Fixed Point Theorems for Self Maps Using Generalized Altering Distance under Contractive Condition of Integral Type

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Abstract — *In this paper, the result of Hosseini[2] is established and extended using generalized altering distance function of five, six and seven variables with weaker hypotheses.*

Keywords — *common fixed point, semi compatible, weakly compatible mappings.*

I. Introduction and Preliminaries

A number of problems in fixed point theory are tackled by many researchers using the concept of 'Altering distance' introduced by M.S.Khan et.al[4]. The existence and uniqueness of common fixed points of two semi-compatible pair of self-maps on a complete metric space, using generalized altering distance function of four real variables under a contractive condition of integral type is established by Hosseini[2]. P.S.Singh[5] has done interesting work on integral type in complete fuzzy metric space.

The main aim of this paper is to establish similar results by using generalized altering distance function of four, five, six and seven real variables with weaker hypotheses.

Further, we point out a loophole in the argument of the main result of Hosseini[2]. He claimed that if $\{y_n\}$ is a sequence in a metric space (X, d) and if $\{a_n\}$, where $a_n = d(y_n, y_{n+1})$ is a decreasing sequence converging to 'a' then $d(y_n, y_{n+2}) \to a$ as $n \to \infty$ and $a = 0$. But this is not true if $y_n = n$ with the usual metric, then $d(y_n, y_{n+1}) \to 1$ as $n \to \infty$ and $d(y_n, y_{n+2}) \to 2$ as $n \to \infty$.

Definition 1.1[3] *An ordered pair of self maps (f, g) on a metric space (X, d) is said to be semi-compatible if and only if (iff) for any sequence $\{x_n\}$ in X with*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z (z \in X)$$

implies

$$\lim_{n\to\infty} d(fgx_n, gz) = 0 (i.e. \lim_{n\to\infty} fgx_n = gz).$$

Definition 1.2 [1] *A pair of self maps $\{f, g\}$ on a metric space (X, d) is said to be weakly compatible if and only if (iff) $fx = gx$ for some $x \in X$ implies $fgx = gfx$.*

Observation 1.3 *An ordered pair of self maps (f, g) is semi-compatible implies that the pair is weakly compatible; but the converse is not true.*

Notation 1.4 *For any integer $k \geq 2$, let Ψ_k denote the set of all functions $\psi : [0, \infty)^k \to [0, \infty)$ such that*

1. ψ is continuous (on its domain),
2. ψ is monotonic increasing in all its variables,
3. for any $t_1, t_2, \cdots, t_k \in [0, \infty)$,
$$\psi(t_1, t_2, \cdots, t_k) = 0 \Leftrightarrow t_1 = t_2 = \cdots = t_k = 0$$
.

(each Ψ is called a generalized altering distance function)·

Now, define $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(x) = \psi(x, x, \cdots, x)$ for all $x \in \mathbb{R}^+$.

(Observe that $\phi(x) = 0 \Leftrightarrow x = 0$).

II. MAIN RESULTS

The central result of Hosseini[2] is the following:

Theorem 2.1 *Let (X, d) be a complete metric space and A, B, S and T be self maps on X such that*

1.
$$\int_0^{\phi(d(Ax,By))} \eta(t)dt \leq \int_0^{\psi_1\left(\begin{matrix} d(Ax,Sx), d(By,Ty), d(Sx,Ty), \\ \frac{1}{2}[d(Ax,Ty)+d(By,Sx)] \end{matrix}\right)} \eta(t)dt - \int_0^{\psi_2\left(\begin{matrix} d(Ax,Sx), d(By,Ty), d(Sx,Ty), \\ \frac{1}{2}[d(Ax,Ty)+d(By,Sx)] \end{matrix}\right)} \eta(t)dt$$

for all $x, y \in X$, where $\psi_1, \psi_2 \in \Psi_4$ and $\phi(u) = \psi_1(u, u, u, u), \forall u \in [0, \infty)$;

2. $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;
3. One of A, B, S and T is a continuous;
4. the ordered pairs (A, S) and (B, T) are semi-compatible;
5. $\eta: [0, \infty) \to [0, \infty)$ is Riemann integrable on any bounded, closed interval of $[0, \infty)$ and $\int_0^{\varepsilon} \eta(t)dt > 0, \forall \varepsilon > 0$.

Then A, B, S and T have a unique common fixed point in X.

Now, we generalize this theorem with weaker conditions.

Theorem 2.2 *Let f, g, L and M be self mappings of a metric space (X, d) such that*

1. $$\int_0^{\phi(d(fx,gy))} \eta(t)\,dt \le \int_0^{\psi_1\left(d(fx,Lx),d(gy,My),d(Lx,My),\frac{1}{2}[d(fx,My)+d(gy,Lx)],\ \frac{1}{3}[d(fx,Lx)+d(gy,My)+d(Lx,My)]\right)} \eta(t)\,dt - \int_0^{\psi_2\left(d(fx,Lx),d(y,My),d(Lx,My),\frac{1}{2}[d(fx,My)+d(gy,Lx)],\ \frac{1}{3}[d(fx,Lx)+d(gy,My)+d(Lx,My)]\right)} \eta(t)\,dt$$
   for all $x, y \in X$, where $\psi_1, \psi_2 \in \Psi_5$ and $\phi(u) = \psi_1(u,u,u,u,u)$, $\forall u \in [0,\infty)$;
2. $f(X) \subseteq M(X)$ and $g(X) \subseteq L(X)$;
3. One of $f(X)$, $g(X)$, $L(X)$ and $M(X)$ is a complete subspace of X;
4. the pairs $\{f, L\}$ and $\{g, M\}$ are weakly compatible;
5. $\eta: [0,\infty) \to [0,\infty)$ is R-integrable on any bounded, closed interval of $[0,\infty)$ and $\int_0^{\varepsilon} \eta(t)\,dt > 0, \forall \varepsilon > 0$.

Then f, g, L and M have a unique common fixed point in X.

Proof:

Let $x_0 \in X$. By (2) we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$fx_{2n} = Mx_{2n+1} = y_{2n}\,(say)$$

and

$$gx_{2n+1} = Lx_{2n+2} = y_{2n+1}(say),\ for\quad n = 0,1,2,\ldots$$

Let $a_n = d(y_n, y_{n+1})$.

Taking $x = x_{2n}$ and $y = x_{2n+1}$ in (1) we get that

$$\int_0^{\phi(d(y_{2n},y_{2n+1}))} \eta(t)\,dt \le \int_0^{\psi_1\left(d(y_{2n},y_{2n-1}),d(y_{2n+1},y_{2n}),d(y_{2n-1},y_{2n}),\frac{1}{2}[d(y_{2n},y_{2n})+d(y_{2n+1},y_{2n-1})],\ \frac{1}{3}[d(y_{2n},y_{2n-1})+d(y_{2n+1},y_{2n})+d(y_{2n-1},y_{2n})]\right)} \eta(t)\,dt$$

$$-\int_0^{\psi_2\left(d(y_{2n},y_{2n-1}),d(y_{2n+1},y_{2n}),d(y_{2n-1},y_{2n}),\frac{1}{2}[d(y_{2n},y_{2n})+d(y_{2n+1},y_{2n-1})],\ \frac{1}{3}[d(y_{2n},y_{2n-1})+d(y_{2n+1},y_{2n+})+d(y_{2n-1},y_{2n})]\right)} \eta(t)\,dt$$

$$i.e., \int_0^{\phi(a_{2n})} \eta(t)\,dt \le \int_0^{\psi_1\left(a_{2n-1},a_{2n},a_{2n-1},\frac{1}{2}[d(y_{2n+1},y_{2n-1})],\ \frac{1}{3}[a_{2n-1}+a_{2n}+a_{2n-1}]\right)} \eta(t)\,dt - \int_0^{\psi_1\left(a_{2n-1},a_{2n},a_{2n-1},\frac{1}{2}[d(y_{2n+1},y_{2n-1})],\ \frac{1}{3}[a_{2n-1}+a_{2n}+a_{2n-1}]\right)} \eta(t)\,dt$$

(2.2.1)

If $a_{2n-1} < a_{2n}$ then

$$\int_0^{\phi(a_{2n})} \eta(t)\,dt \le \int_0^{\psi_1(a_{2n},a_{2n},a_{2n},a_{2n},a_{2n})} \eta(t)\,dt - \int_0^{\psi_2\left(a_{2n-1},a_{2n},a_{2n-1},\frac{1}{2}[d(y_{2n+1},y_{2n-1})],\ \frac{1}{3}[a_{2n-1}+a_{2n}+a_{2n-1}]\right)} \eta(t)\,dt$$

$$\le \int_0^{\psi_1(a_{2n},a_{2n},a_{2n},a_{2n},a_{2n})} \eta(t)\,dt - \int_0^{\psi_2(a_{2n-1},a_{2n},a_{2n-1},0,\frac{1}{3}[a_{2n-1}+a_{2n}+a_{2n-1}])} \eta(t)\,dt$$

$$< \int_0^{\psi(a_{2n})} \eta(t)\,dt \qquad (since\ a_{2n} > 0)$$

which is a contradiction, so $a_{2n} \ge a_{2n-1}$.

Similarly, by taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in (1) we get that $a_{2n+1} \ge a_{2n}$. Hence $a_n \ge a_{n-1}, \forall n$. Thus $\{a_n\}$ is a decreasing sequence of non-negative real numbers and so converges to some $a \ge 0$.

Now, from (2.2.1), we have

$$\int_0^{\phi(a_{2n})} \eta(t)\,dt \le \int_0^{\psi_1\left(a_{2n-1},a_{2n},a_{2n-1},\frac{1}{2}[a_{2n}+a_{2n-1}],\ \frac{1}{3}[a_{2n-1}+a_{2n}+a_{2n-1}]\right)} \eta(t)\,dt - \int_0^{\psi_2(a_{2n-1},a_{2n},a_{2n-1},0,\frac{1}{3}[a_{2n-1}+a_{2n}+a_{2n-1}])} \eta(t)\,dt.$$

Letting $n \to \infty$, we get that

$$\int_0^{\phi(a)} \eta(t)\,dt \le \int_0^{\psi_1(a,a,a,a,a)} \eta(t)\,dt - \int_0^{\psi_2(a,a,a,0,a)} \eta(t)\,dt$$

.

By the property of ψ, we have $\psi_2(a,a,a,0,a) = 0 \Leftrightarrow a = 0$ (by the property of ψ_2).

$$i.e., \lim_{n\to\infty} d(y_n, y_{n+1}) = 0 \qquad (2.2.2)$$

Now, we show that $\{y_n\}$ is a Cauchy sequence (in X); in view of (2.2.2), it is sufficient to show that the subsequence $\{y_{2n}\}$ of $\{y_n\}$ is Cauchy.

Suppose not there exists an $\varepsilon > 0$ and subsequences $\{y_{2n(k)}\}$ and $\{y_{2m(k)}\}$ such that $n(k) > m(k) \ge k$ and

$$d(y_{2m(k)}, y_{2n(k)}) \ge \varepsilon \qquad (2.2.3)$$

Further, we can assume that

$$d\left(y_{2m(k)}, y_{2n(k)-2}\right) < \varepsilon \qquad (2.2.4)$$

(by choosing $n(k)$ to be the smallest number exceeding $m(k)$ for which (2.2.3) holds).

Now,

$$\varepsilon \le d(y_{2m(k)}, y_{2n(k)})$$

$$\le d(y_{2m(k)}, y_{2n(k)-2}) + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)})$$

$$< \varepsilon + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}).$$

Letting $k \to \infty$ and using (2.2.2), we get that

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) = \varepsilon \qquad (2.2.5)$$

Further, we have

$$d(y_{2n(k)}, y_{2m(k)-1}) \le d(y_{2n(k)}, y_{2m(k)}) + d(y_{2m(k)}, y_{2m(k)-1})$$

and

$$d(y_{2n(k)}, y_{2m(k)}) \le d(y_{2n(k)}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}).$$

Letting $k \to \infty$ in the above inequalities, by virtue of (2.2.2) and (2.2.5), we get that

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}) = \varepsilon \qquad (2.2.6)$$

Similarly, we show that

$$\lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1}) = \varepsilon \qquad (2.2.7)$$

and

$$\lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)}) = \varepsilon \qquad (2.2.8).$$

Taking $x = x_{2m(k)}$ and $y = x_{2n(k)+1}$ in (1), we get that

$$\int_0^{\phi(d(y_{2m(k)}, y_{2n(k)+1}))} \eta(t)dt \le \int_0^{\psi_1\left(d(y_{2m(k)}, y_{2m(k)-1}), d(y_{2n(k)+1}, y_{2n(k)}), d(y_{2m(k)-1}, y_{2n(k)}), \frac{1}{2}[d(y_{2m(k)}, y_{2n(k)}) + d(y_{2n(k)+1}, y_{2m(k)-1})], \frac{1}{3}[d(y_{2m(k)}, y_{2m(k)-1}) + d(y_{2n(k)+1}, y_{2n(k)}) + d(y_{2m(k)-1}, y_{2n(k)})]\right)} \eta(t)dt$$

$$- \int_0^{\psi_2\left(d(y_{2m(k)}, y_{2m(k)-1}), d(y_{2n(k)+1}, y_{2n(k)}), d(y_{2m(k)-1}, y_{2n(k)}), \frac{1}{2}[d(y_{2m(k)}, y_{2n(k)}) + d(y_{2n(k)+1}, y_{2m(k)-1})], \frac{1}{3}[d(y_{2m(k)}, y_{2m(k)-1}) + d(y_{2n(k)+1}, y_{2n(k)}) + d(y_{2m(k)-1}, y_{2n(k)})]\right)} \eta(t)dt$$

Letting $k \to \infty$ and using (2.2.2),(2.2.6)(2.2.7) and (2.2.8), we get that

$$\int_0^{\phi(\varepsilon)} \eta(t)dt \le \int_0^{\psi_1(0,0,\varepsilon,\varepsilon,\frac{1}{3}\varepsilon)} \eta(t)dt - \int_0^{\psi_2(0,0,\varepsilon,\varepsilon,\frac{1}{3}\varepsilon)} \eta(t)dt$$

$$\le \int_0^{\psi_1(\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon)} \eta(t)dt - \int_0^{\psi_2(0,0,\varepsilon,\varepsilon,\frac{1}{3}\varepsilon)} \eta(t)dt$$

$$< \int_0^{\psi(\varepsilon)} \eta(t)dt$$

which is a contradiction, since $\varepsilon > 0$. Thus $\{y_{2n}\}$ is a Cauchy sequence and hence $\{y_n\}$ is a Cauchy sequence (in X).

Case I: Suppose $f(X)$ or $M(X)$ is a complete subspace of X.

Since $\{y_{2n}\} \subseteq f(X) (\subseteq M(X))$, there is a $z \in X$ such that $\{y_{2n}\} \to z$ as $n \to \infty$. $\Rightarrow \{y_n\} \to z$ as $n \to \infty$. (further, it follows that $\{y_{2n+1}\} \to z$ as $n \to \infty$).

Since $f(X) \subseteq M(X)$, there is a $v \in X$ such that $z = Mv$.

By taking $x = x_{2n}$ and $y = v$ in (1), we get that

$$\int_0^{\phi(d(y_{2n}, gv))} \eta(t)dt \le \int_0^{\psi_1\left(d(y_{2n}, y_{2n-1}), d(gv, z), d(y_{2n-1}, z), \frac{1}{2}[d(y_{2n}, z) + d(gv, y_{2n-1})], \frac{1}{3}[d(y_{2n}, y_{2n-1}) + d(gv, z) + d(y_{2n-1}, z)]\right)} \eta(t)dt$$

$$- \int_0^{\psi_2\left(d(y_{2n}, y_{2n-1}), d(gv, z), d(y_{2n-1}, z), \frac{1}{2}[d(y_{2n}, z) + d(gv, y_{2n-1})], \frac{1}{3}[d(y_{2n}, y_{2n-1}) + d(gv, z) + d(y_{2n-1}, z)]\right)} \eta(t)dt$$

Letting $n \to \infty$, we get that

$$\int_0^{\phi(d(z, gv))} \eta(t)dt \le \int_0^{\psi_1(0, d(gv,z), 0, \frac{1}{2}d(gv,z), \frac{1}{3}d(gv,z))} \eta(t)dt - \int_0^{\psi_2(0, d(gv,z), 0, \frac{1}{2}d(gv,z), \frac{1}{3}d(gv,z))} \eta(t)dt$$

$$\int_0^{\phi(d(z, gv))} \eta(t)dt \le \int_0^{\psi_1(d(gv,z), d(gv,z), d(gv,z), d(gv,z), d(gv,z))} \eta(t)dt - \int_0^{\psi_2(0, d(gv,z), 0, \frac{1}{2}d(gv,z), \frac{1}{3}d(gv,z))} \eta(t)dt$$

By the properties of ψ_1 and η, it follows that $\psi_2\left(0, d(gv,z), 0, \frac{1}{2}d(gv,z), \frac{1}{3}d(gv,z)\right) = 0$.

Now, by the property of ψ_2, we get that $gv = z$. Thus $gv = z = Mv$.

Since $\{g, M\}$ is weakly compatible, $gMv = Mgv$. i.e, $gz = Mz$.

By taking $x = x_{2n}$ and $y = z$ in (1) we get that

$$\int_0^{\phi(d(y_{2n},gz))} \eta(t)dt \leq \int_0^{\psi_1\left(\begin{array}{l} d(y_{2n},y_{2n-1}),d(gz,gz),d(y_{2n-1},gz),\frac{1}{2}[d(y_{2n},gz)+d(gz,y_{2n-1})], \\ \frac{1}{3}[d(y_{2n},y_{2n-1})+d(gz,gz)+d(y_{2n-1},gz)] \end{array}\right)} \eta(t)dt$$

$$- \int_0^{\psi_2\left(\begin{array}{l} d(y_{2n},y_{2n-1}),d(gz,gz),d(y_{2n-1},gz),\frac{1}{2}[d(y_{2n},gz)+d(gz,y_{2n-1})], \\ \frac{1}{3}[d(y_{2n},y_{2n-1})+d(gz,gz)+d(y_{2n-1},gz)] \end{array}\right)} \eta(t)dt$$

Letting $n \to \infty$, we get that

$$\int_0^{\phi(d(z,gz))} \eta(t)dt \leq \int_0^{\psi_1(0,0,d(gz,z),d(gz,z),\frac{1}{3}d(gz,z))} \eta(t)dt - \int_0^{\psi_2(0,0,d(gz,z),d(gz,z),\frac{1}{3}d(gz,z))} \eta(t)dt$$

$$\leq \int_0^{\psi_1(d(gz,z),d(gz,z),d(gz,z),d(gz,z),d(gz,z))} \eta(t)dt - \int_0^{\psi_2(0,0,d(gz,z),d(gz,z),\frac{1}{3}d(gz,z))} \eta(t)dt$$

$\Rightarrow d(gz,z) = 0 \Rightarrow gz = z$. Thus $gz = Mz = z$.

Since $g(X) \subseteq L(X)$, there is a $w \in X$ such that $z = Lw$.

By taking $x = w$ and $y = x_{2n+1}$ in (1) we get that

$$\int_0^{\phi(d(fw,y_{2n+1}))} \eta(t)dt \leq \int_0^{\psi_1\left(\begin{array}{l} d(fw,z),d(y_{2n+1},y_{2n}),d(z,y_{2n}),\frac{1}{2}[d(fw,y_{2n})+d(y_{2n+1},z)], \\ \frac{1}{3}[d(fw,z)+d(y_{2n+1},y_{2n})+d(z,y_{2n})] \end{array}\right)} \eta(t)dt$$

$$- \int_0^{\psi_2\left(\begin{array}{l} d(fw,z),d(y_{2n+1},y_{2n}),d(z,y_{2n}),\frac{1}{2}[d(fw,y_{2n})+d(y_{2n+1},z)], \\ \frac{1}{3}[d(fw,z)+d(y_{2n+1},y_{2n})+d(z,y_{2n})] \end{array}\right)} \eta(t)dt$$

Letting $n \to \infty$, we get that

$$\int_0^{\phi(d(fw,z))} \eta(t)dt \leq \int_0^{\psi_1(d(fw,z),0,0,\frac{1}{2}d(fw,z),\frac{1}{3}d(fw,z))} \eta(t)dt - \int_0^{\psi_2(d(fw,z),0,0,\frac{1}{2}d(fw,z),\frac{1}{3}d(fw,z))} \eta(t)dt$$

$$\leq \int_0^{\psi_1(d(fw,z),d(fw,z),d(fw,z),d(fw,z),d(fw,z))} \eta(t)dt - \int_0^{\psi_2(d(fw,z),0,0,\frac{1}{2}d(fw,z),\frac{1}{3}d(fw,z))} \eta(t)dt$$

$$\int_0^{\phi(d(fw,z))} \eta(t)dt \leq \int_0^{\psi_1(d(fw,z),0,0,\frac{1}{2}d(fw,z),\frac{1}{3}d(fw,z))} \eta(t)dt - \int_0^{\psi_2(d(fw,z),0,0,\frac{1}{2}d(fw,z),\frac{1}{3}d(fw,z))} \eta(t)dt$$

$$\leq \int_0^{\psi_1(d(fw,z),d(fw,z),d(fw,z),d(fw,z),d(fw,z))} \eta(t)dt - \int_0^{\psi_2(d(fw,z),0,0,\frac{1}{2}d(fw,z),\frac{1}{3}d(fw,z))} \eta(t)dt$$

$\Rightarrow d(fw,z) = 0 \Rightarrow fw = z$. So $fw = z = Lw$.

Since $\{f, L\}$ is weakly compatible, $fLw = Lfw$.

i.e, $fz = Lz$.

By taking $x = z$ and $y = x_{2n+1}$ in (1) we get that

$$\int_0^{\phi(d(fz,y_{2n+1}))} \eta(t)dt \leq \int_0^{\psi_1\left(\begin{array}{l} d(fz,fz),d(y_{2n+1},y_{2n}),d(fz,y_{2n}),\frac{1}{2}[d(fz,y_{2n})+d(y_{2n+1},fz)], \\ \frac{1}{3}[d(fz,fz)+d(y_{2n+1},y_{2n})+d(fz,y_{2n})] \end{array}\right)} \eta(t)dt$$

$$- \int_0^{\psi_2\left(\begin{array}{l} d(fz,fz),d(y_{2n+1},y_{2n}),d(fz,y_{2n}),\frac{1}{2}[d(fz,y_{2n})+d(y_{2n+1},fz)], \\ \frac{1}{3}[d(fz,fz)+d(y_{2n+1},y_{2n})+d(fz,y_{2n})] \end{array}\right)} \eta(t)dt$$

Letting $n \to \infty$, we get that

$$\int_0^{\phi(d(fz,z))} \eta(t)dt \leq \int_0^{\psi_1(0,0,d(fz,z),d(fz,z),\frac{1}{3}d(fz,z))} \eta(t)dt - \int_0^{\psi_2(0,0,d(fz,z),d(fz,z),\frac{1}{3}d(fz,z))} \eta(t)dt$$

$$\leq \int_0^{\psi_1(d(fz,z),d(fz,z),d(fz,z),d(fz,z),d(fz,z))} \eta(t)\,dt - \int_0^{\psi_2\left(0,0,d(fz,z),d(fz,z),\frac{1}{3}d(fz,z)\right)} \eta(t)dt$$

$\Rightarrow \quad d(fz,z) = 0 \Rightarrow fz = z$. Thus $fz = Lz = z$.

Case II: Suppose $g(X)$ or $L(X)$ is a complete subspace of X.

In this case, we first prove that $fz = Lz = z$ and then $gz = Mz = z$.

Thus z is a common fixed point of f, g, L and M in X.

Uniqueness: If z' is also a common fixed point of f, g, L and M in X.

By taking $x = z$ and $y = z'$ in (1) we get that $z' = z$.

Hence the other claim also follows.

This completes the proof of the theorem.

The following results are just extension of Theorem(2.2) and their proofs run on similar lines

Theorem 2.3 *Let f, g, L and M be self mappings of a metric space (X, d) such that*

1. $$\int_0^{\phi(d(fx,gy))} \eta(t)dt \leq \int_0^{\psi_1\left(\begin{array}{l} d(fx,Lx),d(gy,My),d(Lx,My),\frac{1}{2}d(fx,My),\frac{1}{2}d(gy,Lx), \\ \frac{1}{3}[d(fx,Lx)+d(gy,My)+d(Lx,My)] \end{array}\right)} \eta(t)dt - \int_0^{\psi_2\left(\begin{array}{l} d(fx,Lx),d(gy,My),d(Lx,My),\frac{1}{2}d(fx,My),\frac{1}{2}d(gy,Lx), \\ \frac{1}{3}[d(fx,Lx)+d(gy,My)+d(Lx,My)] \end{array}\right)} \eta(t)dt$$

   for all $x, y \in X$, where $\psi_1, \psi_2 \in \Psi_6$ and $\phi(u) = \psi_1(u,u,u,u,u,u)$, $\forall u \in [0,\infty)$;
2. $f(X) \subseteq M(X)$ and $g(X) \subseteq L(X)$;
3. One of $f(X)$, $g(X)$, $L(X)$ and $M(X)$ is a complete subspace of X;
4. the pairs $\{f, L\}$ and $\{g, M\}$ are weakly compatible;
5. $\eta: [0,\infty) \rightarrow [0,\infty)$ is R-integrable on any bounded, closed interval of $[0,\infty)$ and $\int_0^{\varepsilon} \eta(t)dt > 0, \forall \varepsilon > 0$.

Then f, g, L and M have a unique common fixed point in X.

Proof: Similar to Theorem(2.2).

Theorem 2.4 *Let f, g, L and M be self mappings of a metric space (X, d) such that*

1. $$\int_0^{\phi(d(fx,gy))} \eta(t)dt \quad \int_0^{\psi_1\left(\begin{array}{l} d(fx,Lx),d(gy,My),d(Lx,My),\frac{1}{2}d(fx,My),\frac{1}{2}d(gy,Lx),\frac{1}{3}[d(fx,Lx)+d(gy,My)+d(Lx,My)], \\ \frac{1}{5}[d(fx,Lx)+d(gy,My)+d(Lx,My)+\frac{1}{2}(d(fx,My)+d(gy,Lx))] \end{array}\right)} \eta(t)dt - \int_0^{\psi_2\left(\begin{array}{l} d(fx,Lx),d(gy,My),d(Lx,My),\frac{1}{2}d(fx,My),\frac{1}{2}d(gy,Lx),\frac{1}{3}[d(fx,Lx)+d(gy,My)+d(Lx,My)], \\ \frac{1}{5}[d(fx,Lx)+d(gy,My)+d(Lx,My)+\frac{1}{2}(d(fx,My)+d(gy,Lx))] \end{array}\right)} \eta(t)dt$$

   for all $x, y \in X$, where $\psi_1, \psi_2 \in \Psi_7$ and $\phi(u) = \psi_1(u,u,u,u,u,u,u)$, $\forall u \in [0,\infty)$;
2. $f(X) \subseteq M(X)$ and $g(X) \subseteq L(X)$;
3. One of $f(X)$, $g(X)$, $L(X)$ and $M(X)$ is a complete subspace of X;
4. the pairs $\{f, L\}$ and $\{g, M\}$ are weakly compatible;
5. $\eta: [0,\infty) \rightarrow [0,\infty)$ is R-integrable on any bounded, closed interval of $[0,\infty)$ and $\int_0^{\varepsilon} \eta(t)dt > 0, \forall \varepsilon > 0$.

Then f, g, L and M have a unique common fixed point in X.

Proof: Similar to Theorem(2.2).

We conclude our paper with the following example in support of our Theorem (2.2).

Example 2.5 *Let $X = \mathbb{Q}^+ \cup \{0\}$, the set of all non-negative rational numbers, with the usual metric. f, g, L and M be the self maps defined on X by*

$$f(x) = \begin{cases} 0 & \text{if } x \leq 2, \\ 1 & \text{if } x > 2, \end{cases}$$

$$L(x) = \begin{cases} 0 & \text{if } x \leq 2, \\ x^2 & \text{if } x > 2 \end{cases}$$

$gx = 0$ and $Mx = x$ for all $x \in X$.

Define $\psi_1, \psi_2: [0,\infty)^5 \rightarrow [0,\infty)$ by $\psi_1(t_1,t_2,t_3,t_4,t_5)$ =(max $\{t_1,t_2,t_3\} + t_4 + t_5)/3$ and $\psi_2 = \frac{1}{2}\psi_1$.

Define $\phi:[0,\infty)\to[0,\infty)$ by $\phi(t)=t$ and η by $\eta(t)=1, \forall t$.

Case I: $x\le 2$ and $y\in X$.

L.H.S= $\int_0^{\phi(d(fx,gy))}\eta(t)dt=\int_0^{\phi(0)}\eta(t)dt=\int_0^0 1=0$

R.H.S=

$$\int_0^{\psi_1(d(Lx,My),d(Lx,fx),d(gy,My),\frac{1}{2}(d(fx,My)+d(Lx,gy)),\frac{1}{3}(d(Lx,My)+d(Lx,fx)+d(gy,My))}\eta(t)dt - \int_0^{\psi_2(d(Lx,My),d(Lx,fx),d(gy,My),\frac{1}{2}(d(fx,My)+d(Lx,gy)),\frac{1}{3}(d(Lx,My)+d(Lx,fx)+d(gy,My))}\eta(t)dt$$

$$=\int_0^{\psi_1(y,0,y,\frac{1}{2}(y),\frac{1}{3}(y+y))}dt - \int_0^{\psi_2(y,0,y,\frac{1}{2}(y),\frac{1}{3}(y+y))}dt$$

$$=\frac{1}{6}\left[y+\frac{1}{2}(y)+\frac{1}{3}(2y)\right]\ge 0=L.H.S,$$ since $\psi_2=\frac{1}{2}\psi_1$.

Case II: $x>2$ and $y\in X$.

L.H.S= $\int_0^{\phi(d(fx,gy))}\eta(t)dt=\int_0^{\phi(1)}\eta(t)dt=\int_0^1 1=1.$

R.H.S=

$$\int_0^{\psi_1(d(Lx,My),d(Lx,fx),d(gy,My),\frac{1}{2}(d(fx,My)+d(Lx,gy)),\frac{1}{3}(d(Lx,My)+d(Lx,fx)+d(gy,My))}\eta(t)dt - \int_0^{\psi_2(d(Lx,My),d(Lx,fx),d(gy,My),\frac{1}{2}(d(fx,My)+d(Lx,gy)),\frac{1}{3}(d(Lx,My)+d(Lx,fx)+d(gy,My))}\eta(t)dt$$

$$=\int_0^{\psi_1(|x^2-y|,x^2-1,|y|,\frac{1}{2}(|1-y|+x^2),\frac{1}{3}(|x^2-y|+(x^2-1)+|y|)}dt - \int_0^{\psi_2(|x^2-y|,x^2-1,|y|,\frac{1}{2}(|1-y|+x^2),\frac{1}{3}(|x^2-y|+(x^2-1)+|y|)}dt$$

$$=\frac{1}{3}\left[\max\{|x^2-y|,x^2-1,|y|\}+\frac{1}{2}(|1-y|+x^2)+\frac{1}{3}(|x^2-y|+(x^2-1)+|y|)\right] -\frac{1}{6}\left[\max\{|x^2-y|,x^2-1,|y|\}+\frac{1}{2}(|1-y|+x^2)+\frac{1}{3}(|x^2-y|+(x^2-1)+|y|)\right]$$

$$=\frac{1}{6}\left[\max\{|x^2-y|,x^2-1,|y|\}+\frac{1}{2}(|1-y|+x^2)+\frac{1}{3}(|x^2-y|+(x^2-1)+|y|)\right]$$

$$\ge\frac{1}{6}\left[(x^2-1)+\frac{1}{2}(x^2)+\frac{1}{3}(x^2-1)\right]$$

$$=\frac{1}{6}\left[\frac{11}{6}x^2-\frac{4}{3}\right]$$

$$\ge\frac{1}{6}\left[\frac{11}{6}(4)-\frac{4}{3}\right]$$

$$=1$$

Thus L.H.S $\le$ R.H.S.

The other conditions of the Theorem are trivially satisfied.

Clearly '0' is the unique common fixed point of f, g, L and M (in X).

(Observe that X is not complete.)

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