A Common Fixed Point Theorem in Fuzzy Metric Spaces

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***ABSTRACT: In** this paper, we prove a common fixed point theorem for weakly compatible mappings in a fuzzy metric space which generalize and unify the several results.*

KEY Words: - *Fixed point, quasi-contraction, fuzzy metric space, Cauchy sequence, weakly compatible maps.*

AMS SUBJECT CLASSIFICATION: *47H10, 54H25*

1. INTRODUCTION

The notion of fuzzy set was introduced by Zadeh [9]. It was developed extensively by many authors and used in various fields. In this paper we deal with the fuzzy metric space defined by Kramosil and Michalek [6] and modified by George and Veeramani [3].The most interesting references in this direction are Chang [1], Cho [2], Grabiec [4], and Kaleva [5]. In the present paper, we prove a common fixed point theorem for six self mapping by Weakly Compatibility Condition.

2. PRELIMINARIES

DEFINITION 2.1[8]. A binary operation $*$: [0, 1] × [0, 1] → [0, 1] is called a continuous t-norm if([0, 1], $*$) is an abelian topological monoid with the unit 1 such that $a*b \le c*d$ and whenever $a \le c$ and $b \le d$ for all a, b, c, d ∈ [0, 1].

DEFINITION 2.2[6]. The 3-tuple (X, M, $*$) is called a fuzzy metric space (shortly, FM-space) if X is an arbitrary set, $*$ a continuous t-norm and M is a fuzzy set in X × X × [0,∞) satisfying the following conditions:
for all x, y, z ∈ X and s, t > 0.
(FM-1) M(x, y, 0) = 0,
(FM-2) M(x, y, t) = 1 for all t > 0 if and only if x = y,
(FM-3) M(x, y, t) = M(y, x, t)
(FM-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,
(FM-5) M(x, y,.): [0, ∞] → [0, 1] is left continuous,
Note that M(x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify x = y with M(x, y, t) = 1 for all t > 0. The following example shows that every metric space induces a fuzzy metric space.

EXAMPLE 2.3. [3]. Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all x, y ∈ X and all t > 0. Then (X, M, $*$) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

LEMMA 2.4. [4]. For all x, y ∈ X, M(x, y,.) is a non decreasing function.

DEFINITION 2.5 [4]. A sequence $\{x_n\}$ in a fuzzy metric space (X, M, $*$) is said to be a Cauchy sequence if and only if for each ε > 0, t > 0 , there exists $n_0 \in N$, such that $M(x_n, x_m, t) > 1 - \varepsilon$, for all n, $m \ge n_0$. The sequence $\{x_n\}$ is said to converge to a point x in X if and only if for each, ε > 0, t > 0, $n_0 \ge N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \ge n_0$.
A fuzzy metric space (X, M, $*$) is said to be complete if every Cauchy sequence in it converges to a point in it.

REMARK 2.6. Since $*$ is continuous, it follows from (FM-4) that the limit of the sequence in FM-space is uniquely determined. Let (X, M, $*$) be a fuzzy metric space with the following conditions

(FM-6) $\lim_{t \to \infty} M(x, y, t) = 1$ for all x, y ∈ X.

LEMMA 2.7[2]. Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, $*$) with $t*t \ge t$ for all t ∈ [0,1] and condition (FM-6). If there exists a number k ∈ (0,1) such that

$$M(x_{n+2}, x_{n+1}, qt) \ge M(x_{n+1}, x_n, t)$$

for all t □ 0 and n = 1,2 . . . then $\{x_n\}$ is a Cauchy sequence in X.

LEMMA 2.8 [7]. If for all x, y ∈ X, t > 0 with positive number k ∈ (0,1) and

$$M(x, y, kt) \ge M(x, y, t),$$

then x = y.

3. MAIN RESULTS

THEOREM 3.1. Let (X, M, $*$) be a complete fuzzy metric space. Suppose that A, B, S, P, Q and T are mappings from X to itself such that,
(3.1.1) P(X) ⊂ AB(X), Q(X) ⊂ ST(X)
(3.1.2) The pairs (P, ST) and (Q, AB) are weakly compatible.
(3.1.3) There exists a number k ∈ (0, 1) such that

$$M(Px ,Qy ,kt) \ge \min\{ M(STx ,ABy,t), M(Px, STx, t), M(ABy ,Qy ,t),M (ABy, Px, t), M(STx,Qy,t)\}$$

with k ∈ (0, 1), then P,Q,AB and ST have a unique common fixed point.

If the pairs (A,B),(S,T),(Q,B)and(T,P)are commuting mappings then A,B,S,T,P,Q have a unique common fixed point.

PROOF: Let $x_0 \in X$ be any arbitrary point in X. We define sequence $\{ y_n \}$ and $\{x_n\}$ such that

(3.1.4) $y_{2n} = STx_{2n} = Qx_{2n+1}$ and $y_{2n+1} = AB\, x_{2n+1} = Px_{2n}$, n=1,2,3,… This is always possible because of the condition (3.1.1)

Now taking $x=x_{2n}$ and $y = x_{2n+1}$ in (3.1.3) we have

(3.1.5) $M (y_{2n+1}, y2n, kt) = M (Px_{2n}, Qx_{2n+1}, kt)$
$\geq \min \{M (STx_{2n} ,ABx_{2n+1},t),$
$M (Px_{2n}, STx2n, t),$
$M (ABx_{2n+1}, Qx_{2n+1}, t),$
$M (ABx_{2n+1}, Px_{2n}, t),$
$M (STx_{2n}, Qx_{2n+1}, t)\}$
$= \min \{M (y_{2n},y_{2n+1},t), M(y_{2n+1},y_{2n},t),$
$M (y_{2n+1},y_{2n} ,t), M(y_{2n+1},y_{2n+1},t),$
$M (y_{2n},y_{2n},t)\}$

which implies

$M (y_{2n}, y_{2n+1},k t) \geq M (y_{2n},y_{2n+1},t)$

In general

$M (y_n, y_{n+1}, kt) \geq M (y_{n-1}, y_n, t)$

To prove that $\{y_n\}$ is a Cauchy sequence we prove by the method of induction that for all $n \geq n_0$, and for every $m \in N$,

(3.1.6) $M (y_n, y_{n+m}, t) \geq 1-\lambda.$

From (3.1.3) we have

$M (y_n, y_{n+1}, t) \geq M (y_{n-1}, y_n, \frac{t}{k}) \geq M (y_{n-2}, y_{n-1}, \frac{t}{k^2})$
$\geq \; . . . . . \geq M (y_0, y_1, \frac{t}{k^n}) \rightarrow 1$ as $n \rightarrow \infty$.

For $t > 0$, $\lambda \in (0, 1)$, there exist $n_0 \in N$ such that

$M (y_n, y_{n+1}, t) \geq 1-\lambda$

Thus (3.1.6) is true for m=1.Suppose (3.1.6) is true for all m then we will show that it is also true for m+1.

Using the definition of fuzzy metric space, we have

(3.1.7) $M (y_n, y_{n+m+1}, t) \geq \min \{M (y_n, y_{n+m}, \frac{t}{2}), M (y_{n+m}, y_{n+m+1}, \frac{t}{2})\} \geq 1-\lambda$

Hence (3.1.6) is true for m+1.

Thus $\{y_n\}$ is Cauchy sequence. By completeness of (X, M,*), $\{y_n\}$ convergence to some point z in X.

$Px_{2n}, Qx_{2n+1}, ABx_{2n+1}, STx_{2n} \rightarrow z$ as $n\rightarrow \infty$.

Since $P(X) \subset AB(X)$, for a point $u \in X$ such that $ABu = z$

Since $Q(X) \subset ST(X)$,for a point $v \in X$ such that $STv=z$

Putting $x=v$, $y=x_{2n+1}$ in (3.1.3)

(3.1.8) $M(Pv,Qx_{2n+1},kt) \geq \min\{M (STv, Pv, t),$
$M(ABx_{2n+1},Qx_{2n+1},t),$
$M(STv,ABx_{2n+1},t),$
$M (ABx_{2n+1}, Pv, t),$
$M (STv, Qx_{2n+1}, t)\}$

Proceeding limit as $n\rightarrow\infty$, we have

$M(Pv ,z , kt) \geq \min\{M(z , Pv ,t),M(z ,z ,t),M(z ,z ,t),$
$M(z ,Pv ,t),M(z ,z ,t)\}$
$\geq M (z, Pv, t),$

Which gives Pv = z, therefore

(3.1. 9) STv = Pv = z

(P, ST) are weakly compatible, so they commute at coincidence point

Therefore

P (STv) = (ST) Pv that is Pz = STz thus

(3.1. 10) Pz = STz

Putting x =v, y = u in (3.1.3)

(3.1.11) $M(Pv,Qu,kt)\geq\min\{M(STv,Pv,t), M(ABu,Qu,t),$
$M(STv,ABu,t),M(ABu , Pv,t)$
$, M (STv, Qu, t)\}$
$\geq \min\{M(z,z,t),M(z,Qu,t),M(z,z,t),$
$M(z ,z ,t),M(z , Qu, t)\}$

Which gives z = Qu

Therefore Qu = z = ABu

Since (Q, AB) is weakly compatible pair (AB) Qu = Q (ABu) implies ABz = Qz

Thus

(3.1.12) ABz = Qz

Now, we show that z is the fixed point of P by putting $x = x_{2n}$, $y = z$ in (3.1.3)

we have

(3.1. 13) $M (Px_{2n}, Qz, kt) \geq \min\{M(STx_{2n} ,Px_{2n} ,t), M(ABz,Qz,t),M(STx_{2n} ,ABz , t),M(ABz,Px_{2n},t),$
$M (STx_{2n}, Qz, t)\}$

let $n \rightarrow \infty$

$\geq \min\{M(z ,z , t),M(Qz ,Qz ,t), M(z ,Qz ,t),M(Qz ,z ,t),M(z ,Qz ,t)\}$
$\geq M (z, Qz, t)$

which shows z = Qz

(3.1. 14) Thus z = Qz = ABz

Now, we show that z is the fixed point of P by putting x=z, $y=x_{2n+1}$ with $\alpha = 1$ in (3.1.4) we have

$M (Pz, Qx_{2n+1} , kt) \geq \min\{M(STz ,Pz,t),$
$M(ABx_{2n+1},Qx_{2n+1},t) M(STz , ABx_{2n+1},t),$
$M (ABx_{2n+1}, Pz, t) \quad M (STz, Qx_{2n+1}, t)\}$

Let $n\rightarrow \infty$

$M (Pz, z , kt) \geq \min\{M(Pz ,Pz, t)M(z ,z, t)M(Pz ,z, t)M(z, Pz ,t)M(Pz ,z, t)\}$
$\geq M (z ,Pz ,t)$

Which show z = Pz

(3.1.15) Thus Pz = z=STz

Now, we show that z = Tz, by putting x= Tz and $y = x_{2n+1}$ in (3.1.3) and using the commutatively of the pairs (T,P) & (S,T)

(3.1.16) $M (P (Tz), Qx_{2n+1}, kt) \geq$
$\min\{M(ST(Tz),P(Tz),t),M(ABx_{2n+1},Qx_{2n+1},t),M(ST(T$

z),ABx_{2n+1},t), M (ABx_{2n+1}, P (Tz), t) ,M (ST (Tz),Q$x_{2n+1,}$t)}

Let n→ ∞ and using (3.1.15)

(3.1.17) M (Tz, z, kt) ≥ min{M(Tz , Tz ,t),M(z ,z ,t),

M(Tz ,z ,t),M(z ,Tz ,t),M(Tz ,z, t)}

≥ M (Tz, z, t)

Which gives z = Tz.

Since STz = z gives Sz = z,

Finally we have to show that Bz = z.

By putting x= z, y = Bz in (3.1.3) and using the commutatively of the pairs (Q,B) & (A,B)

(3.1.19) M (Pz, QBz, kt) ≥ min{M(STz ,Pz ,t),

M(AB(Bz),Q(Bz),t),M(STz ,AB(Bz) ,t),

M(AB(Bz) ,Pz ,t),M(STz ,Q(Bz),t)}

≥ min{M(z ,z ,t),M(Bz , Bz ,t),

M(z ,Bz ,t),M(Bz ,z ,t),M(z ,Bz ,t)}

M (z, Bz,kt) ≥ M (z, Bz, t)}

Which gives z = Bz.

Since ABz = z implies Az = z

By combination the above results, we have,

(3.1.20) Az = Bz = Sz = Tz = Pz = Qz = z

That is z is the common fixed point of A, B, S, T, P, and Q. For uniqueness, let w (w ≠ z) be another common fixed point of A, B, S, T, P and Q then by (3.1.3),we write

(3.1.21) M (Pz, Qw, kt) ≥ min{M(STz ,Pz ,t),

M(ABw ,Qw ,t),M(STz ,ABw ,t),M(ABw ,Pz ,t),

M(STz ,Qw ,t)}

M (z, w, kt) ≥ M (z, w, t)

Which gives z = w.

If we put B=T=I_x (the identity map on X) in the theorem 3.1 we have the following

COROLLARY (3.2): Let (X, M,*) be a complete fuzzy metric space with a*a ≥ a for all a ∈ [0, 1] and the condition (FM6)

Let A, S, P, Q be mappings from X into itself such that

(3.2.1) P(X) ⊂ A(X), Q(X) ⊂ S(X),

(3.2.2) the pair (P, S) and (Q, A) are weakly compatible,

(3.2.3) There exist a number k ∈ (0, 1) such that

M(Px ,Qy ,kt) ≥ min{M(Sx ,Ay ,t), M(Px ,Sx ,t),

M(Ay ,Qy ,t),M(Ay ,Px ,t),M(Sx, Qy,t)}

for all x, y ∈ X, and t > 0 then P ,S ,A and Q have a unique common fixed point.

If we put P = Q, B = T = I_x in the theorem 3.1 we have the following.

COROLLARY (3.3): Let (X, M,*) be a complete fuzzy metric space with a*a ≥ a, for all a ∈ [0, 1] and

The condition (FM6).Let A, S, T be mapping from X into itself such that

(3.3.1) P(X) ⊂ A(X), P(X) ⊂ S(X),

(3.3.2) The pair (P, A) and (P, S) are weakly compatible,

(3.3.3) There exist a number k ∈ (0, 1) such that

M (Px, Py, kt) ≥ min{M(Sx ,Ay ,t),M(Px ,Sx, t),

M(Ay,Py,t),M(Ay ,Px, t), M(Sx ,Py,t)}

for all x ,y∈ X, and t > 0 then P ,S ,A have a unique common fixed point.

If we put P=Q, A=S and B=T=I_x in the theorem 3.1 we have the following

COROLLARY (3.4): Let (X, M,*) be complete fuzzy metric space with a*a ≥ a for all a ∈ [0, 1] and the

condition (FM6).Let (P, S) be weakly compatible pair of self maps such that,

P(X) ⊂ S(X) and there exist a constant k ∈ (0, 1) such that

M(Px ,Py ,t) ≥ min{M(Sx ,Sy ,t),M(Px ,Sx ,t),M(Sy ,Py ,t),

M(Sy ,Px, t),M(Sx ,Py ,t)}

For all x, y ∈ X, and t> 0, then P and S have a unique common fixed point in X

If we put A=S and B=T=I_x in theorem 3.1 we have the following.

COROLLARY (3.5): Let (X, M, *) be complete fuzzy metric space with a*a ≥ a for all a ∈ [0,1] and the condition (FM6).Let P ,Q ,S be mappings from X to itself such that ,

(3.5.1) P(X) ⊂ S(X), Q(X) ⊂ S(X)

(3.5.2) Either (P, S) or (Q, S) is weakly compatible pair

(3.5.3) M(Px ,Qy ,kt) ≥ min{M(Sx , Sy ,t),M(Px ,Sx ,t),

M(Sy ,Qy ,t),M(Sy ,Px , t),M(Sx ,Qy ,t) }

for all x ,y ∈ X and t > 0 then P ,Q and S have a unique common fixed point in X

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