A Study on unsteady one-dimensional heat flow problem using Rayleigh Ritz, Single-term Walsh series and Leapfrog Method

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Abstract — *In this article presents a study on interesting unsteady one-dimensional heat flow problem is discussed using Rayleigh Ritz, single-term Walsh series (STWS) method [5] and Leapfrog method. The results (approximate solutions) obtained very accurate using the above said methods are compared with the exact solution of that problem. It is found that the solution obtained using Leapfrog method is closer to the exact solution of the unsteady one-dimensional heat flow problem. The high accuracy and the wide applicability of Leapfrog method approach will be demonstrated with numerical example. Solution graphs for discrete exact solutions are presented in a graphical form to show the efficiency of the Leapfrog method. The results obtained show that Leapfrog method is more useful for solving the unsteady one-dimensional heat flow problem and the solution can be obtained for any length of time.*

Keywords — *Unsteady one-dimensional heat flow equations, Rayleigh Ritz method, Single-term Walsh series, Leapfrog method.*

I. INTRODUCTION

A mathematical model is a description of a system using mathematical concepts and language. The process of developing a mathematical model is termed mathematical modelling. Mathematical models are used not only in the natural sciences (such as physics, biology, earth science, meteorology) and engineering disciplines (e.g. computer science, artificial intelligence), but also in the social sciences (such as economics, psychology, sociology and political science), physicists, engineers, statisticians, operations research analysts and economists use mathematical models most extensively.

Mathematical models can take many forms, including but not limited to dynamical systems, statistical models, differential equations, or game theoretic models. These and other types of models can overlap, with a given model involving a variety of abstract structures. In general, mathematical models may include logical models, as far as logic is taken as a part of mathematics. In many cases, the quality of a scientific field depends on how well the mathematical models developed on the theoretical side agree with results of repeatable experiments. Lack of agreement between theoretical mathematical models and experimental measurements often leads to important advances as better theories are developed.

In this paper we developed numerical methods for addressing unsteady one-dimensional heat- flow problem by an application of the Leapfrog method which was studied by Sekar and team of his researchers [1-2, 6-18], which involve two phases. In phase-I, the spatial dependency of the heat flow equation is eliminated by applying the Rayleigh-Ritz method and to determine the suitable initial conditions, the Galerkin Technique is utilized. In phase II, the resulting system of equations is being solved by applying the methods STWS [3] and Leapfrog method to determine the discrete solutions of the unsteady one-dimensional heat- flow problem. Further, to analyse the efficiency of the above-mentioned methods, the discrete solutions obtained are compared with the exact solutions and with the obtained discrete solutions by the methods of Laplace Transform and Leapfrog method. Recently, Sekar *et al*. [5] discussed the unsteady one-dimensional heat- flow problem using STWS. In this paper, the same unsteady one-dimensional heat- flow problem was considered (discussed by Sekar *et al.* [5]) but present a different approach using the Leapfrog Method with more accuracy for unsteady one-dimensional heat- flow problem.

II. UNSTEADY ONE-DIMENSIONAL HEAT FLOW PROBLEM

Let us consider an unsteady one-dimensional heat flow problem (it may be referred as a flow of electricity in cables – the telegraph problem). The governing equation of the flow is given by

$$\frac{\partial T}{\partial t} = \alpha^2 \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < 1 \qquad (1)$$

where T denotes the temperature, t denotes the time, α^2 denotes the thermal diffusivity and x denotes the space coordinate.

The initial and boundary conditions are

$$T(x,0) = 1.0 \qquad (2)$$

And

$$T(0,t) = \frac{\partial T}{\partial x}(1,t) = 0 \qquad (3)$$

III. PHASE I

3.1 RAYLEIGH - RITZ METHOD

This method is used for the elimination of spatial dependency in eq. (1). Assuming that T^* is the weighting function of T, which satisfies the initial and boundary conditions given by eqs. (2) and (3), the following weighted residual equation can be obtained as (Schechter [4])

$$\int_0^1 T^* \left[\frac{\partial T}{\partial t} - \alpha^2 \frac{\partial^2 T}{\partial x^2} \right] dx = 0$$

(4)

After integrating and introducing the boundary conditions (3) we obtain

$$\int_0^1 T^* \frac{\partial T}{\partial t} dx + \alpha^2 \int_0^1 \frac{\partial T^*}{\partial x} \frac{\partial T}{\partial x} dx = 0$$

(5)

Assuming the same function has been applied for T and T^*, then we define

$$T = \sum_{j=1}^{2} C_j(t)\phi_j(x)$$

(6)

$$T^* = \sum_{k=1}^{2} C_k(t)\phi_k(x)$$

(7)

where $\phi_1 = x$ and $\phi_2 = x^2$. Substituting eqs. (6) and (7) into eq. (5) we obtain

$$\int_0^1 \phi_k \left[\sum_{j=1}^{2} \frac{\partial C_j}{\partial t} \phi_j \right] dx + \alpha^2 \int_0^1 \left[\sum_{\substack{k=1 \\ j=1}}^{2} C_j \frac{\partial \phi_k}{\partial x} \frac{\partial \phi_j}{\partial x} \right] dx = 0$$

(8)

Eq. (8) can be expressed as

$$A\,C'(t) + \alpha^2 B\,C(t) = 0 \qquad (9)$$

where $A = \int_0^1 \phi_k \, \phi_j \, dx$, $\qquad B = \int_0^1 \frac{\partial \phi_k}{\partial x} \frac{\partial \phi_j}{\partial x} dx$,

$$C'(t) = \begin{bmatrix} C_1'(t) & C_2'(t) \end{bmatrix} \text{ and } C(t) = \begin{bmatrix} C_1(t) & C_2(t) \end{bmatrix}$$

Evaluating the indicated integration, we get

$$\begin{bmatrix} 20 & 15 \\ 15 & 12 \end{bmatrix} \begin{bmatrix} C_1'(t) \\ C_2'(t) \end{bmatrix} + \alpha^2 \begin{bmatrix} 60 & 60 \\ 60 & 80 \end{bmatrix} \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(10)

3.2 GALERKIN METHOD

To solve the system, we need some initial conditions for C_1 and C_2, since in the present approximation, the initial condition T(X, 0) = 1 cannot be satisfied. We then represent the residual of the approximation with the initial condition as (Schechter [4])

$$E_1 = T(x,0) - 1 = xC_1(0) + x^2C_2(0) - 1$$

(11)

Now, employing the Galerkin method, we get

$$\int_0^1 \left[x\,C_1(0) + x^2 C_2(0) - 1 \right] x\,dx = 0 \qquad (12)$$

$$\int_0^1 \left[x\,C_1(0) + x^2 C_2(0) - 1 \right] x^2\,dx = 0 \qquad (13)$$

Solving eqs. (12) and (13), we obtain

$$C_1(0) = 4, \quad C_2(0) = -10/3 \qquad (14)$$

Hence, for the problem (1), the spatial dependency of the heat flow has been eliminated by applying the Rayleigh-Ritz method thereby reducing the problem to a system of linear first order differential equations (10) whose initial conditions are given in (14).

IV. PHASE II

Here, numerical methods namely STWS and Leapfrog method, have been introduced to calculate $C_1(t)$ and $C_2(t)$ for the system (10).

4.1 SINGLE TERM WALSH SERIES (STWS)

Consider the system of linear differential equations

$$K\,x'(t) = A\,x(t) + B\,u(t) \qquad (15)$$

with $x(0) = x_0$.

where K and A are n x n matrices, B is an n x r matrix, x(t) is an n-state vector, and u(t) is an r-input vector. In this technique, the given function is expanded as a single - term Walsh series in the normalized interval $\tau \in [0.1)$, which corresponds to $t \in [0.1/m)$ by defining $t = \tau/m$, m being any integer. The following are the recursive relations, in STWS method, to determine the discrete solution for the system (15).

$$R_i = \left[K - \frac{A}{2m} \right]^{-1} S_i$$

$$P_i = \frac{R_i}{2} + x(i-1) \qquad (16)$$

$$x(i) = R_i + x(i-1)$$

where $S_i = \frac{A}{m} x(i-1) + \frac{B}{m} u_i$; i = 1, 2, 3, ….

Then, x(i) will give the discrete values of the state and P_i gives the Block Pulse Function (BPF) values of the state to any length of time. The main advantage of this method is that if the matrix K in (15) is singular, this difference $\left[K - \frac{A}{2m} \right]$ turns out to be non-singular. Hence, the inverse of the matrix can be computed.

The state – space equation (10) is

$$\begin{bmatrix} 20 & 15 \\ 15 & 12 \end{bmatrix} \begin{bmatrix} C_1'(t) \\ C_2'(t) \end{bmatrix} + \alpha^2 \begin{bmatrix} 60 & 60 \\ 60 & 80 \end{bmatrix} \begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(17)

with $C(0) = [C_1(0)\ \ C_2(0)]^T = [4\ \ -10/3]^T$.

If $A = \begin{bmatrix} 20 & 15 \\ 15 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 60 & 60 \\ 60 & 80 \end{bmatrix}$ then the eq. (17) becomes

$$A\,C'(t) = -\alpha^2 B\,C(t)$$

(18)

with $C(0) = [\,4\ \ -10/3]^T$. Applying the STWS approach, the following recursive relationship is obtained.

$$R_i = \left[A + \frac{\alpha^2 B}{2m}\right]^{-1} S_i$$

$$P_i = \frac{R_i}{2} + C(i-1)$$

(19)

$$C(i) = R_i + C(i-1)$$

where $S_i = \frac{\alpha^2}{m}.B\,C(i-1)$ and $i = 1, 2, 3, \ldots$ the interval number.

The discrete and Block Pulse Function (BPF) values of C(t) are obtained from C(i) and P_i, to any length of time.

To obtain the discrete solutions, via extended RK methods, we write the system of eqs. (10) explicitly as :

$$C_1' = \alpha^2 \left[12\,C_1 + 32\,C_2\right]$$

$$C_2' = -\alpha^2 \left[20\,C_1 + \frac{700}{15} C_2\right]$$

(20)

4.2 LEAPFROG METHOD

In mathematics Leapfrog integration is a simple method for numerically integrating differential equations of the form $\ddot{x} = F(x)$, or equivalently of the form $\dot{v} = F(x), \dot{x} \equiv v$, particularly in the case of a dynamical system of classical mechanics. Such problems often take the form $\ddot{x} = -\nabla V(x)$, with energy function $E(x,v) = \frac{1}{2}|v|^2 + V(x)$, where V is the potential energy of the system. The method is known by different names in different disciplines. In particular, it is similar to the Velocity Verlet method, which is a variant of Verlet integration. Leapfrog integration is equivalent to updating positions $x(t)$ and velocities $v(t) = \dot{x}(t)$ at interleaved time points, staggered in such a way that they 'Leapfrog' over each other. For example, the position is updated at integer time steps and the velocity is updated at integer-plus-a-half time steps.

Leapfrog integration is a second order method, in contrast to Euler integration, which is only first order, yet requires the same number of function evaluations per step. Unlike Euler integration, it is stable for oscillatory motion, as long as the time-step Δt is constant, and $\Delta t \le 2/w$. In Leapfrog integration, the equations for updating position and velocity are

$$x_i = x_{i-1} + v_{i-1/2}\Delta t, a_i = F(x_i), v_{i+1/2} = v_{i-1/2} + a_i \Delta t,$$

where x_i is position at step $i, v_{i+1/2}$, is the velocity, or first derivative of x, at step $i+1/2, a_i = F(x_i)$ is the acceleration, or second derivative of x, at step i and Δt is the size of each time step. These equations can be expressed in a form which gives velocity at integer steps as well. However, even in this synchronized form, the time-step Δt must be constant to maintain stability.

$$x_{i+1} = x_i + v_i \Delta t + \frac{1}{2} a_i \Delta t^2,$$

$$v_{i+1} = v_i + \frac{1}{2}(a_i + a_{i+1})\Delta t.$$

One use of this equation is in gravity simulations, since in that case the acceleration depends only on the positions of the gravitating masses, although higher order integrators (such as Runge–Kutta methods) are more frequently used. There are two primary strengths to Leapfrog integration when applied to mechanics problems. The first is the time-reversibility of the Leapfrog method. One can integrate forward n steps, and then reverse the direction of integration and integrate backwards n steps to arrive at the same starting position. The second strength of Leapfrog integration is its symplectic nature, which implies that it conserves the (slightly modified) energy of dynamical systems. This is especially useful when computing orbital dynamics, as other integration schemes, such as the Runge-Kutta method, do not conserve energy and allow the system to drift substantially over time.

V. DISCUSSION

Solving eq.(20) by the Laplace – Transform, the analytic expressions for $C_1(t)$ and $C_2(t)$ are

$$C_1(t) = 1.6408\,e^{-32.1807\alpha^2 t} + 2.3592\,e^{-2.486\alpha^2 t}$$

$$C_2(t) = -\left[2.265\,e^{-32.1807\alpha^2 t} + 1.068\,e^{-2.486\alpha^2 t}\right]$$

The exact solution of eqs. (1) which satisfies the initial and boundary conditions given by eqs. (2) and (3) is obtained as (refer Ritger and Rose [115]).

$$T(x,t) = 2\sum_{n=0}^{\infty} \frac{e^{-\lambda_n^2 \alpha^2 t} \sin(\lambda_n x)}{\lambda_n}$$

where $\lambda_n = (2n+1)(\pi/2)$

The numerical values of T(x,t), with different values of α^2 = 0.5, 0.75, 1.0 and 2.0 based on the value of x = 0.5 and 0.1, have been obtained by the methods of Ritz-Laplace Transform, Ritz-STWS and the Ritz-Leapfrog, and are respectively shown in Tables 1 – 4, together with their corresponding exact solutions.

Also, the discrete solutions obtained by the methods STWS and Leapfrog, for the values of $C_1(t)$ and $C_2(t)$ of the eqs. (10), coincide well with the solutions obtained by the Laplace Transform. The numerical values of T(x,t), with different values of α^2 = 0.5, 0.75, 1.0 and 2.0 based on the value of x = 0.5 and 0.1, have been obtained by these two methods and are in good agreement with the exact solution (20).

The obtained absolute error using the methods of Ritz-Laplace Transform, Ritz-STWS and Ritz-Leapfrog are given in Tables 5 – 12. For a sample, an error graph for x = 1 at time t = 1.4 is shown in Figure 1.

Table 1 Variation of *T(x,t)* for $\alpha^2 = 0.5$

Value of x = 0.5				
Time	**Exact**	**Laplace**	**STWS**	**Leapfrog**
0.20	0.7354	0.7220	0.7201	0.7220
0.40	0.5529	0.5555	0.5553	0.5555
0.60	0.4295	0.4329	0.4328	0.4329
0.80	0.3353	0.3376	0.3375	0.3376
1.00	0.2619	0.2633	0.2632	0.2633
1.20	0.2046	0.2053	0.2053	0.2053
1.40	0.1598	0.1601	0.1601	0.1601
Value of x = 1.0				
0.20	0.9488	0.9820	0.9863	0.9820
0.40	0.7717	0.7843	0.7846	0.7843
0.60	0.6062	0.6124	0.6123	0.6124
0.80	0.4739	0.4777	0.4775	0.4777
1.00	0.3703	0.3725	0.3724	0.3725
1.20	0.2892	0.2905	0.2904	0.2905
1.40	0.2260	0.2266	0.2265	0.2266

Table 2 Variation of *T(x,t)* for $\alpha^2 = 0.75$

Value of x = 0.5				
Time	**Exact**	**Laplace**	**STWS**	**Leapfrog**
0.20	0.6322	0.6306	0.6293	0.6306
0.40	0.4295	0.4330	0.4327	0.4330
0.60	0.2963	0.2982	0.2979	0.2982
0.80	0.2046	0.2055	0.2051	0.2055
1.00	0.1413	0.1414	0.1412	0.1414
1.20	0.0975	0.0974	0.0973	0.0974
1.40	0.0673	0.0671	0.0670	0.0671
Value of x = 1.0				
0.20	0.8637	0.8843	0.8867	0.8843
0.40	0.6062	0.6124	0.6122	0.6124
0.60	0.4189	0.4218	0.4215	0.4218
0.80	0.2892	0.2905	0.2902	0.2905
1.00	0.1997	0.2001	0.1998	0.2001
1.20	0.1379	0.1378	0.1376	0.1378
1.40	0.0952	0.0950	0.0947	0.0950

Table 3 Variation of *T(x,t)* for $\alpha^2 = 1.0$

Value of x = 0.5				
Time	**Exact**	**Laplace**	**STWS**	**Leapfrog**
0.20	0.5529	0.5555	0.5548	0.5555
0.40	0.3353	0.3376	0.3372	0.3376
0.60	0.2046	0.2054	0.2050	0.2054
0.80	0.1249	0.1249	0.1246	0.1249
1.00	0.0762	0.0760	0.0757	0.0760
1.20	0.0465	0.0463	0.0460	0.0463
1.40	0.0284	0.0281	0.0280	0.0281
Value of x = 1.0				
0.20	0.7717	0.7844	0.7848	0.7844
0.40	0.4739	0.4777	0.4771	0.4777
0.60	0.2892	0.2906	0.2900	0.2906
0.80	0.1765	0.1767	0.1763	0.1767
1.00	0.1077	0.1075	0.1071	0.1075
1.20	0.0657	0.0654	0.0651	0.0654
1.40	0.0401	0.0398	0.0396	0.0398

Table 4 Variation of *T(x,t)* for $\alpha^2 = 2.0$

Value of x = 0.5				
Time	**Exact**	**Laplace**	**STWS**	**Leapfrog**
0.20	0.3353	0.3376	0.3366	0.3376
0.40	0.1249	0.1249	0.1236	0.1249
0.60	0.0465	0.0462	0.0455	0.0462
0.80	0.0173	0.0171	0.0167	0.0171
1.00	0.0064	0.0063	0.0062	0.0063
1.20	0.0024	0.0023	0.0023	0.0023
1.40	0.0009	0.0009	0.0008	0.0009
Value of x = 1.0				
0.20	0.4739	0.4777	0.4734	0.4777
0.40	0.1765	0.1767	0.1749	0.1767
0.60	0.0657	0.0654	0.0644	0.0654
0.80	0.0245	0.0242	0.0237	0.0242
1.00	0.0091	0.0090	0.0087	0.0090
1.20	0.0034	0.0033	0.0032	0.0033
1.40	0.0013	0.0012	0.0012	0.0012

Table 5 Absolute Error in *T(x,t)* for $\alpha^2 = 0.5$

	Time	**Laplace**	**STWS**	**Leapfrog**
When x = 0.5	0.20	0.0134	0.0135	0.0135
	0.40	0.0026	0.0026	0.0026
	0.60	0.0034	0.0034	0.0034
	0.80	0.0023	0.0023	0.0023

	1.00	0.0014	0.0014	0.0014
	1.20	0.0007	0.0008	0.0008
	1.40	0.0003	0.0003	0.0003
When x = 1.0	0.20	0.3376	0.3366	0.3376
	0.40	0.1249	0.1236	0.1249
	0.60	0.0462	0.0455	0.0462
	0.80	0.0171	0.0167	0.0171
	1.00	0.0063	0.0062	0.0063
	1.20	0.0023	0.0023	0.0023
	1.40	0.0009	0.0008	0.0009

Table 6 Absolute Error in *T(x,t)* for $\alpha^2 = 0.75$

	Time	**Laplace**	**STWS**	**Leapfrog**
When x = 0.5	0.20	0.0016	0.0017	0.0016
	0.40	0.0035	0.0034	0.0035
	0.60	0.0019	0.0018	0.0019
	0.80	0.0009	0.0008	0.0009
	1.00	0.0001	0.0002	0.0001
	1.20	0.0001	0.0001	0.0001
	1.40	0.0002	0.0003	0.0002
When x = 1.0	0.20	0.0206	0.0207	0.0206
	0.40	0.0062	0.0063	0.0062
	0.60	0.0029	0.0029	0.0029
	0.80	0.0013	0.0013	0.0013
	1.00	0.0004	0.0004	0.0004
	1.20	0.0001	0.0001	0.0001
	1.40	0.0002	0.0003	0.0002

Table 7 Absolute Error in *T(x,t)* for $\alpha^2 = 1.0$

	Time	**Laplace**	**STWS**	**Leapfrog**
When x = 0.5	0.20	0.0026	0.0026	0.0026
	0.40	0.0023	0.0023	0.0023
	0.60	0.0008	0.0007	0.0008
	0.80	0.0000	0.0000	0.0000
	1.00	0.0002	0.0002	0.0002
	1.20	0.0002	0.0003	0.0002
	1.40	0.0003	0.0003	0.0003
When x = 1.0	0.20	0.0127	0.0127	0.0127
	0.40	0.0038	0.0038	0.0038
	0.60	0.0014	0.0013	0.0014
	0.80	0.0002	0.0002	0.0002
	1.00	0.0002	0.0002	0.0002
	1.20	0.0003	0.0004	0.0003
	1.40	0.0003	0.0004	0.0003

Table 8 Absolute Error in *T(x,t)* for $\alpha^2 = 2.0$

	Time	**Laplace**	**STWS**	**Leapfrog**
When x = 0.5	0.20	0.0023	0.0023	0.0023
	0.40	0.0000	0.0000	0.0000
	0.60	0.0003	0.0003	0.0003
	0.80	0.0002	0.0002	0.0002
	1.00	0.0001	0.0001	0.0001
	1.20	0.0001	0.0001	0.0001
	1.40	0.0000	0.0000	0.0000
When x = 1.0	0.20	0.0038	0.0037	0.0038
	0.40	0.0002	0.0002	0.0002
	0.60	0.0003	0.0004	0.0003
	0.80	0.0003	0.0003	0.0003
	1.00	0.0001	0.0002	0.0001
	1.20	0.0001	0.0001	0.0001
	1.40	0.0000	0.0000	0.0000

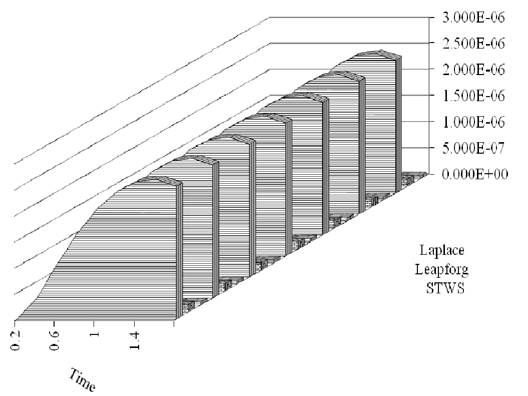

Fig. 1 Error graph for $x = 1.0$ at $t = 1.4$

VI. Conclusions

As an outcome of this study, new methods have been proposed for the investigation of unsteady one-dimensional heat-flow problem. The novel features of the present numerical schemes are the adoption of the Rayleigh – Ritz technique for the elimination of spatial dependency in the heat flow equation, the STWS and Leapfrog methods for solving the resulting system of first order linear equations in time, and the Galerkin method for determining the initial conditions.

It is observed that Ritz-Laplace Transform, Ritz-STWS and Ritz-Leapfrog yield similar results. Reviewing these methods, applied for the unsteady one-dimensional heat-flow problem, it is clearly noticeable that Ritz-STWS, Ritz-Leapfrog methods involve less number of computations and the complexity of these methods are very simple.

It is also to be noted that from Figure 1, the analytical method of Laplace Transform stands first, in respect to accuracy. However, Leapfrog method is found to yield better results among the STWS technique.

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REFERENCES

[1] S. Karunanithi, S. Chakravarthy and S. Sekar, “Comparison of Leapfrog and single term Haar wavelet series method to solve the second order linear system with singular-A”, *Journal of Mathematical and Computational Sciences*, vol. 4, no. 4, 2014, pp. 804-816.

[2] S. Karunanithi, S. Chakravarthy and S. Sekar, “A Study on Second-Order Linear Singular Systems using Leapfrog Method”, *International Journal of Scientific & Engineering Research*, vol. 5, issue 8, August-2014, pp. 747-750.

[3] K. Murugesan, Paul Dhayabaran and D.J. Evans, *“Analysis of different second order multivariable linear system using single term Walsh series technique and Runge-Kutta method”*, *Int. J. Comp. Math.*, vol. 72, 1999, pp. 367-374.

[4] R. S. Schechter, “*The variational methods in engineering”*, Mc-Graw Hill, New York, 1967.

[5] S. Sekar, R. Muthukrishnan and S. Subbulakshmi, “Numerical investigation of the heat flow problem using Rayleigh Ritz, STWS and Runge-Kutta methods based on various means”, *International Journal of Current Research*, vol. 3, issue 12, 2011, pp. 142-148.

[6] S. Sekar and K. Prabhavathi, “Numerical solution of first order linear fuzzy differential equations using Leapfrog method”, *IOSR Journal of Mathematics*, vol. 10, no. 5 Ver. I, (Sep-Oct. 2014), pp. 07-12.

[7] S. Sekar and K. Prabhavathi, “Numerical Solution of Second Order Fuzzy Differential Equations by Leapfrog Method”, *International Journal of Mathematics Trends and Technology*, vol. 16, no. 2, 2014, pp. 74-78.

[8] S. Sekar and K. Prabhavathi, “Numerical Strategies for the n^{th} –order fuzzy differential equations by Leapfrog Method”, *International Journal of Mathematical Archive*, vol. 6, no. 1, 2014, pp. 162-168.

[9] S. Sekar and K. Prabhavathi, “Numerical Strategies for the n^{th} –order fuzzy differential equations by Leapfrog Method”, *International Journal of Mathematical Archive*, vol. 6, no. 1, 2014, pp. 162-168.

[10] S. Sekar and K. Prabhavathi, “Numerical aspects of Fuzzy Differential Inclusions using Leapfrog Method”, *Global Journal of Pure and Applied Mathematics*, vol. 11, no. 1, 2015, pp. 52-55.

[11] S. Sekar and K. Prabhavathi, “Numerical treatment for the Nonlinear Fuzzy Differential Equations using Leapfrog Method”, *International Journal of Mathematics Trends and Technology*, vol. 26, no. 1, October 2015, pp. 35-39.

[12] S. Sekar and K. Prabhavathi, “Numerical investigation of the hybrid fuzzy differential equations using Leapfrog Method”, *International Journal of Pure and Applied Mathematics*, vol. 103, no. 3, 2015, pp. 385-394.

[13] S. Sekar and M. Vijayarakavan, “Numerical Investigation of first order linear Singular Systems using Leapfrog Method”, *International Journal of Mathematics Trends and Technology*, vol. 12, no. 2, 2014, pp. 89-93.

[14] S. Sekar and M. Vijayarakavan, “Numerical Solution of Stiff Delay and Singular Delay Systems using Leapfrog Method”, *International Journal of Scientific & Engineering Research*, vol. 5, no. 12, December-2014, pp. 1250-1253.

[15] S. Sekar and M. Vijayarakavan, “Observer design of Singular Systems (Robot Arm Model) using Leapfrog Method”, *Global Journal of Pure and Applied Mathematics*, vol. 11, no. 1, 2015, pp. 68-71.

[16] S. Sekar and M. Vijayarakavan, “Numerical approach to the CNN Based Hole-Filler Template Design Using Leapfrog Method”, *IOSR Journal of Mathematics*, vol. 11, issue 2 Ver. V, (Jul-Aug. 2015), pp. 62-67.

[17] S. Sekar and M. Vijayarakavan, “Numerical treatment of Periodic and Oscillatory Problems Using Leapfrog Method”, *International Journal of Mathematics Trends and Technology*, vol. 26, no. 1, October 2015, pp. 24-28.

[18] S. Sekar and G. Balaji, “Analysis of the differential equations on the sphere using single-term Haar wavelet series”, *International Journal of Computer, Mathematical Sciences and Applications*, vol. 4, nos.. 3-4, 2010, pp. 387-393.