On Modified Near Algebras

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Abstract **:** *The charm of the mathematics in normed algebras [1] lies in the fine interplay between the algebraic and topological properties of the algebras [2]. To see how the introduction of a norm on a strong near module bestows nice properties to a strong near module, we introduce the concept of a normed strong near module in this paper*

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1. Introduction

Definition 1.1. A **near algebra** A is a linear space over R on which a multiplication is defined such that (1) A forms a semigroup under multiplication;

(2) multiplication is right distributive with respect to addition;

(3) $\alpha(xy)=(\alpha x)y$ for all $x,y\in A$ and $\alpha\in R$.

Definition1.2. [3] A near algebra B is called a **normed near algebra** provided that there is associated with each $x\in B$ a real number $\|x\|$ called the norm of x, with the following properties:

(1) $\|x\|\geq 0$ and $\|x\|=0$ if and only if $x=0$(the additive identity of B);

(2) $\|x+y\|\leq\|x\|+\|y\|$;

(3) $\|\alpha x\|=|\alpha|\ \|x\|$;

(4) $\|xy\|\leq\|x\|\|y\|$;

(5) $\|xy-xz\|\leq\|x\|\ \|y-z\|$, for all $x,y,z\in B$ and $\alpha\in R$;

(6) If B has an identity e, then $\|e\|=1$.

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2. MAIN RESULTS

Let $(M,+)$ be a group and let N be a near ring and suppose `$\cdot$' is a mapping of $N\times M$ into M.

Definition 2.1. $(M,+,\cdot)$ is called a **strong near module** over N if

(1) $(n_1+n_2)m=n_1m+n_2m$ for all $n_1,n_2\in N$ and $m\in M$;

(2) $n(m_1+m_2)=nm_1+nm_2$ for all $n\in N$ and $m_1,m_2\in M$;

(3) $(n_1n_2)m=n_1(n_2m)$ For all $n_1,n_2\in N$ and $m\in M$.

Definition 2.2. A **normed strong near moldule** is a strong near module M over the field of reals on which there is defined a norm i.e., a function which assigns to each element m in the space a real number $\|m\|$ in such a manner that

(1) $\|m\|\geq 0$ and $\|m\|=0$ if and only if $m=0$;

(2) $\|m_1+m_2\|\leq\|m_1\|+\|m_2\|$;

(3) $\|\alpha m\|=|\alpha|\ \|m\|$ for all $m,m_1,m_2\in M$ and $\alpha\in R$.

Definition 2.3. A **Modified strong near algebra** M is a strong near module $(M,+,\cdot)$ over a near ring $(N,+,\cdot)$ on which multiplication $*$ is defined such that

(1) $(M,*)$ is a semigroup;

(2) $(m_1+m_2)*m_3=m_1*m_3+m_2*m_3$;

(3) $n(m_1*m_2)=(nm_1)*m_2$ for all $m_1,m_2,m_3\in M$ and $n\in N$.

Definition 2.4. Let M be a normed strong near module. We call a mapping $f:M\to R$ a **semilinear map** if for every m_1,m_2 in M,

$$f(f(m_1)m_2)=f(m_1)f(m_2).$$

Lemma 2.5. Suppose M is a strong near module over the real field R. Suppose m_0 is a non zero element of M and

$M_0 = \{rm_0 / r \in R\}$. Then M_0 is a one-dimensional vector space over R.
Proof. Routine.

Theorem 2.6. Let f be a nonconstant semilinear map on a normed strong module M. If $f(m) < 0$ for some $m \in M$ then $f(M) = R$. If $f(m) \geq 0$ for all m in M then $f(M) = [0, \infty)$.

Proof. If $f(0) \neq 0$ then for any m in M, $f(m)f(0) = f(f(m)0) = f(0)$ $\Rightarrow f(0)[f(m) - 1] = 0 \Rightarrow f(m) = 1$ for all $m \in M$ which is a contradiction.
Hence, we have proved that $f(0) = 0$.
Case(i): Suppose that $f(m) \geq 0$ for all m.
Since f is nonconstant, there exists an element m_0 in M such that $0 < f(m_0)$.
Let $f(m_0) = \alpha$ and M_0 be the one dimensional subspace of M generated by m_0.
By the connectedness of M_0, we see that $f(M_0)$ is an interval in R that contains two different elements $0, \alpha$ with $\alpha > 0$.
Therefore $f(M_0)$ is a non-degenerate interval.
Hence $f(M_0)$ has an element $b > 0$ such that $b \neq 1$. Let $b = f(m)$. Then, by an easy inductive argument, we can show that
$f(b^n m) = b^{n+1}$ …………(1).
Also, we have

$$b = f(m) = f\left(b\frac{1}{b}m\right) = bf\left(\frac{1}{b}m\right) \text{ so that}$$

$$f\left(\frac{1}{b}m\right) = 1.$$

$$\text{Also } 1 = f\left(b\frac{1}{b^2}m\right) = bf\left(\frac{1}{b^2}m\right) \text{ implies}$$

$$f\left(\frac{1}{b^2}m\right) = \frac{1}{b}.$$

Let $m_1 = \frac{1}{b^2}m$ and $\frac{1}{b} = c$. Then $f(m_1) = c$ and an appeal to (1) shows that
$f(c^n m_1) = c^{n+1}$

i.e., $f\left(\frac{1}{b^n}m_1\right) = \frac{1}{b^{n+1}}$.

Thus, $\{b^n, b^{-n} / n \in N\} \subseteq f(M_0)$.
Since one of b, b^{-1} is greater than 1, it now follows that $f(M_0) = [0, \infty)$,
since $(-\infty, 0) \cap f(M_0) = \phi$, it follows that $f(M) = [0, \infty)$.
Case(ii): Suppose that $f(M)$ contains a negative number. Let $f(m) = c < 0$ for some $m \in M$ and
M_0 be the subspace generated by m.
Since $0 \in f(M_0)$, $f(M_0)$ contains two distinct elements. Since $f(M_0)$ is an interval of

$$\text{Also, } c = f(m) = f\left(c\frac{1}{c}m\right) = cf\left(\frac{1}{c}m\right) \text{ which}$$

$$\text{shows } f\left(\frac{1}{c}m\right) = 1.$$

$$\text{Now } 1 = f\left(\frac{1}{c}\right)m = f\left(c\frac{1}{c^2}m\right) = cf\left(\frac{1}{c^2}m\right)$$

$$\text{shows that } f\left(\frac{1}{c^2}\right)m = \frac{1}{c}.$$

As above, we get $f\left(\frac{1}{c^{n+1}}m\right) = \frac{1}{c^n}$ for all positive integers $n > 0$.
Hence,

$$\{c, c^2, c^3, \ldots, c^n, \ldots\} \cup \left\{\frac{1}{c}, \frac{1}{c^2}, \ldots, \frac{1}{c^n}, \ldots\right\} \subseteq f(M_0)$$

.

Since one of $c, \frac{1}{c}$ is less than – 1,
the connected subset $f(M_0)$ of R contains both positive and negative elements of arbitrarily large absolute value.
Hence, $f(M_0) = R$. Thus, we derive that $f(M) = R$ in this case.

Definition 2.7. A modified strong near algebra $(M, +, \cdot, *)$ is said to be **unitary** if $1m = m$ for all $m \in M$.

Theorem 2.8. In a normed unitary modified strong near algebra M_f where f is a nonconstant semilinear map there exists an element e in M such that $f(e)=1$ and any such e is a right identity.

Proof. Since f is nonconstant, by Theorem [2.6] $f(M)=[0,\infty)$ or R.

In either case, $1\in f(M)\Rightarrow 1=f(e)$ for some $e\in M$.

For any $m*e=f(e)m=1m=m$ for all $m\in M$.

This shows that e is a right identity.

Theorem 2.9. If M_f is a normed unitary modified strong near algebra where f is a nonconstant semilinear map and one-one, then M_f is commutative.

Proof. For any $m_1, m_2 \in M$,

$m_1 * m_2 = f(m_2)m_1$ and $m_2 * m_1 = f(m_1)m_2$.

Now

$f(m_1 * m_2) = f(f(m_2)m_1) = f(m_2)f(m_1)$.

Also

$f(m_2 * m_1) = f(f(m_1)m_2) = f(m_1)f(m_2)$.

Since $(R,\cdot)$ is commutative, we have

$f(m_1 * m_2) = f(m_2 * m_1)$.

Since f is one-one, we have

$m_1 * m_2 = m_2 * m_1$.

So `$*$' is commutative on M and hence $(M_f,+,\cdot,*)$ is commutative.

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