No Measurable Cardinals in ZFC System

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Abstract — ***There** is no measurable cardinal number in Zermelo-Fraenkel-Axiom of Choice system. This result is proved in the article.*

Keywords — *measurable cardinals, ZFCsystem, lexicographic order*

I. Introduction

A nonempty set $X$is said to have a measurable cardinal number, if there is a non zero countably additive measure $m: P(X) \to \{0,1\}$ on the power set $P(X)$ of X such that $P(\{x\}) = 0, \forall x \in X$. If there is such a measure $m;$ then $P(X) = P_0(X) \cup P_1(X);$ when $P0(X) = \{A \subseteq X : m(A) = 0\}$ contains all singleton subsets $\{x\}$ of X and it is closed under countable unions; and when $P1(X) = \{A \subseteq X : m(A) = 1\}$ contains X and it is closed under countable intersections. The non existence of measurable cardinals has been unknown for a long time since 1904 (see [2]).

Let us first recall a known verification of the fact that the closed-open interval $[0,1)$ of the real line does not have a measurable cardinal number. Let $X = [0,1)$. Then $X \in P1(X)$; in the previous notations. So, only one subinterval $[x, x+\frac{1}{2})$, say, among the subintervals $[0,\frac{1}{2})$ and $[\frac{1}{2}, 1)$ belongs to $P1(X)$; when the other one belongs to $P0(X)$. Again, only one subinterval among the subintervals $[x,\ x+\frac{1}{4})$ and $[x+\frac{1}{4},\ x+\frac{1}{2})$ belongs to $P1(X)$, when the other one belongs to $P0(X)$. Thus it is possible to find a sequence of closed- open intervals $(In)_{n=1}^{\infty}$ such that $[0,1) \supseteq I1 \supseteq I2 \supseteq I3 \cdots$, $In \in P1(X), \forall n$, and such that length of In is 2^{-n}; $\forall n$. Then $\bigcap_{n=1}^{\infty} In \in P1(X)$, when $\bigcap_{n=1}^{\infty} In$ is either an empty set or a singleton set. This is a contradiction, which leads to the conclusion that the set $[0,1)$ does not have a measurable cardinal number. This classical argument is to be used to prove the non existence of measurable cardinal numbers in ZFC system. Note that the cardinality of $[0,1)$ is the cardinality of $In, \forall n = 1,2,\cdots$.

II. Construction of a Subdivision

Let $M = \{0,1,2,3,\cdots\}$, which is the well ordered set ω. Let $X = \{0,1\}^M - \{(1,1,1,\cdots)\}$, where $\{0,1\}^M = \{(x_0, x_1, x_2, \cdots) : x_i = 0\ or\ 1;\ \forall i\}$. Then the cardinal number of $[0,1)$ is the cardinality of X. It is first to be proved that $\{0,\ 1,\ 2,\ 3,\ \cdots\}$ does not have a measurable cardinal number, by using the technique discussed in the previous paragraph. Let us consider the following lexicographic order relation in $\{0;\ 1\}^M$ (see p.87 in [1]). $(x_0,\ x_1,\ x_2, \cdots) \le (y_0,\ y_1,\ y_2, \cdots)$ in $\{0,1\}^M$ if $x_i \le y_i;\ \forall i = 0,\ 1,\ 2, \cdots, or\ x_j < y_j$ for some j and $x_i = y_i$ for $i \le j-1$. Let us write $(x_0,\ x_1,\ x_2, \cdots) < (y_0,\ y_1,\ y_2, \cdots)$ if $(x_0,\ x_1,\ x_2,\ \cdots) \le (y_0,\ y_1,\ y_2,\ \cdots)$ and $(x_0,\ x_1,\ x_2,\ \cdots) \ne (y_0,\ y_1,\ y_2, \cdots)$. There are two "end points" $(0,0,0,\cdots)$ and $(1,1,1,\cdots)$ in $\{0,1\}^M$. The "point" $(1,0,1,0,1,0,\cdots)$ is considered as the "midpoint" of the two end points $(0,0,0,\cdots)$ and $(1,1,1,\cdots)$. The rule to construct "midpoint" of two sequences $(x_0, x_1, x_2, \cdots)$ and $(y_0, y_1, y_2, \cdots)$ satisfying $x_i \le y_i, \forall i$, in $\{0,1\}^M$, is the following. The first coordinate, which has a change in end points from 0 in one end point to 1 in another end point, receives 1 for midpoint. The first coordinate means first among coordinates with changes. Similarly, the third coordinate, the fifth coordinate, and so on, which have changes in end points from 0 to 1, receive 1 for midpoint. The second coordinate, the fourth coordinate, the sixth coordinate, and so on, which have changes in end points from 0 to 1, receive 0 for midpoint. The other coordinates, which do not have changes in end points from 0 to 1, receive same unchanged value for midpoint. Thus, the midpoint of the sequences $(0,0,0,\cdots)$ and $(1,0,1,0,1,0,\cdots)$ is $(1,0,0,0,1,0,0,0,1,0,0,0,1,\cdots)$. The midpoint of the sequences $(1,0,1,0,1,0,\cdots)$ and $(1,1,1,\cdots)$ is $(1,1,1,0,1,1,1,0,1,1,1,0,\cdots)$. The midpoint of the sequences $(0,0,0,\cdots)$ and $(1,0,0,0,1,0,0,0,1,0,0,\ 0,\cdots)$ is $(1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,\cdots)$. The midpoint of the sequences $(1,0,0,0,1,0,0,0,1,0,0,0,1,\cdots)$ and $(1,0,1,0,1,0,1,0,\cdots)$ is $(1,0,1,0,1,0,0,0,1,0,1,0,1,0,0,0,1,0,1,0,1,0,0,0,\cdots)$. Thus, for $x^{(0)} = (0,0,0\cdots)$ and $x^{(1)} = (1,1,1,\cdots)$, we can construct a sequence $x^{(2)}, x^{(3)}, x^{(4)}\cdots$, such that $x^{(2)}$ is the midpoint of $x^{(0)}$ and $x^{(1)}$, $x^{(3)}$ is the midpoint of $x^{(0)}$ and $x^{(2)}$, and $x^{(4)}$ is the midpoint of $x^{(2)}$ and $x^{(1)}$; and the points $x^{(5)}, x^{(6)}, x^{(7)}$, and $x^{(8)}$ are the midpoints of the respective pairs $(x^{(0)},\ x^{(3)}), (x^{(3)}, x^{(2)})$, $(x^{(2)},\ x^{(4)})$, and $(x^{(4)},\ x^{(1)})$; and so on. For the points x, y in $\{0;\ 1\}^M$ satisfying $x < y$, let us use the notation $[x, y)$ to denote the set $\{(x_n)_{n=0}^{\infty} \in X : x \le (x_n)_{n=0}^{\infty} < y\}$. Thus $x^{(2)}$ is the "midpoint" of

$[x^{(0)}, x^{(1)})$; $x^{(3)}$ is the "midpoint" of $[x^{(0)}, x^{(2)})$; $x^{(4)}$ is the "midpoint" of $[x^{(2)}, x^{(1)})$; $x^{(5)}$ is the "midpoint" of $[x^{(0)}, x^{(3)})$; and so on. Let $M0 = \{(1,0,0,0,\cdots),(1,1,0,0\cdots),$ $(1,1,1,0,\cdots),\cdots\}$.Then for any strictly decreasing sequence of closed-open subintervals $I1 \supset I2 \supset I3 \supset \cdots$ formed by midpoints constructed above starting from $x^{(0)}$ and $x^{(1)}$, the set $M0 \cap (\bigcap_{n=1}^{\infty} In)$ is an empty set or a finite set. More specifically, the following hold. The first element of $M0$ is in the intersection of all intervals containing the point $x^{(0)}$. The next two elements in $M0$ are in the intersection of all intervals containing the point $x^{(2)}$. The next four elements in $M0$ are in the intersection of all intervals containing the point $x^{(4)}$. The next eight elements in $M0$ are in the intersection of all intervals containing the point $x^{(8)}$,and this pattern continues. Now, the arguments of the previous section imply that $M0$ does not have a measurable cardinal number; when the arguments are applied to the intervals mentioned above.

Now, let $(M, \leq)$ denote a (general) well ordered set without least upper bound of M in M, and let us consider M as a set of ordinals in the form $[0,\lambda)$ under the natural identification. Let $X = \{(x_\alpha)_{\alpha\in M} : x_\alpha = 0 \text{ or } 1, \forall\alpha \in M\} - \{(y_\alpha)_{\alpha\in M} : y_\alpha = 1; \forall\alpha \in M\}$ be endowed with the lexicographic order $\leq$ of $\{(x_\alpha)_{\alpha\in M} : x_\alpha = 0$ or $1, \forall\alpha \in M\}$ (see p.87 in[1]). Let us start with $x^{(0)} = (x_\alpha)_{\alpha\in M}$ for which $x_\alpha = 0,\ \forall\alpha \in M$, and with $x^{(1)} = (y_\alpha)_{\alpha\in M}$ for which $y_\alpha = 1,\ \forall\alpha \in M$. Let us use the earlier interval notation. To find a midpoint, the previous rule is applied simultaneously on every well ordered subset $\{\alpha,\ \alpha+1,\ \alpha+2,\cdots\}$ of $M = [0,\lambda)$, for every limit ordinal α, when the rule is applied on the initial subset $\{0,1,2,3,\cdots\}$of M. Thus, we can again construct a sequence $x^{(2)}, x^{(3)}, x^{(4)},\cdots$ such that $x^{(2)}$ is the "midpoint" of $[x^{(0)}, x^{(1)})$; $x^{(3)}$ is the "midpoint" of $[x^{(0)}, x^{(2)})$; $x^{(4)}$ is the "midpoint" of $[x^{(2)}, x^{(1)})$; $x^{(5)}$ is the "midpoint" of $[x^{(0)}, x^{(3)})$; and so on. Let $M0 = \{({z_\alpha}^{(\beta)})\alpha\epsilon M : {z_\alpha}^{(\beta)} = 1, \forall\alpha \leq \beta$, and 0 for $\alpha > \beta$, for every fixed $\beta \in M\}$. Since M is well ordered, the set $M0$ is also a well ordered set in a natural way. One may find the following with respect to this order. The first element of $M0$ is in the intersection of all intervals containing the point $x^{(0)}$. The next two elements in $M0$ are in the intersection of all in-tervals containing the point $x^{(2)}$. The next four elements in $M0$ are in the intersection of all intervals containing the point $x^{(4)}$. The next eight elements in $M0$ are in the intersection of all intervals containing the point $x^{(8)}$, and this pattern continues. The points other than these countably many points of $M0$ do not belong to any intersection of any strictly decreasing sequence of closed-open intervals constructed through midpoints. Again, the earlier arguments can be applied to the intervals $[x^{(0)}. x^{(1)}), [x^{(0)}. x^{(2)}),$ $[x^{(2)}. x^{(1)}), [x^{(0)}. x^{(3)}), [x^{(3)}. x^{(2)}), [x^{(2)}. x^{(4)}),$ $[x^{(4)}. x^{(1)}),\cdots$ to conclude that $M0$ does not have a measurable cardinal number. So, M does not have a measurable cardinal number. Thus the following theorem has been proved.

Theorem: There is no set having a measurable cardinal number in ZFC system.

Proof: Every infinite set can be identified as a well ordered set M without least upper bound of M.

REFERENCES

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[2] K. Kuratowski and A. Mostowski, Set theory with an introduction to descriptive set theory, Second edition, North-Holland Publ.,Warszawa,1976.