A note on mathematical programming problems

Rashi Khubnani,Tarunika Sharma

Assistance Professor ,Mathematics Department ,New horizon college of Engineering ,Marathalli outer ring Road,Bangalore,560103

Abstract

A review has been done on the growth and improvement of solution of non linear programming problems under various factors. The mathematical work of Kuhn-Tucker, Fritz John, Mangasarian, Wolfe, Vond, Kanniappan has been traced and compared.

Key words
Objective function, Optimal solution, Pareto optimality, Duality.

I. INTRODUCTION

Mathematical Programming is a branch of optimization. It is used in various fields of man's activity where it is necessary to choose one course of action from several possible courses. Due to the rapid increase in size and complexity of problems the use of systematic approach to problem solving is stimulated. Nonlinear programming presents different perspective on mathematical programming problems in which the objective function and the constraint functions are not necessarily linear. The optimization problems in optimal control, structural design, mechanical design, electrical circuits, and stochastic resource allocation can be formulated as nonlinear programs. There are many real world problems which have more than one conflicting objective functions. Such programming problems are called multi objective programming problems. In mathematical programming it is customary to distinguish linear and convex programming. In nonlinear programming the objective function becomes nonlinear if one or more of the constraints inequalities have non-linear relationship or both. Non-linear programming which has the problem of minimizing a convex objective function in the convex set of points is called convex programming where the constraints may taken to be non-linear.

II. HYPOTHESES FORMULATION

(a) The general non linear problem can be defined as under

Minimize f(x)

Subject to $g_i(x) \leq 0$ for i=1,2...m

$h_i(x) = 0$ for i=1,2...l

$x \in X$

Where f, $g_1, g_2 ... g_m, h_1, h_2 ... h_l$ are the functions defined on R^n ,x is a vector of n components $x_1, x_2 ... x_n$.

The above problem must be solved for the values of variable $x_1, x_2 ... x_n$, that satisfy the restrictions and mean while minimize the function f.

The function f is usually called as objective function , $g_i(x) \leq 0$ for i=1,2...m are inequality constraint and $h_i(x) = 0$ for i=1,2...l are called equality constraint.

A vector $x \in X$ satisfying all constraints is feasible solution to the problem. All such solutions form feasible region .The non linear programming problem then is to find out feasible point x such that $f(x) \geq f(x)$ for each feasible point x. Such a point x is called optimal solution to the problem.

(b) A general multi objective programming problem having k(≥2) objectives is of the form:

min $f(x) = (f_1(x), f_2(x), \ldots f_k(x))$

Subject to $g_j(x) \leq 0$, j=1,2,…,m and $x \in S$

where f_i,i=1,2,…,k and g_j,j=1,2,…,m are real valued functions defined on $S \subseteq R^n$.

(c) In non-linear fractional programming we maximize(minimize) the ratio of two non- linear functions subject to linear or non-linear constraints. It is of the form

(FP) maximize $\frac{f(x)}{g(x)}$

Subject to $h_j(x) \leq 0$,j=1,2,…,m and $x \in S$

(FP) is said to be concave-convex fractional program, if f(x) is concave, g(x) is convex set S, if g is non-affine, then f is required to be non-negative .If f and g are differentiable, then concave convex fractional program has a pseudo concave objective function.

III. ANALYSIS

The best-known necessary optimality criterion for a mathematical programming problem is the Kuhn-Tucker criterion [2]. In order for the Kuhn-Tucker criterion to hold, one must impose a constraint-qualification on the constraints of the problem. However, the Fritz-John criterion [3] , is more general. In order for the Kuhn-Tucker criterion to hold, one must impose a constraint-qualification on the constraints of the problem. Moreover, the Fritz John criterion itself can be used to derive a form of the constraint qualification for the Kuhn-Tucker criterion, so no such constraint qualification is required.

Originally, Fritz John derived his conditions for the case of inequality constraints alone and if equality constraints are present, they are merely replaced by two inequality constraints. The new generalization of Fritz John's conditions derived in this work [1] treats equalities as equalities and does not convert them to inequalities, hence can be treated together.

Duality is the most important topic in optimization. In relation to primal nonlinear programming several dual programs are defined Wolfe formulated one of the most known dual in literature and established Weak and strong duality results under hypothesis that the functions occurring in problem are convex.

The following pair of programming has been studied by Wolfe [S]

(P) Minimize f(x) subject to $h_i(x) \geq 0$, i=1,…,m,
(D) Maximize f(x) - $\sum_i u_i h_i(x)$ subject to
$\nabla f(x) = \sum_i u_i \nabla h_i(x)$, $u \geq 0$.

Here f is a convex function on R^n and h_i's are concave functions .f and h_i are of course assumed differentiable. Furthermore a constraint qualification is assumed satisfied. Wolfe then proves the following:

Theorem: If x^0 is optimal for (P) then there exists a vector u^0 such that (x^0, u^0) is optimal for (D).Furthermore, the two problems have the same value.

Mangasarian gives a set of examples to show that Wolfe duality results may not longer hold if objective function is only pseudo convex but not convex even if constraints are linear. Mangasarian obtained necessary and sufficient conditions of optimality for nonlinear programming problems without assuming differentiability of the functions involved. He further derived Kuhn-Tucker's necessary optimality conditions under the weaker constraint qualification for pseudo-convex objective function and quasi-convex constraints.

A duality theorem of P. Wolfe for nonlinear differentiable programming is extended to the non differentiable case by replacing gradients by sub gradients [5].The dual pair is further simplified in the case that non differentiability enters only in the objective functions and then only through a positively homogeneous convex function.

The given programming problem was considered by BERTRAM VOND [6]

PRIMAL (P)
MINIMIZE $F(x) = f(x) + (x^t Bx)^{1/2}$
SUBJECT TO $g(x) \geq 0$

Where f and g are differentiable functions from R^n into R and R^m respectively, and B is an n x n symmetric positive semi definite matrix. If B = 0 then F is differentiable and (P) is the usual nonlinear programming problem. Special cases of (P), with F not differentiable, were also discussed.

The necessary and sufficient conditions were given for the existence of an M optimal solution, dual problem to (P) was formulated and appropriate duality theorems were established.

To do so Fritz-John conditions rather than Kuhn-Tucker conditions were used to discuss the advantages of using Fritz-John conditions rather than Kuhn-Tucker conditions to prove converse duality.

The results of this paper are easily extended to programming problems with objective function of the form

$$f(x) + \sum_{i=1}^{k} (x^t B^i x)^{1/2}$$

Where $B^i, i = 1, \dots, k,$ is positive semi-definite and symmetric. Extensions to complex programming are also possible.

In [7] SHINJI TANIMOTO discusses a more general class of non differentiable functions and duality theory for the programming problems involving such functions is considered.
The objective functions we treat here are of the type

$$f(x) = sup_{y \in Y} \phi(x, y), \qquad (1)$$

Where $\phi(.,.): R^n \times R^m \to R$ is a continuous function with continuous derivative with respect to x and Y is a specified compact subset in R^m.notice that f is not differentiable in general and that $f(.)$ is a convex function whenever $\phi(.,v)$ is a convex function of x for each v .

The primal problem is as follows:

PROBLEM (P)
MINIMIZE $f(x)$
SUBJECT TO $x \in X = \{x \in R^n | g(x) \leq 0\}$,

Where f is the function defined by (1) and $g(.): R^n \to R^p$ is a convex and differentiable function. We assume that X is non empty.

The purpose of this paper is to formulate two dual problems to Problem (P) and then to establish duality relationships. One of them has an extra restrictive condition, while the other does not. We obtain a duality theorem for each dual problem and converse duality is also discussed .

Later the concept of pareto optimality was used for multiobjective programming problem[8].Optimization problems with vector valued cost criteria or programming problem with several objectives conflicting with one another have been of considerable interest particularly in economics .In (necessary conditions for optimality of non differentiable convex multiobjective programming by P.Kanniappan) multiobjective programming problem which is also known as vector minimization problem isreduced to system of scalar minimization problems and then using the known results in convex programming necessary condition of fritz & Kuhn tucker type for pareto optimality were derived by introducing constraint qualification of slaters type ,the theorem of Schechter.

For scalar values objective functions was used for this purpose. Later on by using a distinct approach, Lai and Liu[9] established the necessary and sufficient conditions for Pareto optimality of the multiobjective function without constraint qualification of Salter's condition.

Minami worked on topological space in his paper [10], he discussed the non differentiable multiobjective convex program on a locally convex linear topological space in the case where the objective functions and the constraint functions are continuous and convex, but not necessarily Geateaux differentiable.

Applying the Dubovitskii-Milyutin separation lemma, he showed that the generalized Kuhn-Tucker conditions given by a sub differentiable formula are necessary and sufficient for weak Pareto-optimal solutions.

Furthermore, he discussed the non differentiable multiobjective program on a Banach space in the case where the objective functions and the constraint functions are locally Lipschitzian, but not necessarily convex; he gave Kuhn-Tucker forms, given by Clarke's generalized gradients, as necessary conditions for weak Pareto-optimal solutions.

And then using the results he worked on the non differentiable multiobjective program with equality and inequality constraints on a Banach space and give necessary conditions for weak Pareto-optimal solutions.

IV REFERENCES

[1] MANGASARIAN AND S. FROMOVITZ ,The Fritz John Necessary Optimality Conditions in the Presence of Equality and Inequality Constraints, *journal of mathematical analysis and applications 17, 37-47 (1967)*

[2] H. W. KUHN AND A. W. TUCKER. Nonlinear programming. *In "Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability" (J. Neyman, ed.), pp. 481-492. Univ. of Calif. Press, Berkeley, Calif., 1951.*

[3] F. JOHN. Extremum problems with inequalities as side conditions. In *"Studies and Essays, Courant Anniversary Volume" (K. 0. Friedrichs, 0. E. Neugebauer and J. J. Stoker, eds.), pp. 187-204. Wiley (Interscience), New York, 1948.*

[4] P. WOLFE, a duality theorem for nonlinear programming, *quart. appl. math. 19(1961), 239-244.*

[5] MURRAY SCHECHTER,A Subgradient Duality Theorem, *journal of mathematical analysis and applications 61, 850-855 (1977)*

[6] BERTRAM VOND, A Class of Nondifferentiable Mathematical Programming Problems, *Journal of mathematical analysis and applications 46, 169-174 (1974)*

[7] SHINJI TANIMOTO, Duality for a Class of Nondifferentiable Mathematical Programming Problems, *Journal of mathematical analysis and applications 79, 286-294 (1981)*

[8] P.KANNIAPPAN,Nondifferentiable Convex MultiobjectiveProgramming I *Journal of optimization theory and applications: vol. 40, no. 2, june 1983*

[9] LAI, H. C., and LIu, J. C., Necessary and Sufficient Conditions for Pareto Optimality in Nondifferentiable Convex Multiobjective Programming, *NationalTsing Hua University, Taiwan, Prepdnt, 1984*

[10] M1NAMI, M., Weak Pareto Optimality of Muhiobjective Problems in a Locally Convex Linear Topological Space, *Journal of Optimization Theory and Applications, Vol. 34, pp. 469-484, 1981.*

[11] TANINO, T., and SAWARAKI, Y., *Duality Theory in Multiobjective Programming*, Journal of Optimization Theory *on Sub gradient Duality, Journal of Mathematical Analysis* and Applications, Vol. 71, pp. 251-262, 1979.

[12] . ROCKAFELLAR, R. T., *Convex Analysis,* Princeton University Press, Princeton, New Jersey, 1970.nd Applications, Vol. 27, pp. 509-529, 1979.