Domination Uniform Subdivision Number of Graph

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Abstract - *Let $G = (V, E)$ be a simple undirected graph. A subset D of $V(G)$ is said to be dominating set if every vertex of $V(G) - D$ is adjacent to at least one vertex in D. The minimum cardinality taken over all minimal dominating sets of G is the domination number of G and is denoted by $\gamma(G)$. The domination uniform subdivision number of G is the least positive integer k such that the subdivision of any k edges from G results in a graph having domination number greater than that of G and is denoted by $usd_\gamma(G)$. In this paper, we investigate the domination uniform subdivision number of some standard graphs. Also we determine the bounds of usd_γ and characterize the extremal graphs.*

Keywords - *domination number, domination uniform subdivision number.*
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I. Introduction

Let $G = (V, E)$ be a simple undirected graph of order n and size m. If $v \in V(G)$, then the *neighbourhood* of v is the set $N(v)$ consisting of all vertices u which are adjacent to v. The *closed neighbourhood* is $N[v] = N(v) \cup \{v\}$. The *degree* of v in G is $|N(v)|$ and is denoted by $deg(v)$. The *maximum degree* of G is max $\{deg(v) : v \in V(G)\}$ and is denoted by (G). A vertex v is said to be *full vertex* if $deg(v) = n - 1$. A vertex v is said to be *pendant vertex* if $deg(v) = 1$. A vertex is called *support* if it is adjacent to a pendant. A support is said to be *strong support* if it is adjacent to more than one pendent. A subgraph F of a graph G is called an *induced subgraph* of G if whenever u and v are vertices of F and uv is an edge of G, then uv is an edge of F as well. If F is induced by a vertex set $V' \subset V(G)$, then F can be represented as $< V' >$.

A path, a cycle, and a complete graph on n vertices are denoted by P_n, C_n and K_n respectively. A graph is said to be *connected* if there exists a path between any pair of vertices. Otherwise it is said to be *disconnected*. *Tree* is a connected acyclic graph. A tree T is said to be *caterpillar* if removel of leaves from T gives path. A graph G is a *k-partite* graph if $V(G)$ can be partitioned into k subsets $V_1, V_2, \ldots, V_k$ such that uv is an edge of G if u and v belong to different partite sets. If, in addition, every two vertices in different partite sets are joined by an edge, then G is a *complete k-partite* graph. If $|Vi| = n_i$ for $1 \leq i \leq k$, then we denote such graphs by $K_{n_1, n_2, \ldots, n_k}$. It is also known as complete multipartite graph. A bipar-tite graph is denoted by $K_{r,s}$.

The *corona* of two graphs G_1 and G_2, is the graph $G = G_1 \circ G_2$, formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.

A set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V - S$ is adjacent to at least one vertex in S. The minimum cardinality taken over all dominating sets of G is called the *domination number* of $G$and is denoted by $\gamma(G)$.

The domination subdivision number introduced by Arumugua Velammal in [10]. It's bound was obtained in [2] and several authors characterised trees according to their domination subdivision number. Also many results have also been obtained on the parameters sd_{dd}, $sd_{\gamma c}$ and $sd_{\gamma t}$.

An edge $uv \in E(G)$ is subdivided if the edge uv is deleted, a new vertex x (called a subdivision vertex) is added, along the new edges: ux and xv. A derived graph G' of G by subdividing all the edges of $E \subseteq E(G)$ is denoted by $G \parallel E'$. A subdivision graph $S(G)$ of a graph G is obtained from G by subdividing all the edges exactly once. Subdivision graph of star graph $K_{1,r}$ is said to be spider. A graph is said to be wounded spider if it is obtained by subdividing at most r - 1 edges of

$K_{1,r}$. The domination subdivision number is defined by $sd_\gamma(G) = min\{|E'| : \gamma(G||E') > \gamma(G)\}$. In [7] and [8], two different graph theoretical concepts were studied on subdivision graphs. In this paper, generalized definition of domination uniform subdivision number.

II. EXACT VALUE FOR SOME STANDARD GRAPHS

In this section, we define domination uniform subdivision number and obtained exact value for some standard graphs.

Definition 2.1. A domination uniform subdivision number of G is the least positive integer k such that the sub division of any k edges from G results in a graph having domination number greater than that of G and is denoted by $usd_\gamma(G)$. If it is not exists, then $usd_\gamma(G) = 0$.

Definition 2.2. A subset $S \subseteq E(G)$ is said to be stable subdivision set if $\gamma(G \| S) = \gamma(G)$. A stable subdivision set S is said to be maximum stable subdivision set if there is no stable subdivision set S' such that $|S'| > |S|$.

Remark 2.3. $usd_\gamma(G) = |S| + 1$, where S is a maximum stable subdivision set of G.

Theorem 2.4. *If* $\Delta(G) = n - 1$, *then* $usd_\gamma(G) = 1$.

Proof. Since $\Delta(G) = n-1$, $\gamma(G) = 1$. Let $e \in E(G)$. Then $\Delta(G \|\{e\}) = n - 2$ and so $\gamma(G\|\{e\}) \geq 2$. Hence $\gamma(G) < \gamma(G\|\{e\})$ for every $e \in E(G)$. Thus $usd_\gamma(G) = 1$.

Theorem 2.5. *For* $n \geq 3$, $usd_\gamma(P_n) = \begin{cases} 1 & \text{if } n \cong 0 \text{ mod } 3 \\ 3 & \text{if } n \cong 1 \text{ mod } 3 \\ 2 & \text{if } n \cong 2 \text{ mod } 3 \end{cases}$

Proof. **Case (i)** : $n \cong 0$ mod 3

Then $n = 3k$, $k \in N$. Therefore $\gamma(P_n) = k$. Let $e \in E(P_n)$. Then $P_n\|\{e\} = P_{n+1}$ and hence $\gamma(P_n\|\{e\}) = k + 1$. Therefore $\gamma(P_n\|\{e\}) > \gamma(P_n)$ for any edge $e \in E(Pn)$.Hence $usd_\gamma(P_n) = 1$.

Case (ii) : $n \cong 1$ mod 3

Then $n = 3k + 1$, $k \in N$. Therefore $\gamma(Pn) = k + 1$. Let $e \in (Pn)$. Then $P_n \|\{e\} = P_{n+1}$ and so $\gamma(P_n \| \{e\}) = k + 1$. Let $e, e' \in E(P_n)$. Then $P_n \|\{e, e'\} = P_{n+2}$ and hence $\gamma(P_n\|\{e, e'\}) = k + 1$. Let $e'' \in E(P_n)$. Then $P_n\|\{e, e', e''\} = P_{n+3}$ and so $\gamma(P_n\|\{e, e', e''\}) = k + 2$. Hence $\gamma(P_n\|\{e, e', e''\}) > \gamma(P_n)$ for any three edges e, e', e''. Thus $usd_\gamma(P_n) = 3$.

Case (iii): $n \cong 2$ mod 3

Then $n = 3k + 2$, $k \in N$. Therefore $\gamma(Pn) = k + 1$. Let $e \in E(P_n)$. Then $P_n\|\{e\} = P_{n+1}$ and hence $\gamma(P_n\|\{e\}) = k + 1$. Let $e, e' \in E(P_n)$. Then $P_n\|\{e, e'\} = P_{n+2}$ and so $\gamma(P_n\|\{e, e'\}) = k +2$. Hence $\gamma(Pn\|\{e, e'\}) > \gamma(Pn)$ for any two edges e, e' of P_n. Thus $usd_\gamma(P_n) = 2$.

Theorem 2.5. $usd_\gamma(C_n) = \begin{cases} 1 & \text{if } n \cong 0 \text{ mod } 3 \\ 3 & \text{if } n \cong 1 \text{ mod } 3 \\ 2 & \text{if } n \cong 2 \text{ mod } 3 \end{cases}$

Proof. Proof is similar to the proof of Theorem 2.5.

Theorem 2.7. $usd_\gamma\left(K_{n_1,n_2,\ldots,n_r}\right) = 1$ *for some* $n_i = 1$.

Proof. Since $\Delta\left(K_{n_1,n_2,\ldots,n_r}\right) = n-1$, where $n = n_1 + n_2 + \ldots + n_r$, by Theorem 2.4 $usd_\gamma\left(K_{n_1,n_2,\ldots,n_r}\right) = 1$.

Corollary 2.8. $usd_\gamma(K_{1,r}) = 1$, *for all* $r \geq 2$.

Theorem 2.9. $usd_\gamma\left(K_{n_1,n_2,\ldots,n_r}\right) = \sum_{i=1}^{r} n_i - 1$ *with* $n_k = 2$ *and* $n_i > 1$ *for all* i.

Proof. Let $G = K_{n_1,n_2,\ldots,n_r}$. Then $\gamma(G) = 2$. Let $V_1, V_2, \ldots, V_r$ be partition of vertex set $V(G)$, $|V_i| = n_i$ and $V_k = \{u_1, u_2\}$. Let $E' = \{e \in E(G) \,/\, e = u_1v_i$ or $e = u_2v_j$ and $v_i \neq v_j\}$. Take $G' = G \| E'$. Then $|E'| = n_1 + n_2 + \ldots + n_{k-1} + n_{k+1} + \ldots + n_r = \sum_{i=1}^{r} n_i - 2$. In G', u_1 is adjacent to all the new vertices correspond to the subdivided edges which are incident with u_1 in G and all the vertices in $N_G(u_1)$ which are not adjacent to u_2 in G'. Similarly u_2 is adjacent to all the new vertices correspond to the subdivided edges which are incident with u_2 in G and all the vertices in $N_G(u_2)$ which are not adjacent to u_1 in G'. Therefore $\gamma(G') = \gamma(G) = 2$. Therefore, E' is a stable subdivision set of G. Now, we are going to prove that E' is a maximum stable subdivided set of G. Suppose there exists a stable subdivision set E'' of G such that $|E''| > |E'|$. If $E'' \supset E'$, then E'' can not be a stable subdivision set since $\gamma(G \|\{E' \cup \{e\}\}) > \gamma(G)$ for any $e \in E(G) - E'$. Therefore $E'' \neq E'$. Then E'' contains at least two edges which are not in E'.

Case (i): $E' \cap E'' = \phi$.

Since E'' is stable subdivision set, E'' does not contain independent set of cardinality more than 2. Therefore there exists two vertices x and y such that every edges

of E'' adjacent to either x or y. By the definition of E'', x
and y do not belong to the same partite set. Therefore
x and y belong to the two different partite sets V_i and V_j respectively. Without loss of generality assume that
$i < j < k$. Then $|E''| = n_1 + n_2 + \ldots + n_{i-1} + n_{i+1} + \ldots +$
$n_{j-1} + n_{j+1} + \ldots + n_{k-1} + n_{k+1} < \sum_{i=1}^{r} n_i - 2 = |E'|$
which is contradiction.

Case (ii): $E' \cap E'' \neq \phi$.

Since E'' contains at least two edges which are not in E', E'' has three independent edges. Therefore E'' is not stable subdivision set which is a contradiction.
Hence E' is the maximum stable subdivision et of G. Thus $usd_{\gamma}(G) = \sum_{i=1}^{r} n_i - 2 + 1 = \sum_{i=1}^{r} n_i - 1$.

Corollary 2.10. *$usd_{\gamma}(K_{2,r}) = r + 1$ for $r \geq 2$.*

Theorem 2.11. *$usd_{\gamma}\left(K_{n_1,n_2,\ldots,n_r}\right) = \sum_{i=1}^{r-2} n_i + 2$ where $n_i \geq n_{i+1} > 2$ for all i.*

Proof. Let $G = K_{n_1,n_2,\ldots,n_r}$, where $n_1 \geq n_2 \geq \ldots n_r > 2$ and $V_1, V_2, \ldots, V_r$ be partition of $V(G)$ with $|V_i| = n_i$. We have $\gamma(G) = 2$. Let $V_r = \{u_1, u_2, \ldots, u_{n_r}\}$ and $V_{r-1} = \{v_1, v_2, \ldots, v_{n_{r-1}}\}$. Fix $u \in V_r$ and $v \in V_{r-1}$. Now consider a set $E' = \{e \in E(G) \,/\, e = ux_j$ where $x_j \in V_k$, $k \neq n_{r-1}$ or $e = vy_j$ where $y_j \in V_k$, $k \neq n_r$ and $x_j \neq y_j\} \cup \{uv\}$. Then $|E'| = n_1 + n_2 + \ldots + n_{r-2} + 1$. Let us take $G' = G||E'$. Then u is adjacent to all the new vertices which correspond to subdivided edge having u as an end vertex in G and all the y_j's. Also v is adjacent to all the new vertices which correspond to subdivided edge having v as an end vertex in G and all the x_j's. Hence $\gamma(G') = 2 = \gamma(G)$. Let $E^* = E' \cup \{e\}$, $e \in E(G) \setminus E'$. Then
e is any one of the following form
(i) $e = u_iv_j$, where $u_i \neq u$ and $v_j \neq v$
(ii) $e = uy_j$, where $y_j \in V_k \cap N(v)$ and $k \neq n_r$
(iii) $e = vx_j$, where $x_j \in V_k \cap N(u)$ and $k \neq n_{r-1}$
(iv) e is incident with neither u nor v.
In the above four possibilities, there is no two element dominating set of $G||E^*$ and hence $\gamma(G||E^*) > \gamma(G)$. Therefore E' is the maximal set having the property that $\gamma(G||E') = \gamma(G)$. Thus $usd_{\gamma}(G) \geq \sum_{i=1}^{r-2} n_i + 2$. Suppose $E'' \subseteq E(G)$ such that $\gamma(G||E'') = \gamma(G)$ and $|E''| > |E'|$. Since E' is maximal, $E' \not\subset E''$. Also E'' contains at least two edges e' and e'' which are not in E'. Let $E^{**} = E' \setminus \{e\} \cup \{e', e''\}$, where $e \in E'$. Then e' and e'' are combination any two the above mentioned form. Therefore $G||E^{**}$ has no dominating set of cardinality two. If $E^{**} = E''$, then E'' is not possible. Proceeding like this finally we get a set E'' such that $E'' \cap E' = \phi$ with $\gamma(G||E'') > \gamma(G) = 2$. Hence E' is a maximal set with maximum cardinality. Thus

$$usd_{\gamma}(G) = |E^*| = \sum_{i=1}^{r-2} n_i + 2.$$

Corollary 2.12. *$usd_{\gamma}(K_{m,n}) = 2$ where $m, n > 2$.*

III. RESULTS ON TREES

In this section we obtain exact value for some special trees and determine bound for tree in terms of maximum degree.

Theorem 3.1. *If T is a spider with k leaves, then $usd_{\gamma}(T) = k$.*

Proof. Let x be vertex of T with $deg(x) = \Delta(T)$, $\{u_1, u_2, \ldots, u_k\}$ be set of all support vertices and $\{v_1, v_2, \ldots, v_k\}$ be set of all pendant vertices such that $u_iv_i \in E(T)$. Take $E' = \{u_iv_i : 1 \leq i \leq k-1\}$. Then $|E'| = k - 1$ and $\gamma(T||E') = k = \gamma(T)$. Therefore E' is a stable subdivision set. Suppose there exist stable subdivision set E'' such that $|E'| > |E|$. Since for every $e \notin E'$, $E' \cup \{e\}$ is not a stable set, $E' \not\subset E''$. Then E'' contains at least two edges e' and e'' which are not in E'. Both e' and e'' are of the form u_ix, where $1 \leq i \leq k$ or u_kv_k. Let $E^{**} = E' \setminus \{e\} \cup \{e', e''\}$, where $e \in E'$. Therefore $G||E^{**}$ has no dominating set of cardinality two. If $E^{**} = E''$, then E'' is not possible. Proceeding like this finally we get a set E'' such that $E'' \cap E' = \phi$ with $\gamma(T||E'') > \gamma(T) = 2$. Hence E' is a maximum stable subdivision set. Thus $usd_{\gamma}(T) = k$.

Theorem 3.2. *For $n \geq 3$, $usd\gamma(T) \leq 2k - 1$, where $k = n - \Delta(T)$ and $k \in N$.*

Proof. Let v be vertex of T such that $deg(v) = \Delta(T)$. If $V(T) - N[v] = \phi$, then $T \cong K_{1,n-1}$ and hence $usd_{\gamma}(T) = 1 = 2k - 1$. Next we consider tree T such that $V(T) - N[v] \neq \phi$.

Case (i): $< V(T) - N[v] > \cong \overline{K}_{k-1}$.

Since T is tree, no two vertices of $N(v)$ are adjacent.

Subcase (i): $N(v)$ contains no strong support.

If $\langle V(T) - \{v\}\rangle \cong rK_1 \cup (k-1)K_2$, where $r \geq 1$, then $usd_{\gamma}(T) = 2(k-1) + 1 = 2k - 1$. Otherwise T is a spider and so by Theorem 3.1 $usd_{\gamma}(T) = k - 1$.

Subcase (ii): $N(v)$ contains at least one strong support.

If $\langle V(T) - \{v\}\rangle$ has r stars, i^{th} star contains l_i

leaves $(1 \le i \le r)$ and if it has isolated vertices, then

$$usd_{\gamma}(T) = 2[(k\text{-}1) - \sum_{i=1}^{r} l_i] + 2 = 2k - 2\sum_{i=1}^{r} l_i <$$

Otherwise

$$usd_{\gamma}(T) = (k\text{-}1) - \sum_{i=1}^{r} l_i + 1 = k - \sum_{i=1}^{r} l_i < 2k - 1.$$

Case (ii): $\langle V(T) - \{v\} \rangle \ncong \overline{K}_{k-1}$.

Then $\langle V(T) - \{v\} \rangle$ contains at least one component with two vertices. Since T is tree, exactly one vertex of each component of $\langle V(T) - \{v\}\rangle$ is adjacent to exactly one vertex of $N(v)$. Therefore we can not find a stable subdivision set which is constructed by two edges for each vertex of $\langle V(T) - \{v\} \rangle$. Hence $usd_{\gamma}(T) < 2(k-1) + 1 = k - 1$.

Theorem 3.3. *Let T be a tree. Then* $usd_{\gamma}(T) = 2k-1$, *where* $k = n - \Delta(T)$ *if and only if T is either wounded spider or star.*

Proof. Assume that T is either wounded spider or star. If T is a wounded spider, then $usd_{\gamma}(T) = 2k-1$. Otherwise $T \cong K_{1,n-1}$ and so $usd_{\gamma}(T) = 1 = 2k - 1$.

Conversely, assume that $usd_{\gamma}(T) = 2k - 1$. Suppose T is neither wounded spider nor star. Since T is not a star, there is no tree such that $V(T) - N[v] = \phi$. Now assume that $V(T) - N[v] \neq \phi$.

Case (i) : $\langle V(T) - \{v\}\rangle \cong \overline{K}_{k-1}$.

If $N(v)$ contains no strong support, then T is a spider and hence by Theorem 3.1 $usd_{\gamma}(T) = k$. Otherwise $N(v)$ contains at least one strong support and hence by the proof of Theorem 3.2, $usd_{\gamma}(T) < 2k - 1$.

Case (ii) : $\langle V(T) - \{v\}\rangle \ncong \overline{K}_{k-1}$.

By case (ii) in the proof of the Theorem 3.2, $usd_{\gamma}(T) < 2k - 1$.

Theorem 3.4. *If T is a caterpillar, then* $usd_{\gamma}(T) \le \lceil 3n/4 \rceil$.

Proof. Let D be a dominating set of T. Without loss of generality we assume that D contains all the supports of T.

Case (i): Every internal vertex is a support.

Let us first consider graph T having no strong support. Let $u_1, u_2, \ldots, u_k$ be supports and $v_1, v_2, \ldots, v_k$ be corresponding pendant vertices. Let $E' = \{u_iu_j \in E(T)\} \cup \{u_iv_i : \text{no two } u_i\text{'s are adjacent}\}$. Then $|E'| \le k - 1 + \lceil k/2 \rceil = \lceil 3k/2 \rceil - 1$. We can easily verify that $\gamma(T\|E') = \gamma(T)$ and hence E' is a stable subdivision set of T.

Claim : E' is a maximum stable subdivision set of T.

For $n \le 20$, we can verify that E' is a maximum stable subdivision set of T. Now we prove this for general case. Suppose there exists a stable subdivision set $E'' \subseteq E(T)$ such that $|E''| > |E'|$.

subcase (i): $E'' \supset E'$.

Let us take $E'' \supseteq E' \cup \{u_jv_j\}$, $u_jv_j \notin E'$. Then $u_{j'}v_{j'} \in E'$, where $u_{j'}$ is adjacent to u_j. Since the internal edge $u_ju_{j'}$, $u_{j'}v_{j'}$ and u_jv_j are subdivided in $T\|E''$, $\gamma(T\|E'') = \gamma(T\|E')+1 > \gamma(T)$. Therefore E'' is not a maximum stable subdivision set.

Subcase (ii): $E'' \not\supseteq E'$.

If $E' \cap E'' = \phi$, then $|E''| \le (n-1) - |E'| \le 2k - 1 - \lceil 3k/2 \rceil + 1 = \lceil k/2 \rceil < |E'|$ which contradicts the definition of E''. Therefore we assume that $E' \cap E'' \neq \phi$. Suppose $|E' \backslash E''| = 1$. Let $e \in E'$ and $e \notin E''$. Therefore $e = u_iu_j$ or u_iv_i . Then E'' cotains at least two leaves u_jv_j and u_kv_k which are not in E'. Therefore $u_{j'}v_{j'}$, $u_{k'}v_{k'} \in E'$, where $u_{j'}$ is adjacent to u_j and $u_{k'}$ is adjacent to u_k. Let $E_1 = \{u_jv_j, u_ju_{j'}, u_{j'}v_{j'}\}$. Then $\gamma(T\|E_1) > \gamma(T)$ . . . . (1). Hence $\gamma(T\|E'') > \gamma(T)$. Suppose $|E' \backslash E''| = r$, then E'' contains at least $r + 1$ leaves which are not in E'. Since by (1), subdivision of E'' increases the domination number. By both the subcases we get contradiction. Hence E' is a maximum stable subdivision set of T. Now $usd_{\gamma}(T) = |E'| + 1 \le \lceil 3k/2 \rceil = \lceil 3n/4 \rceil$.

If T has strong support, then the number of independent leaves is less than $\lceil n/2 \rceil$. Hence $usd_{\gamma}(T) < \lceil 3n/4 \rceil$.

case (ii): There exists an internal vertex which is not support.

Since there exists at least two supports which are not adjacent, we can not take all the internal edges in maximum stable subdivision set M and the number of leaves is less than $\lceil n/2 \rceil$, $|M| < \lceil 3n/4 \rceil$.

Corollary 3.5. *If T is a columb graph, then* $usd_{\gamma}(T) = \lceil 3n/4 \rceil$.

IV. LOWER AND UPPER BOUNDS

In this section we obtain the lower and upper bounds for domination uniform subdivision number and we characterize the extremal graphs.

Theorem 4.1. *Let G be a graph with components* $G_1, G_2, \ldots, G_k$. *Then*

$$usd_{\gamma}(G) = \sum_{i=1}^{k} usd_{\gamma}(G_i) - k + 1.$$

Proof. Let $d_i = usd_{\gamma}(G_i)$ and $E_i \subseteq E(Gi)$ such that $\gamma(G_i \|E_i) = \gamma(G_i)$ and $|E_i| = d_i - 1$. Therefore E_i is a maximal set with the above condition. Take

$$E' = \bigcup_{i=1}^{k} E_i.$$ Then

$$|E'| = \sum_{i=1}^{k} (d_i - 1) = \sum_{i=1}^{k} d_i - k.$$

Now

$$\gamma(G \| E') = \sum_{i=1}^{k} \gamma(G_i \| E_i) = \sum_{i=1}^{k} \gamma(G_i) = \gamma(G).$$

To prove E' is a maximal set having maximum cardinality with respect to the above condition. Consider $E^* = \bigcup_{i=1}^{k} E_i \cup e$, where $e \in E(G)\backslash E'$. Then $e \in E(G_i)$ for some i. $\gamma(G \| E^*) = \sum_{j \neq i} \gamma(G_j \| E_j) + \gamma(G_i \| (E_i \cup e)) > \sum_{j=1}^{k} \gamma(G_j)$

Therefore E' is a maximal set. Suppose $E'' \subseteq E$, $\gamma(G\|E'') = \gamma(G)$ and $|E''| > |E|$. If $E'' \supseteq E'$, then E'' contains $E_i \cup \{e_i\}$, $e_i \in E(Gi)\backslash Ei$. Then $\gamma(G_i \|(E_i \cup \{ei\})) > \gamma(G_i)$ and hence $\gamma(G\|E'') > \gamma(G)$ which is a contradiction. If $E'' \not\supseteq E'$, then there exists at least one component G_j such that E'' contains more number of elements of G_j than that E' contains. Let us take $E'' \supseteq E'_j$. Then $|E'_j| > |E_j| = d_i$ and $\gamma(G_j\|E'_j) > \gamma(G_j)$. Therefore $\gamma(G\|E'') > \gamma(G)$ which is a contradiction. Hence $usd_\gamma\ G = |E^*| = \sum_{i=1}^{k} d_i - k + 1.$

Theorem 4.2. *For any graph G, $0 \leq usdg(G) \leq m$. Also the bounds are sharp.*

Proof. If $\gamma(G\|E(G)) = \gamma(G)$, then $usd_\gamma(G) = 0$. For example $usd_\gamma(P_2) = 0$. If $\gamma(G\|E(G)) > \gamma(G)$, then $usd_\gamma(G) \leq m$. Consider $usd_\gamma(P_3 \circ K_1) = 6 =$ size of $P_3 \circ K_1$. Hence the bounds are sharp.

Theorem 4.3. *For any graph G, $usd_\gamma(G) = 0$ if and only if $G \cong k_1P_1 \cup k_2P_2$ where $k_1 \geq 0$ and $k_2 \geq 0$.*

Proof. Assume that $G \cong k_1P_1 \cup k_2P_2$, $k_1 \geq 0$ and $k_2 \geq 0$. Clearly $usd_\gamma(G) = 0$. Conversely, assume that $usd_\gamma(G) = 0$. Suppose $G \not\cong k_1P_1 \cup k_2P_2$. Then G contains a component G_i with at least three vertices and $sd_\gamma(G_i)$ exists. Hence $usd_\gamma(G) > 0$.

Theorem 4.4. *For any connected graph G, $usd_\gamma(G) = m$ if and only if $G \cong K_{1,r} \circ K_1$, for all $r \geq 1$.*

Proof. Assume that $G \cong K_{1,r} \circ K_1$ for some $r \geq 1$. Then $E(G) - \{e\}$, where e is leaf which is incident with full vertex of $K_{1,r}$, is maximum stable subdivision set of G and hence $usd_\gamma(G) = m$.

Conversely assume that $usd\gamma(G) = m$. Suppose G ® $K_{1,r} \circ K_1$ for all $r \geq 1$.

Case (i) : $\gamma(G) = n/2$

Then by theorem $G \cong H \circ K_1$. Also H ® $K_{1,r}$, for some $r \geq 1$. Since $usd_\gamma > 0$, H ® K_1. Let $v \in V\ H$ such that $deg(v) = \Delta(H)$. Let $N_H(v) = \{v_1, v_2, \ldots, v_k\}$, $u_1, u_2, \ldots, u_k$ be corresponding pendent vertices of $v_1, v_2, \ldots, v_k$ respectively and uv be leaf in G. Then $\langle N(v)\rangle$ contains at least one edge. Let $E' = \{vv_1, \ldots, vv_r, v_1u_1, \ldots, v_ru_r\}$. Then E' is stable subdivision set. Any edge set consists of E' and at least one edge of $\langle NH(v)\rangle$ or uv whose subdivision increase the domination number. Also any maximum stable subdivision set S containing v_iv_j does not contain either v_iu_i or v_ju_j and S does not contain uv. Therefore by the above argument $usd_\gamma(G) \leq m - 1$.

Case (ii) : $\gamma(G) < n/2$.

Then any minimum dominating set D contains at least one vertex which has at least two private neighbors in $V(G) - D$. Let $v \in D$. Then v has at least two neighbors say, v_1 and v_2 in $V(G) - D$. Let $e_1 = vv_1$ and $e_2 = vv_2$. Then $E_1 = E(G) - \{e_1\}$ and $E_2 = E(G) - \{e_2\}$ whose subdivision increase the domination number and $|E_1| = |E_2| = m - 1$. Therefore, for any minimum dominating set D, subdivision of any $m - 1$ edges increases the domination number. Since D is arbitrary, $usd_\gamma(G) \leq m - 1$.

Corollary 4.5. *For any graph G, $usd_\gamma(G) = m$ if and only if $G \cong k_1(K_{1,r}) \cup k_2P_1 \cup k_3P_2$.*

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