Numerical solution to fifth order linear differential equation using sixth degree B-spline collocation method

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Abstract

This paper presents the comparison of numerical solutions which are obtained by using B-spline based collocation method of linear differential equation with constant coefficients. Different degree's of B-spline basis are employed as bases in this B-spline based collocation method .The higher degree B-spline base function which is employed in collocation method gives the best approximate solution to the considered differential equation instead of using the same degree B-spline base function to equal order differential equation with boundary conditions. Increasing of the degree of B- spline base function improves the numerical solution and also it alternate of increasing of the number of collocation points in the problem domain to get best solution to given differential equation. Numerical results show that higher degree B-pline basis function which are employed in collocation method as basis are best choice to achieve appropriate solution to the differential equation.

Keywords

Collocation method, B-splines, Linear differential equations with constant coefficients

1. Introduction

B-splines were first introduced by Schoenberg in [1, 2] who revealed that spline have powerful approximation properties. Subsequently, many approximation methods have been used [3].Most properties and efficient construction of B-splines can be found in [4].In particular, after de Boor's results [4], spline techniques became popular for a broad range of applications [5]. The efficiency of the method in explicit form has been proved by many researchers [6]-[12].In particular, recursive form of B-spline is employed in this paper to the higher order linear differential equations with constant coefficients. Considering the fifth order linear differential equations with constant coefficients

$$k_1\frac{d^5U}{dx^5}+k_2\frac{d^4U}{dx^4}+k_3\frac{d^3U}{dx^3}+k_4\frac{d^2U}{dx^2}+k_5\frac{dU}{dx}+k_6U=Q(x) \quad , \quad a<x<b$$

(1) with the boundary conditions

i) $U(a)=d1, U(b)=d2$

or ii)

$$U(a)=d1, \frac{dU(b)}{dx}=d3$$

where $a,b,d_1,d_2,k_1,k_2,k_3,k_4,k_5 \; and \; k_6$ are the constants. $Q(x)$ is a function of x Let

$U^h(x)=\sum_{i=-2}^{n+2} C_i N_{i,p}(x)$ (2) ,where C_i 's are constants to be determined and $N_{i,p}(x)$ are B-spline functions , be the approximate global solution to the exact solution $U(x)$ of the considered second order linear differential equation with constant coefficients.

2.1B-splines

In this section, definition and properties of B-spline basis functions are given in detail. A zero degree and other than zero degree B-spline basis functions are defined at x_i recursively over the knot vector

$X = \{x_1, x_2, x_3 .... x_{n-1}, x_n\}$ as

$$i) \quad if \quad p = 0$$

$$N_{i,p}(x) = 1 \qquad if \quad x \in (x_i, x_{i+i})$$
$$N_{i,p}(x) = 0 \qquad if \quad x \notin (x_i, x_{i+i})$$

$$ii) \quad if \quad p \geq 1$$

$$N_{i,p}(x) = \frac{x - x_i}{x_{i+p} - x_i} N_{i,p-1}(x) + \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} N_{i+1,p-1}(x)$$

where p is the degree of the B-spline basis function and x is the parameter belongs to X .When evaluating these functions, ratios of the form 0/0 are defined as zero

2.2 Derivatives of B-splines

If p=2, we have

$$N'_{i,p}(x) = \frac{x - x_i}{x_{i+p} - x_i} N'_{i,p-1}(x) + \frac{N_{i,p-1}(x)}{x_{i+p} - x_i} + \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} N'_{i+1,p-1}(x) - \frac{N_{i+1,p-1}(x)}{x_{i+p+1} - x_{i+1}}$$

$$N'_{i,p}(x) = 2\frac{N'_{i,p-1}(x)}{x_{i+p} - x_i} - 2\frac{N'_{i+1,p-1}(x)}{x_{i+p+1} - x_{i+1}}$$

In the above equations, the basis functions are defined as recursively in terms of previous degree basis function i.e. the pth degree basis function is the combination of ratios of knots and (p-1) degree basis function. Again (p-1) th degree basis function is defined as the combination ratios of knots and (p-2) degree basis function. In a similar way every B-spline basis function of degree up to (p-(p-2)) is expressed as the combination of the ratios of knots and its previous B-spline basis functions. The zero degree B-spline bases are C^0 –continuous. First degree B-spline bases are C^1-continuous and first derivative exist and second derivative exists for second degree B-spline basis functions. B-spline basis functions are defined on knot vectors. Knots are real quantities. Knot vector is a non decreasing set of knots. Knot vectors are classified as non-uniform knot vectors, uniform knot vector and open uniform knot vectors. Uniform knot vector in which difference of any two consecutive knots is constant is used for test problems in this paper. Two knots are required to define the zero degree basis function .In a similar way, the required number elements in a knot vector to define a pth degree B-spline basis function at a knot is always more than the two of the degree of the basis function.B-spline basis functions of degree zero, degree one and degree two over uniform knot vector are shown graphically below in the following figures (i), (ii) and (iii)

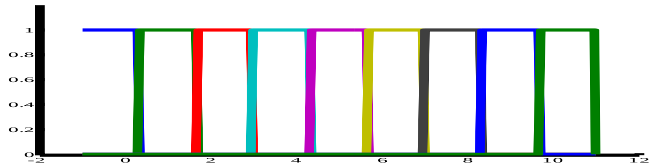

Fig (i), Zero degree B-spline basis function over uniform knot vector X= {- 1, 0 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

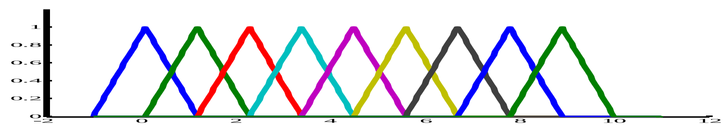

Fig (ii), First degree B-spline basis function with uniform Knot vector X= {-1,0,1,2,3,4,5,6,7,8,9,10,11,12}

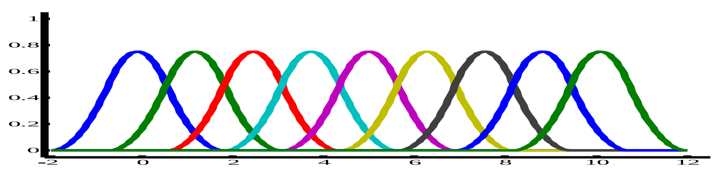

Fig (iii), Second degree B-spline basis function with uniform Knot vector X= {-2,-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

2.3 B-spline collocation method

Collocation method is used widely in approximation theory particularly to solve differential equations .In collocation method, the assumed approximate solution is exact solution at some nodal points. B-spline basis functions are used as the basis in B-spline collocation method whereas the base functions which are used in collocation method are the polynomials vanishes at the boundary values. Residue which is obtained by substituting equation (2) in equation (1) is made equal to zero at nodes in the given domain to determine unknowns in (2).Let $[a,b]$ be the domain of the governing differential equation and is partitioned as $X=\{a=x_0, x_1, x_2 ....x_{n-1}, x_n=b\}$ with equal length $h=\dfrac{b-a}{n}$ of n sub domains. The x_i's are known as nodes, the nodes are treated as knots in collocation B-spline method where the B-spline basis functions are defined and these nodes are used to make the residue equal to zero to determine unknowns C_i's in (2).Five extra knots are taken into consideration beside the domain of problem both side when evaluating the fifth degree B-spline basis functions at the nodes which are within the considered domain. Six extra knots are considered when implementing the six degree B-spline collocation method.

3. Numerical Experiments

$$\frac{d^5U}{dx^5} + U = -4\exp(x)(x^2\cos(x) - 2x\cos(x) - 9\cos(x) - \exp(x)(3x^2\sin(x) + 34x\sin(x) + 3\sin(x))$$

$$with\ \ U(0) = 0, U(1) = 0, U'(0) = 1, U'(1) = 0\ \ and\ U''(0) = -2$$

The analytical solution for the above problem is

$$y = e^{x} * \sin x * (1 - x)^{2}$$

Table1 :The different degree B-spline collocation solutions and analytical solutions at various points

X	*0*	*.1*	*.2*	*.3*	*.4*	*.5*	*.6*	*.7*	*.8*	*.9*	*1*
Numerical Solution by Fifth degree B-spline	0	.0895	.1562	.1978	.2132	.2031	.1707	.1222	.0674	.0205	0
Numerical Solution by SIX degree B-spline	0	.0894	.1553	.1955	.2091	.1976	.1646	.1168	.0639	.0193	0
Analytical solution	0	.0894	.1553	.1955	.2091	.1976	.1646	.1168	.0639	.0193	0

Table 1 presents values of fifth degree B-spline collocation solution, sixth degree B-spline collocation solution and analytical solution. Fig(iv) shows the over all behavior of the solution fifth B-spline collocation solution whereas Fig(v) gives the sixth degree B-spline collocation solution.

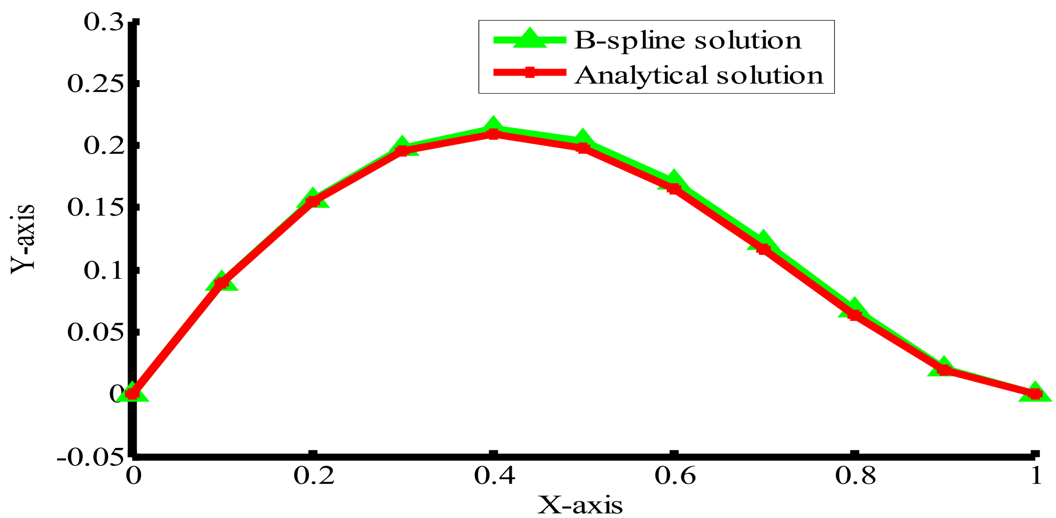

Fig (iv), Comparison of Fifth degree B-spline collocation solution with the analytical solution

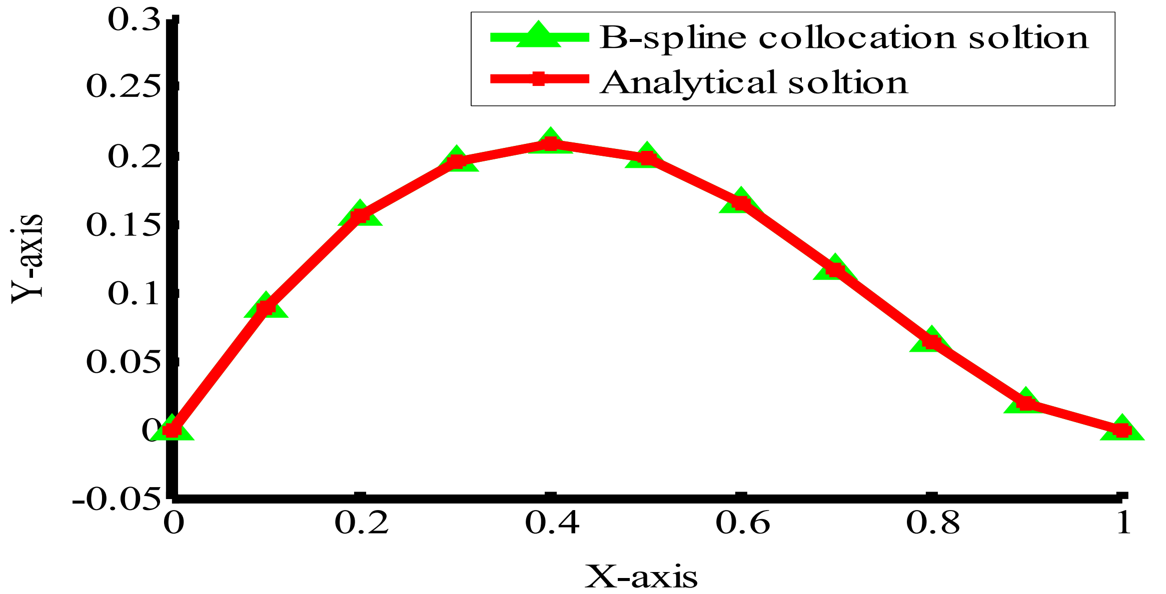

Fig (v),Comparison of Six degree B-spline collocation solution with the analytical solution

It is observed from the Table1that the values obtained by using the sixth degree B-spline collocation solution are more close to analytic solution when compared with the fifth degree B-spline collocation solution. Improvement in the solution is occurred when degree of the solution is increased instead of increasing the number collocation points in the fifth degree B-spline collocation method.

4. Conclusions

The rate of convergence of the sixth degree B-spline collocation solution is tested by comparing the fifth degree B-spline collocation solution by considering the fifth order linear differential equation with constant coefficients. The convergence rate is high when the high degree B-spline collocation is employed than the order of differential equation. This is proved by taking the numerical example.

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