

\mathcal{K} - Quasi Normal Topological Spaces

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Abstract: The purpose of this paper is to introduce a weaker version of \mathcal{K} -normality ($\mathcal{K} = \alpha, p, s, b \& \beta$) called \mathcal{K} - Quasi normality, which surely lies between strongly \mathcal{K} -normality & \mathcal{K} - normality. It contains the fact that \mathcal{K} - Quasi normality is topological property as well as hereditary property with respect to open subspaces. The characterization & preservation theorem in the context are presented in order to continue the study of the class of normal spaces, namely \mathcal{K} - Quasi normal spaces.

Key words: \mathcal{K} - Quasi normal spaces, \mathcal{K} -open sets & \mathcal{K} -closed sets.

I. INTRODUCTION

1982 is the year for projection & extensive research of the concept of pre-open sets alongwith pre-continuous & research of the concept of pre-open sets alongwith pre-continuous mappings in general topology by A.S.Mashhour et.al[1]. C.Kuratowski mentioned that a set in a space is said to be regular open set or an open domain if it is the interior of its closure.

As a recall, in 1963 semi-open sets were defined as the subset A of the space (X, T) is semi-open, denoted by $A \in SO(X, T)$, iff there exists $O \in T$ such that $O \subset A \subset Cl(O)$.

In 1978 α -sets were introduced as the subset A is an α -open set, denoted by $A \in \alpha O(X, T)$, iff $A \subset \text{int}(cl(\text{int}(A)))$. Then in 1981, pre-open sets were introduced. The subset A is preopen, denoted by $A \in PO(X, T)$, iff $A \subset \text{int}(cl(A))$.

The semi-pre open set, called by D. Andrijevic was conceptualized under the name β -open by M.E. Abd El - Monsef etc in 1983 as A is a β -open set, denoted by $A \in \beta O(X, T)$, iff $A \subset cl(\text{int}(cl(A)))$.

Andrijevic[4] introduced a new class of generalized open sets in a topological space, the so called b -open sets, which is contained in the class of semi-pre-open sets and contains all semi-open sets and pre-open sets. Also, such a class of b -open sets generates the same topology as the class of pre-open sets. Extensive research on generalizing of closeness was done in the recent years and many topologists have utilized these concepts to the various notions of subsets, weak separation axioms, weak regularity, and weaker and stronger forms of covering axioms in the literature.

In this paper we utilize these sets to define and study the new classes of spaces, called \mathcal{K} - quasi normal spaces in topology. Also we characterize their basic properties along with already existing weaker forms of normality.

The characterization as well as the preservation theorems for \mathcal{K} -quasi-normal spaces where $\mathcal{K} = \alpha, p, s, b \& \beta$ with common basic properties has been focused and prepared as a ready reckoner for the researchers in this paper.

Any other notion and symbol, not defined in this paper, may be found in the appropriate reference.

II. \mathcal{K} -QUASI-NORMAL SPACES

($\mathcal{K} = \alpha, p, s, b \& \beta$)

Definition (2.1): A space (X, T) is said to be \mathcal{K} -quasi -normal (briefly, $\mathcal{K}q$ -normal) if for each pair of disjoint closed set A and \mathcal{K} -closed set B of X there exist \mathcal{K} -open sets $U \& V$ in the manner $A \subseteq U$ and $B \subseteq V$ such that $U \cap V = \emptyset$, where $\mathcal{K} = \alpha, p, s, b \& \beta$.

As a recall of strongly \mathcal{K} -normality & \mathcal{K} -normality, it is obvious that $\mathcal{K}q$ -normality properly fits in between strongly \mathcal{K} -normality & \mathcal{K} -normality & the implication diagram is given as below.

Strongly \mathcal{K} -normality $\rightarrow \mathcal{K}q$ -normality \rightarrow
 \mathcal{K} -normality

None of these implication is reversible as shown by the following examples:

Example (2.1): (i) let $X = \{a, b, c\}$, $T = \{\emptyset, \{b\}, \{b, c\}, X\}$,

then, $T^c =$ set of all T -closed sets $= \{\emptyset, \{a\}, \{a, c\}, X\}$

Now, $PO(X, T) = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$ & $PC(X, T) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$.

Obviously, (a) (X, T) is p -normal.

(b) (X, T) is not strongly p -normal.

(c) (X, T) is also not quasi- p -normal.

(ii) Let $X = \{a, b, c\}$, & $T = \{\emptyset, \{b\}, \{b, c\}, X\}$, $T^c = \{\emptyset, \{a\}, \{a, c\}, X\}$

Then $\alpha O(X) = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\} = \beta O(X)$ & $\alpha C(X) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\} = \beta C(X)$.

- (a) The space (X,T) is normal as well as α -normal.
- (b) The space (X,T) is neither strongly α -normal nor α -quasi normal.
- (c) The space (X,T) is normal as well as β -normal.
- (d) The space (X,T) is neither strongly β -normal nor β -quasi normal.

(iii) Let $X = \{a,b,c,d\}$, & $T = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$,
 $T^c = \{\emptyset, \{a,d\}, \{a,b,d\}, \{a,c,d\}, X\}$,
 Then $\alpha O(X) = \{\emptyset, \{b\}, \{c\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}, X\}$ & $\alpha C(X) = \{\emptyset, \{a\}, \{d\}, \{a,d\}, \{a,b,d\}, \{a,c,d\}, X\}$.

- (a) The space (X,T) is not strongly α -normal.
- (b) The space (X,T) is normal, α -normal & α q-normal.

The characterization of \mathcal{K} -quasi normality (i.e. $\mathcal{K}q$ -normality) has been enunciated as:

Theorem (2.1): A topological space (X,T) is $\mathcal{K}q$ -normal iff every closed set A and a \mathcal{K} -open set V containing A there is a \mathcal{K} -open set U such that $A \subseteq U \subseteq \mathcal{K}\text{-cl}(U) \subseteq V$.

Proof: Suppose that (X,T) be a $\mathcal{K}q$ -normal space, in which A is a closed set and V is a \mathcal{K} -open set containing A . Then V^c is a \mathcal{K} -closed set such that $A \cap V^c = \emptyset$.

Since, (X,T) is a $\mathcal{K}q$ -normal, hence, there exist \mathcal{K} -open sets U & W such that $A \subseteq U$, $V^c \subseteq W$ and $U \cap W = \emptyset$.

This means that $A \subseteq U \subseteq W^c \subseteq V$.

Since, W^c is a \mathcal{K} -closed set containing the \mathcal{K} -open set U , hence, $U \subseteq \mathcal{K}\text{-cl}(U) \subseteq W^c$.

Therefore, $A \subseteq U \subseteq \mathcal{K}\text{-cl}(U) \subseteq V$.

Conversely, assume that the prefixed condition holds good.

Let A & B be disjoint pair of a closed and a \mathcal{K} -closed set respectively. Then $A \cap B = \emptyset \Rightarrow A \subseteq B^c$ and B^c is \mathcal{K} -open.

Now, according to the prefixed condition there exist a \mathcal{K} -open set U such that $A \subseteq U \subseteq \mathcal{K}\text{-cl}(U) \subseteq B^c$. This provides that

$A \subseteq U$, $B \subseteq [\mathcal{K}\text{-cl}(U)]^c$ and $U \cap [\mathcal{K}\text{-cl}(U)]^c = \emptyset$.

Hence, (X,T) is $\mathcal{K}q$ -normal.

\mathcal{K} -topological property:

Theorem (2.2): \mathcal{K} -quasi-normality is a \mathcal{K} -topological property i.e. \mathcal{K} -homeomorphic image of a \mathcal{K} -quasi-normal space is a \mathcal{K} -quasi-normal space.

Proof:

As a recall, a mapping $f : (X,T) \rightarrow (Y,\sigma)$ from a space (X,T) to another space (Y,σ) is known to be a \mathcal{K} -homeomorphism iff f is one-one and onto, M -pre-open and \mathcal{K} -irresolute where $\mathcal{K} = \alpha, p, s, b \& \beta$.

Let (X,T) be a \mathcal{K} -quasi-normal (i.e. $\mathcal{K}q$ -normal) space and let (Y,σ) be its \mathcal{K} -homeomorphic image under the function f . Then f is bijective, M - \mathcal{K} open and \mathcal{K} -irresolute.

Now, we show that (Y,σ) is \mathcal{K} -quasi normal space:

Suppose E is closed and F , \mathcal{K} -closed disjoint sets of Y . As f is \mathcal{K} -irresolute, $f^{-1}(E)$ and $f^{-1}(F)$ be two disjoint \mathcal{K} -closed sets in X . Since (X,T) is \mathcal{K} -quasi normal, there exist disjoint \mathcal{K} -open sets U and V such that $f^{-1}(E) \subseteq U$ and $f^{-1}(F) \subseteq V$. As $f^{-1}(E) \subseteq U$ implies $E \subseteq f(U)$ and $f^{-1}(F) \subseteq V$ implies $F \subseteq f(V)$. As f is M - \mathcal{K} open & bijective, $f(U)$, $f(V) \in \mathcal{K} O(Y)$ such that $f(U) \cap f(V) = \emptyset$. This shows that (Y,σ) is \mathcal{K} -quasi-normal space.

Hence, the theorem.

Hereditary Criteria : We, now, highlight the specific hereditary property posses by \mathcal{K} -normal space.

Theorem (2.3): An open subspace of \mathcal{K} -quasi-normal space is \mathcal{K} -quasi-normal where $\mathcal{K} = \alpha, p, s, b \& \beta$.

Proof: Let (Y, T_Y) be an open subspace of \mathcal{K} -quasi-normal space (X,T) .

Let A and B be disjoint closed & \mathcal{K} -closed subsets of Y respectively. Consequently, A & B are disjoint closed & \mathcal{K} -closed subsets of X respectively. So, by \mathcal{K} -quasi-normality of X , there exist disjoint \mathcal{K} -open sets U & V in X such that $A \subseteq U$ & $B \subseteq V$. As Y is open, $U \cap Y$ & $V \cap Y$ are \mathcal{K} -open in Y such that $A \subseteq U \cap Y$ & $B \subseteq V \cap Y$. Hence, (Y, T_Y) is \mathcal{K} -quasi-normal.

Hence, the theorem.

III. CONCLUSION

\mathcal{K} -quasi-normality, being a weaker version of \mathcal{K} -normality, has been introduced where $\mathcal{K} = \alpha, p, s, b \& \beta$. It has been shown that \mathcal{K} -quasi-normality is a topological property as well as hereditary property with regard to open sub spaces. Characterization as well as preservation theorem for \mathcal{K} -quasi-normality has been established. Some counter examples have been cited to exhibit that \mathcal{K} -quasi-normality lies in between strongly \mathcal{K} -normality & \mathcal{K} -normality.

Surely the literature content for the \mathcal{K} -quasi-normality is a tool to analyse the concept with fundamental properties at a time and at one place

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