fuzzy sets in E. Then their Hausdroff D : E × E → $R_+ \cup \{0\}$,

$$D(u,v) = \sup_{\alpha\in[0,1]} \max\left\{\left|\underline{u}(\alpha) - \underline{v}(\alpha)\right|, \left|\bar{u}(\alpha) - \bar{v}(\alpha)\right|\right\},$$

i.e.,D(u, v) is maximal distance between α-level sets of u and v.

In this section, we study the fuzzy initial value problem for a second-order linear fuzzy differential equation.

$$\left.\begin{aligned} &x''(t) + a(t)x'(t) + b(t)x(t) = \omega(t), \\ &x(0) = c_1, \\ &x'(0) = c_2, \end{aligned}\right\} \qquad (1)$$

where $c_1,\ c_2 \in R_f,\ a(t), b(t), \omega(t) \in R$. In this paper, we suppose $a(t), b(t) > 0$. Our strategy of solving (1) is based on the selection of derivative type in the fuzzy differential equation. We first give the following definition for the solutions of (1).

Definition

Let $x : [a,b] \to R_f$ be fuzzy-valued function and n,m = 1,2. One says x is an (n,m)-solution for problem (4.1). If $D_n^{(1)} x(t),\ D_{n,m}^{(2)} x(t)$ exist and

$$D_{n,m}^{(2)} x(t) + a(t) D_n^{(1)} x(t) + b(t)x(t) = \omega(t),\ x(0) = c_1,\ D_n^{(1)} x(0) = c_2 .$$

IV. Numerical Example of Nonlinear Fuzzy Differential Equations

Example 4.1

Consider the following nonlinear fuzzy differential equation with fuzzy initial value is given by M. Rostami *et al.* [44].

$$\left.\begin{aligned} &x'(t) = t^2 x(t) - 4tx(t) + 3x(t), \quad t \in [0,2] \\ &x(0) = (0,1,1) \end{aligned}\right\}$$

Example 4.2

Consider the following nonlinear fuzzy differential equation with fuzzy initial value is given by M. Z. Ahmad *et al.* [43].

$$\left.\begin{aligned} &x'(t) = \cos(tx), \quad t \in [0,3] \\ &x(0) = (0, \pi/4, 3\pi/4, \pi) \end{aligned}\right\}$$

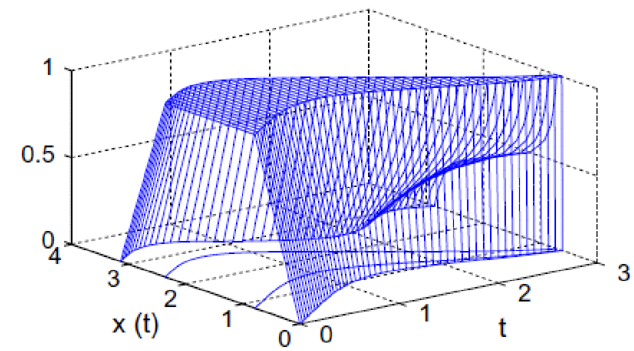

Fig. 1. Solution Graph for Example 4.1 using Runge-Kutta Method

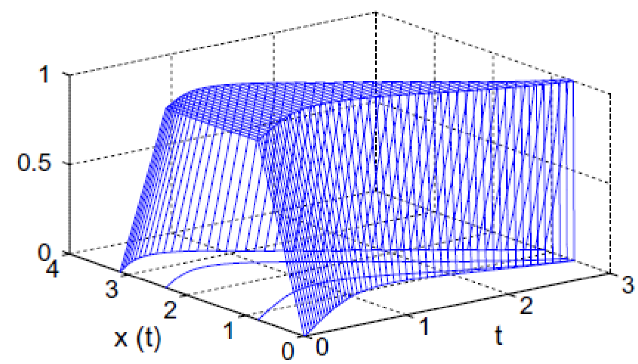

Fig. 2. Solution Graph for Example 4.1 using Leapfrog Method

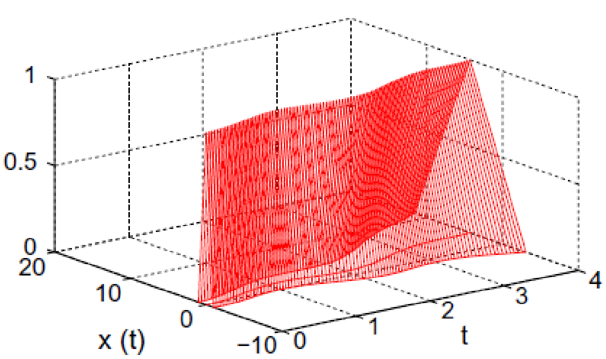

Fig. 3. Solution Graph for Example 4.2 using extension principle

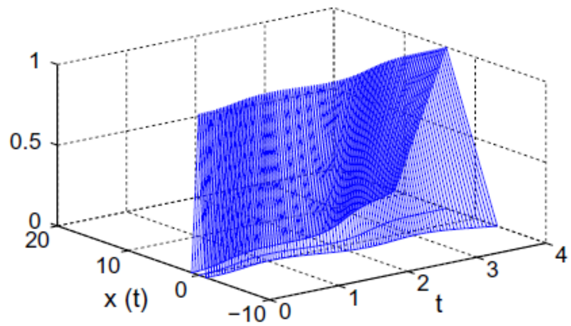

Fig. 4. Solution Graph for Example 4.2 using Leapfrog Method

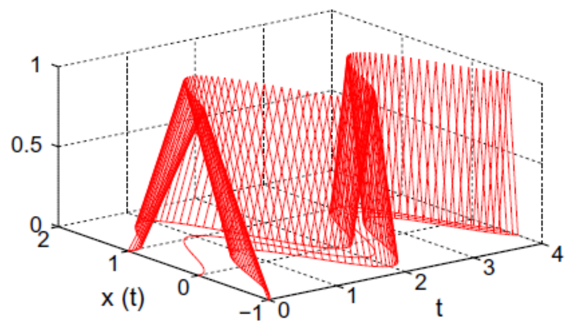

Fig. 5. Solution Graph for Example 4.2 using extension principle

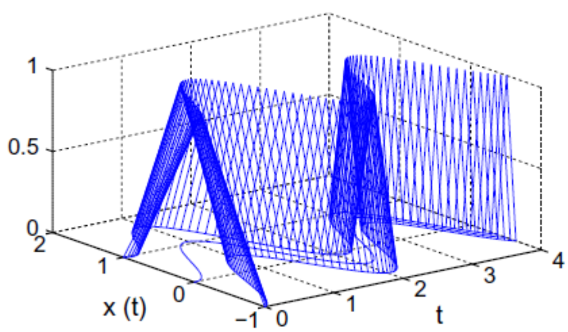

Fig. 6. Solution Graph for Example 4.2 using Leapfrog Method

Since the exact solution cannot be found analytically, we use the numerical method proposed in this study. The results are shown in Fig. 1-6, where we can see that the diameter of the approximate solution shows a non-monotone behaviour as t increases. This illustrates that the numerical method proposed in this paper is capable of generating periodic solutions, while the other two numerical methods do not have this capability.

V. Conclusions

In this paper we introduce a new numerical method for solving nonlinear fuzzy differential equations. The efficiency and the accuracy of the Leapfrog method have been illustrated by suitable examples. The solutions obtained are coincide well with the solutions of the nonlinear fuzzy differential equations and other methods.

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References

[1] S. Abbasbandy, T. Allahviranloo, Numerical solutions of fuzzy differential equations by Taylor method, Computational Methods in Applied Mathematics 2 (2002) 113–124.

[2] S. Abbasbandy, T. Allahviranloo, Numerical solution of fuzzy differential equation by Runge–Kutta method, Nonlinear Studies 11 (2004) 117–129.

[3] O. Akin, T. Khaniyev, O. Oruc, I.B. Turks_en, An algorithm for the solution of second order fuzzy initial value problems, Expert Systems with Applications 40 (2013) 953–957.

[4] B. Bede, T.G. Bhaskar, V. Lakshmikantham, Perspectives of fuzzy initial value problems, Communications in Applied Analysis 11 (2007) 339–358.

[5] B. Bede, S.G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, Fuzzy Sets and Systems 151 (2005) 581–599.

[6] B. Bede, I.J. Rudas, A.L. Bencsik, First order linear fuzzy differential equations under generalized differentiability, Information Sciences 177 (2007) 1648–1662.

[7] T.G. Bhaskar, V. Lakshmikantham, V. Devi, Revisiting fuzzy differential equations, Nonlinear Analysis 58 (2004) 351–358.

[8] A. Bonarini, G. Bontempi, A qualitative simulation approach for fuzzy dynamical models, ACM Transactions on Modelling and Computer Simulation 4 (1994) 285–313.

[9] J.J. Buckley, T. Feuring, Fuzzy differential equations, Fuzzy Sets and Systems 110 (2000) 43–54.

[10] Y. Chalco-Cano, H. Roman-Flores, On new solutions of fuzzy differential equations, Chaos, Solitons & Fractals 38 (2008) 112–119.

[11] Y. Chalco-Cano, H. Roman-Flores, Comparation between some approaches to solve fuzzy differential equations, Fuzzy Sets and Systems 160 (2009) 1517–1527.

[12] P. Diamond, Stability and periodicity in fuzzy differential equations, IEEE Transactions on Fuzzy Systems 8 (2000) 583–590.

[13] Z. Ding, M. Ma, A. Kandel, Existence of the solutions of fuzzy differential equations with parameters, Information Sciences 99 (1997) 205–217.

[14] W.M. Dong, F.S. Wong, Fuzzy weighted average and implementation of the extension principle, Fuzzy Sets and Systems 21 (1987) 183–199.

[15] N. Gasilov, S.E. Amrahov, A.G. Fatullayev, A geometric approach to solve fuzzy linear systems of differential equations, Applied Mathematics and Information Sciences 5 (2011) 484–499.

[16] B. Ghazanfari, A. Shakerami, Numerical solutions of fuzzy differential equations by extended Runge–Kutta-like formulae of order 4, Fuzzy Sets and Systems 189 (2011) 74–91.

[17] R. Goetschel, W. Voxman, Elementary fuzzy calculus, Fuzzy Sets and Systems 18 (1986) 31–43.

[18] M. Hanss, The transformation method for the simulation and analysis of systems with uncertain parameters, Fuzzy Sets and Systems 130 (2002) 277–289.

[19] E. Hullermeier, An approach to modelling and simulation of uncertain dynamical systems, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 5 (1997) 117–137.

[20] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems 24 (1987) 301–317.

[21] O. Kaleva, A note on fuzzy differential equations, Nonlinear Analysis 64 (2006) 895–900.

[22] A. Khastan, K. Ivaz, Numerical solution of fuzzy differential equations by Nystrom method, Chaos, Solitons & Fractals 41 (2008) 859–868.

[23] A. Khastan, J.J. Nieto, R. Rodrı´guez-Lopez, Periodic boundary value problems for first-order linear differential equations with uncertainty under generalized differentiability, Information Sciences 222 (2013) 544–558.

[24] G.J. Klir, Fuzzy arithmetic with requisite constraints, Fuzzy Sets and Systems 91 (1997) 165–175.

[25] W.A. Lodwick, K.D. Jamison, Special issue: interfaces between fuzzy set theory and interval analysis, Fuzzy Sets and Systems 135 (2003) 1–3.

[26] M. Ma, M. Friedman, A. Kandel, Numerical solution of fuzzy differential equations, Fuzzy Sets and Systems 105 (1999) 133–138.

[27] M.T. Mizukoshi, L.C. Barros, Y. Chalco-Cano, H. Roman-Flores, R.C. Bassanezi, Fuzzy differential equations and the extension principle, Information Sciences 177 (17) (2007) 3627–3635.

[28] R.E. Moore, W.A. Lodwick, Interval analysis and fuzzy set theory, Fuzzy Sets and Systems 135 (2003) 5–9.

[29] J.J. Nieto, A. Khastan, K. Ivaz, Numerical solution of fuzzy differential equations under generalized differentiability, Nonlinear Analysis: Hybrid Systems 3 (2009) 700–707.

[30] S.C. Palligkinis, G. Papageorgiou, I.T. Famelis, Runge–Kutta methods for fuzzy differential equations, Applied Mathematics and Computation 209 (2008) 97–105.

[31] M.L. Puri, D.A. Ralescu, Differentials of fuzzy functions, Journal of Mathematical Analysis and Applications 91 (1983) 552–558.

[32] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems 24 (1987) 319–330.

[33] S. Song, C. Wu, Existence and uniqueness of solutions to the Cauchy problem of fuzzy differential equations, Fuzzy Sets and Systems 110 (2000) 55–67.

[34] D. Vorobiev, S. Seikkala, Towards the theory of fuzzy differential equations, Fuzzy Sets and Systems 125 (2002) 231–237.

[35] K.L. Wood, K.N. Otto, E.K. Antonsoon, Engineering design calculation with fuzzy parameters, Fuzzy Sets and Systems 52 (1992) 1–20.

[36] C. Wu, S. Song, Existence theorem to the Cauchy problem of fuzzy differential equations under compactness-type conditions, Information Sciences 108 (1998) 123–134.

[37] C. Wu, S. Song, E.S. Lee, Approximate solutions, existence and uniqueness of the Cauchy problem of fuzzy differential equations, Journal of Mathematical Analysis and Applications 202 (1996) 629–644.

H.Q. Yang, H. Yao, J.D. Jones, Calculating functions on fuzzy numbers, Fuzzy Sets and Systems 55 (1993) 273–283.

[38] S. Sekar and K. Prabhavathi, "Numerical solution of first order linear fuzzy differential equations using Leapfrog method", *IOSR Journal of Mathematics*, vol. 10, no. 5 Ver. I, (Sep-Oct. 2014), pp. 07-12.

[39] S. Sekar and K. Prabhavathi, "Numerical Solution of Second Order Fuzzy Differential Equations by Leapfrog Method", *International Journal of Mathematics Trends and Technology*, vol. 16, no. 2, 2014, pp. 74-78.

[40] S. Sekar and K. Prabhavathi, "Numerical Strategies for the n^{th} –order fuzzy differential equations by Leapfrog Method", *International Journal of Mathematical Archive*, vol. 6, no. 1, 2014, pp. 162-168.

[41] S. Sekar and S. Senthilkumar, "Numerical Solution for hybrid fuzzy systems by single term Haar wavelet series technique", *International Journal of Mathematical Archive*, vol. 4, no. 11, 2013, pp. 23-29.

[42] S. Sekar and M. Vijayarakavan, "Numerical Investigation of first order linear Singular Systems using Leapfrog Method", *International Journal of Mathematics Trends and Technology*, vol. 12, no. 2, 2014, pp. 89-93.

[43] S. Sekar and M. Vijayarakavan, "Numerical Solution of Stiff Delay and Singular Delay Systems using Leapfrog Method", *International Journal of Scientific & Engineering Research*, vol. 5, no. 12, December-2014, pp. 1250-1253.

[44] M.Z. Ahmad, M.K. Hasan and B. De Baets, "Analytical and numerical solutions of fuzzy differential equations", *Information Sciences*, vol. 236 (2013) 156-167.

M. Rostami, M.Kianpour and E.Bashdoust, "A numerical algorithm for solving nonlinear fuzzy differential equations", *The Journal of Mathematics and Computer Science*, vol. 2, no. 4 (2011) 667-671.