Sum of Orthogonal Bimatrices in R_{nxn}

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Abstract

Let $F \in \{R, C, H\}$. Let $\mathcal{U}_{n\times n}$ be the set of unitary bimatrices in $F_{n\times n}$, and let $O_{n\times n}$ be the set of orthogonal bimatrices in $F_{n\times n}$. Suppose $n \geq 2$. we show that every $A_B \in F_{n\times n}$ can be written as a sum of bimatrices in $\mathcal{U}_{n\times n}$ and of bimatrices in $O_{n\times n}$. let $A_B \in F_{n\times n}$ be given that and let $k \geq 2$ be the least integer that is a least upper bound of the singular values of A_B. When F=R, we show that if $k \leq 3$, then A_B can be written as a sum of 6 orthogonal bimatrices; if $k \geq 4$, we show that A_B can be written as a sum of $k+2$ orthogonal bimatrices.

Keywords: Orthogonal matrix, bimatrix, orthogonal bimatrix, unitary bimatrix, sum of orthogonal bimatrices, sum of unitary bimatrices.

AMS classification: 15A09, 15A15, 15A57.

1. Introduction

Matrices provide a very powerful tool for dealing with linear models. Bimatrices are still a powerful and an advanced tool which can handle over one linear model at a time. Bimatrices are useful when time bound comparisons are needed in the analysis of a model. Bimatrices are of several types. We denote the space of *nxn* complex matrices by $\mathcal{C}_{nxn}$. For $A \in \mathcal{C}_{nxn}$, $A^T, A^{-1}, A^{\dagger}$ and $\det(A)$ denote transpose, inverse, Moore-Penrose inverse and determinant of A respectively. If $AA^T = A^T A = I$ then A is an orthogonal matrix, where I is the identity matrix. In this paper we study orthogonal bimatrices as a generalization of orthogonal matrices. Some of the properties of orthogonal matrices are extended to orthogonal bimatrices. Some important results of orthogonal matrices are generalized to orthogonal bimatrices.

Basic Definitions and Results

Definition 1.1 [7]

A bimatrix A_B is defined as the union of two rectangular array of numbers A_1 and A_2 arranged into rows and columns. It is written as $A_B = A_1 \cup A_2$ with $A_1 \neq A_2$ (except zero and unit bimatrices) where,

$$A_1 = \begin{bmatrix} a_{11}^1 & a_{12}^1 & \cdots & a_{1n}^1 \\ a_{21}^1 & a_{22}^1 & \cdots & a_{2n}^1 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^1 & a_{m2}^1 & \cdots & a_{mn}^1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} a_{11}^2 & a_{12}^2 & \cdots & a_{1n}^2 \\ a_{21}^2 & a_{22}^2 & \cdots & a_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^2 & a_{m2}^2 & \cdots & a_{mn}^2 \end{bmatrix}$$

'$\cup$' is just for the notational convenience (symbol) only.

Definition 1.2 [7]

Let $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be any two *m* x *n* bimatrices. The sum D_B of the bimatrices A_B and C_B is defined as

$$D_B = A_B + C_B = (A_1 \cup A_2) + (C_1 \cup C_2)$$

$$= (A_1 + C_1) \cup (A_2 + C_2)$$

Where $A_1 + C_1$ and $A_2 + C_2$ are the usual addition of matrices.

Definition 1.3 [8]

If $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be two bimatrices, then A_B and C_B are said to be equal (written as $A_B = C_B$) if and only if A_1 and C_1 are identical and A_2 and C_2 are identical. (That is, $A_1 = C_1$ and $A_2 = C_2$).

Definition 1.4 [8]

Given a bimatrix $A_B = A_1 \cup A_2$ and a scalar λ, the product of λ and A_B written as λA_B is defined to be

$$\lambda A_B = \begin{bmatrix} \lambda a_{11}^1 & \lambda a_{12}^1 & \cdots & \lambda a_{1n}^1 \\ \lambda a_{21}^1 & \lambda a_{22}^1 & \cdots & \lambda a_{2n}^1 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1}^1 & \lambda a_{m2}^1 & \cdots & \lambda a_{mn}^1 \end{bmatrix} \cup \begin{bmatrix} \lambda a_{11}^2 & \lambda a_{12}^2 & \cdots & \lambda a_{1n}^2 \\ \lambda a_{21}^2 & \lambda a_{22}^2 & \cdots & \lambda a_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1}^2 & \lambda a_{m2}^2 & \cdots & \lambda a_{mn}^2 \end{bmatrix}$$

$$= \lambda A_1 \cup \lambda A_2 \ .$$

That is, each element of A_1 and A_2 are multiplied by λ.

Remark 1.5 [8]

If $A_B = A_1 \cup A_2$ be a bimatrix, then we call A_1 and A_2 as the component matrices of the bimatrix A_B.

Definition 1.6 [7]

If $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ are both *n* x *n* square bimatrices then, the bimatrix multiplication is defined as, $A_B \times C_B = A_1 C_1 \cup A_2 C_2$.

Definition 1.7 [7]

Let $A_B^{m\times m} = A_1 \cup A_2$ be a *mxm* square bimatrix. We define $I_B^{m\times m} = I^{m\times m} \cup I^{m\times m} = I_1^{m\times m} \cup I_2^{m\times m}$ to be the identity bimatrix.

Definition 1.8 [7]

Let $A_B^{m\times m} = A_1 \cup A_2$ be a square bimatrix, A_B is a symmetric bimatrix if the component matrices A_1 and A_2 are symmetric matrices. i.e, $A_1 = A_1^T$ and $A_2 = A_2^T$.

Definition 1.9 [7]

Let $A_B^{m\times m} = A_1 \cup A_2$ be a *mxm* square bimatrix i.e, A_1 and A_2 are *mxm* square matrices. A skew-symmetric bimatrix is a bimatrix A_B for which $A_B = -A_B^T$, where $-A_B^T = -A_1^T \cup -A_2^T$ i.e, the component matrices A_1 and A_2 are skew-symmetric.

2. Orthogonal and Unitary Bimatrices

Definition 2.1 [6]

A bimatrix $A_B = A_1 \cup A_2$ is said to be orthogonal bimatrix, if $A_B A_B^T = A_B^T A_B = I_B$ (or) $A_1 A_1^T \cup A_2 A_2^T = A_1^T A_1 \cup A_2^T A_2 = I_1 \cup I_2$.

(That is, the component matrices of A_B are orthogonal.)

That is, $A_B^T = A_B^{-1}$ (or) $A_1^T \cup A_2^T = A_1^{-1} \cup A_2^{-1}$.

Remark 2.2

Let $A_B = A_1 \cup A_2$ be a orthogonal bimatrix. If A_1 and A_2 are square and posses the same order then A_B is called square orthogonal bimatrix, and if A_1 and A_2 are of different orders then A_B is called mixed square orthogonal bimatrix.

Example 2.3

(1) $$A_B = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{2} & 1 & -\sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & \sqrt{3} \end{bmatrix} \cup \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$ is a square orthogonal bimatrix.

(2) $$A_B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cup \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & -\cos\theta \end{bmatrix}$$ is a mixed square orthogonal bimatrix.

Definition 2.4 [5]

Let $A_B = A_1 \cup A_2$ be an $n \times n$ complex bimatrix. (A bimatrix A_B is said to be complex if it takes entries from the complex field). A_B is called a unitary bimatrix if $A_B A_B^* = A_B^* A_B = I_B$ (or) $\overline{A}_B^{\,T} = A_B^{-1}$.

That is, $A_1 A_1^* \cup A_2 A_2^* = A_1^* A_1 \cup A_2^* A_2 = I_1 \cup I_2$.

Example 2.5

$$A_B = A_1 \cup A_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} i & i \\ i & -i \end{bmatrix} \cup \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$ is a unitary bimatrix.

In this paper, we have determined which bimatrices (if any) in $R_{n\times n}$ can be written as a sum of unitary or orthogonal bimatrices. Also we have obtained that if $k \le 3,$ then A_B can be written as a sum of 6 orthogonal bimatrices, and if $k \ge 4,$ then A_B can be written as a sum of $k+2$ orthogonal bimatrices, where k be the least integer that is a least upper bound of the singular values of A_B. We let $\mathcal{U}_{n\times n}$ and $O_{n\times n}$ are the set of unitary and orthogonal bimatrices in the complex field. We begin with the following observation.

Lemma 2.6

Let n be a given positive integer. Let $G \subset F_{n\times n}$ be a group under multiplication. Then $A_B \in F_{n\times n}$ can be written as a sum of bimatrices in G if and only if for every $Q_B, P_B \in G,$ the bimatrix $Q_B A_B P_B$ can be written as a sum of bimatrices in G.

Notice that both $\mathcal{U}_{n\times n}$ and $O_{n\times n}$ are groups under multiplication.

Let $\alpha_1,\alpha_2 \in F$ be given. Then lemma 2.6 guarantees that for each $Q_B \in G,$ we have that $\alpha_1 Q_1 \cup \alpha_2 Q_2$ can be written as a sum of bimatrices from *G* if and only if $\alpha_1 I_1 \cup \alpha_2 I_2$ can be written as a sum of bimatrices from *G*.

Lemma 2.7

Let $n \geq 2$ be a given integer. Let $G \subset F_{n\times n}$ be a group under multiplication. Suppose that *G* contains $K_B \equiv diag\ 1,-1,...,-1$ and the permutation bimatrices. Then every $A_B \in F_{n\times n}$ can be written as a sum of bimatrices in *G* if and only if for each $\alpha_1,\alpha_2 \in F,\ \alpha_1 I_1 \cup \alpha_2 I_2$ can be written as a sum of bimatrices from *G*.

3. Sum of orthogonal bimatrices in R_{nxn}

The only bimatrices in the set of all orthogonal bimatrices of order 1 are ±1. Hence, not every element of $F_{1\times 1}$ can be written as a sum of elements in the set of all orthogonal bimatrices of order 1. In fact, only the integers can be written as a sum of elements of the set of all orthogonal bimatrices of order 1.

Notice that O_n □ $= u_n$ □ . When *n*=1, only the integers can be written as a sum of elements of O_1 □ . Suppose that *n*=2. We mimic the computations done in the case when $F =$ □ .

Let $\theta_1,\theta_2 \in$ □ be given, set $\alpha_1 = Cos\ \theta_1;\ \alpha_2 = Cos\ \theta_2$ and set $\beta_1 = Sin\ \theta_1;\ \beta_2 = Sin\ \theta_2$

Then $\left[A_1\ \alpha_1,\beta_1\ \cup A_2\ \alpha_2,\beta_2\ \right]$ in equation (2) of **[4]** is an element of O_2 □ .

Moreover, $\left[\ A_1^I + A_1^{II}\ \cup\ A_2^I + A_2^{II}\ \right] = 2\left[Cos\theta_1 I_1^{II} \cup Cos\theta_2 I_2^{II}\right].$

Now, for every $\delta_1,\delta_2 \in$ □ there exist a positive integer *m* and $\theta_1,\theta_2 \in$ □ such that $2m\ Cos\ \theta_1 = \delta_1;\ 2m\ Cos\ \theta_2 = \delta_2.$

We conclude that every $A_1 \cup A_2\ \in$ □ $_{n\times n}$ can be written as a sum of an even number of bimatrices from O_2 □ .

When *n*=3, we again mimic the computations done in the case when $F =$ □ using $\alpha_1 = Cos\ \theta_1;\ \alpha_2 = Cos\ \theta_2$ and $\beta_1 = Sin\ \theta_1;\ \beta_2 = Sin\ \theta_2$ to show that for every $\delta_1,\delta_2 \in$ □ the bimatrix $\delta_1 I_1^{III} \cup \delta_2 I_2^{III}$ can be written as a sum of an even number of bimatrices from O_3 □ .

Let $n \geq 4$ be a given integer. If *n*=2*k* is even, then write $\delta_1 I_1^{2k} \cup \delta_2 I_2^{2k}\ =\ \delta_1 I_1^{II} \cup \delta_2 I_2^{II}\ \oplus ... \oplus\ \delta_1 I_1^{II} \cup \delta_2 I_2^{II}$ (k copies), and note that each $\delta_1 I_1^{II} \cup \delta_2 I_2^{II}$ can be written as a sum of an even number of bimatrices from O_2 □ .

If *n*=2*k*+1 is odd, then write $\delta_1 I_1^{2k+1} \cup \delta_2 I_2^{2k+1}\ =\ \delta_1 I_1^{2n-2} \cup \delta_2 I_2^{2n-2}\ \oplus\ \delta_1 I_1^{III} \cup \delta_2 I_2^{III}$.

Now, $\delta_1 I_1^{2n-2} \cup \delta_2 I_2^{2n-2}$ can be written as a sum of an even number of bimatrices from O_{2n-2} □ and $\delta_1 I_1^{III} \cup \delta_2 I_2^{III}$ can be written as a sum of an even number of matrices from

O_{2n-2} □ and $\delta_1 I_1^{III} \cup \delta_2 I_2^{III}$ can be written as a sum of an even number of bimatrices from O_3 □ . We conclude that $\delta_1 I_1^{2k+1} \cup \delta_2 I_2^{2k+1}$ can be written as a sum of an even number of bimatrices from O_{2k+1} □ .

Hence, Lemma 3.2 of **[4]** guarantees that for every integer $n \geq 2,$ every $A_1 \cup A_2 \in$ □ $_{n\times n}$ can be written as a sum of bimatrices from O_n □ .

Theorem 3.1

Let $n \geq 2$ be a given integer. Every $A_1 \cup A_2 \in$ □ $_{n\times n}$ can be written as a sum of bimatrices from O_n □ $= \mathcal{U}_n$ □ .

Proof

Let $n \geq 2$ be a given integer and let $U_1 \cup U_2 \in \mathcal{U}_n$ □ be given.

Then $U_1 \cup U_2 \in \mathcal{U}_n$ □ $\cap O_n$ □ , that is, a real orthogonal bimatrix is both complex unitary bimatrix and complex orthogonal bimatrix.

Hence, $A_1 \cup A_2 \in$ □ $_{n\times n}$ which a sum of matrices is in $\mathcal{U}_n$ □ is both a sum of complex unitary bimatrices and a sum of complex orthogonal bimatrices. Thus, the restrictions on these cases apply. It *k* is a positive integer such that $\sigma_1^1 \; A_1 > k; \; \sigma_2^1 \; A_2 > k,$ then $A_1 \cup A_2$ cannot be written as a sum of *k* real orthogonal bimatrices.

Let *m* be a positive integer. Theorem 3.9 of **[4]** guarantees that $I_1 \cup I_2 \in$ □ $_{2m+1}$ cannot be written as a sum of two bimatrices in O_{2m+1} □ **.**

Now, we cannot be written as a sum of two bimatrices from O_{2m+1} □ $\subset O_{2m+1}$ □ .

In general, if $\alpha_1, \alpha_2 \notin \; -2,0,2$ and if $Q_1 \cup Q_2 \in O_{2m+1}$ □ , then $\alpha_1 Q_1 \cup \alpha_2 Q_2$ cannot be written as a sum of two bimatrices from O_{2m+1} □ .

Let $n \geq 2$ be a given integer, and let $A_1 \cup A_2 \in$ □ $_{n\times n}$ be given. We now look at the bimatrices in O_n □ that make up the sum $A_1 \cup A_2$.

Definition 3.2

Let $\theta_1, \theta_2 \in$ □ be given. We define

$$\left[A_1 \; \theta_1 \; \cup A_2 \; \theta_2 \; \right] \equiv \begin{bmatrix} Cos \; \theta_1 & Sin \; \theta_1 \\ -Sin \; \theta_1 & Cos \; \theta_1 \end{bmatrix} \cup \begin{bmatrix} Cos \; \theta_2 & Sin \; \theta_2 \\ -Sin \; \theta_2 & Cos \; \theta_2 \end{bmatrix} \text{ and}$$

$$\left[B_1 \; \theta_1 \; \cup B_2 \; \theta_2 \; \right] \equiv \begin{bmatrix} Cos \; \theta_1 & Sin \; \theta_1 \\ Sin \; \theta_1 & Cos \; \theta_1 \end{bmatrix} \cup \begin{bmatrix} Cos \; \theta_2 & Sin \; \theta_2 \\ Sin \; \theta_2 & Cos \; \theta_2 \end{bmatrix} \qquad (1)$$

Remark 3.3

Set $K_1^{II} \cup K_2^{II} \; \equiv \left[B_1 \; 0 \; \cup B_2 \; 0 \; \right]$ and notice that $\left[A_1 \; 0 \; \cup A_2 \; 0 \; \right] = \; I_1^{II} \cup I_2^{II}$.

Let $0 \le r, s \in \mathbb{R}$ be given, and let $k \ge 2$ be an integer. If $r, s \le k,$ then Lemma 3.1 of **[6]** and taking the real and imaginary parts of the equation

$$e^{i\theta_1^1} + \ldots + e^{i\theta_k^1} = \alpha_1;$$
$$e^{i\theta_1^2} + \ldots + e^{i\theta_k^2} = \alpha_2 \qquad (2)$$

Show that there exist $\theta_1^1, \theta_2^1, \ldots, \theta_k^1 \in \mathbb{R};\ \theta_1^2, \theta_2^2, \ldots, \theta_k^2 \in \mathbb{R}$ such that $\left[A_1(\theta_1^1) + \ldots + A_1(\theta_k^1)\right] \cup \left[A_2(\theta_1^2) + \ldots + A_2(\theta_k^2)\right] = r\left[I_1^{II} \cup I_2^{II}\right].$ Moreover, there exist $\beta_1^1, \ldots, \beta_k^1 \in \mathbb{R};\ \beta_1^2, \ldots, \beta_k^2 \in \mathbb{R}$ such that $\left[B_1(\beta_1^1) + \ldots + B_1(\beta_k^1)\right] \cup \left[B_2(\beta_1^2) + \ldots + B_2(\beta_k^2)\right] = S\left[K_1^{II} \cup K_2^{II}\right]$

Theorem 3.4

Let a positive integer *n* and let $(A_1 \cup A_2) \in \mathbb{R}_{2n}$ be given. Suppose that $k \ge 2$ is an integer such that $\sigma_1^1(A_1) \le k;\ \sigma_2^1(A_2) \le k$. Then $A_1 \cup A_2$ can be written as a sum of 2*k* matrices in $O_{2n}(\mathbb{R})$. Moreover, for every integer $m \ge 2k$ the matrix $A_1 \cup A_2$ can be written as a sum of *m* matrices in $O_{2n}(\mathbb{R})$.

Proof

Let $(A_1 \cup A_2) = (U_1 \cup U_2)(\Sigma_1 \cup \Sigma_2)(V_1 \cup V_2)$ be a singular value decomposition of $(A_1 \cup A_2)$.

Then Lemma 2.6 guarantees that we only need to concern ourselves with $\varepsilon.$ For *n*=1, notice that $diag_B(\sigma_1^1, \sigma_1^2) \cup diag_B(\sigma_2^1, \sigma_2^2) = s\left[I_1^{II} \cup I_2^{II}\right] + r\left[K_1^{II} \cup k_2^{II}\right],$ where $s = \frac{1}{2}(\sigma_1^1 + \sigma_1^2) = \frac{1}{2}(\sigma_2^1 + \sigma_2^2)$ and $t = \frac{1}{2}(\sigma_1^1 - \sigma_1^2) = \frac{1}{2}(\sigma_2^1 - \sigma_2^2)$.

Now, $0 \le t \le s \le k.$ Hence, $s(I_1^{II} \cup I_2^{II})$ and $t(K_1^{II} \cup k_2^{II})$ can each be written as a sum of *k* orthogonal bimatices. Moreover, for each integer $p \ge k,$ notice that $s(I_1^{II} \cup I_2^{II})$ can be written as a sum of *p* orthogonal bimatrices. Hence, $\left[(sI_1^{II} + rK_1^{II}) \cup (sI_2^{II} + rK_2^{II})\right]$ can be written as a sum of *p*+*k* orthogonal bimatrices.

For $n > 1$ write

$$\Sigma_1 \cup \Sigma_2 = diag(\sigma_1^1, \sigma_2^1, \ldots, \sigma_{2n-1}^1, \sigma_{2n}^1) \cup diag(\sigma_1^2, \sigma_2^2, \ldots, \sigma_{2n-1}^2, \sigma_{2n}^2)$$
$$= \left(diag(\sigma_1^1, \sigma_2^1) \oplus \ldots \oplus diag(\sigma_{2n-1}^2, \sigma_{2n}^1)\right) \cup \left(diag(\sigma_1^2, \sigma_2^2) \oplus \ldots \oplus diag(\sigma_{2n-1}^2, \sigma_{2n}^2)\right)$$

Notice now that for each $j = 1, \ldots, n,\ diag(\sigma_{2j-1}^1, \sigma_{2j}^1) \cup diag(\sigma_{2j-1}^2, \sigma_{2j}^2)$ can be written as a fun of 2*k* orthogonal bimatrices, say $(P_{j1}^1 \cup P_{j1}^2), \ldots, (P_{j(2k)}^1 \cup P_{j(2k)}^2)$

For each $l = 1, \ldots, 2k,$ set $(Q_l^1 \cup Q_l^2) \equiv (P_{1l}^1 \cup P_{1l}^2) \oplus \ldots \oplus (P_{nl}^1 \cup P_{nl}^2)$, and notice that $\Sigma = (Q_1^1 + \ldots + Q_{2k}^1) \cup (Q_1^2 + \ldots + Q_{2k}^2)$

Finally, notice that for each integer $m \geq 2k$ and for each $j = 1,...,n$, the matrix $diag\ \sigma^1_{2j-1}, \sigma^1_{2j} \cup diag\ \sigma^2_{2j-1}, \sigma^2_{2j}$ can be written as a sum of *m* orthogonal bimatrices.

Remark 3.5

Consider $C^1_0 \cup C^2_0 \equiv \left[diag\ b_1, a_1 \cup diag\ b_2, a_2 \right]$ with real numbers $b_1, b_2 \geq a_1, a_2 \geq 0$.

If $b_1, b_2 \geq 2$, then Theorem 3.4 ensures that $C^1_0 \cup C^2_0$ can be written as a sum of 4 real orthogonal bimatrices. Moreover, for each integer $t \geq 4$, $C^1_0 \cup C^2_0$ can be written as a sum of t real orthogonal bimatrices.

Suppose that $b_1, b_2 \leq 3$ if $0 \leq b_1 \leq 2;\ 0 \leq b_1 \leq 2$, then Theorem 3.4 guarantees that $C^1_0 \cup C^2_0$ can be written as a sum of 4 real orthogonal bimatrices. Moreover, $C^1_0 \cup C^2_0$ can also be written as a sum of 5 real orthogonal bimatrices.

If $2 < b_1 \leq 3;\ 2 < b_2 \leq 3$, then we look at two cases:

(i) $0 \leq a_1 \leq 1;\ 0 \leq a_2 \leq 1$ and

(ii) $1 \leq a_1 \leq 3;\ 1 \leq a_2 \leq 3$

In the first case, set $C^1_1 \cup C^1_2 \equiv C^0_1 \cup C^0_2 - K^2_1 \cup K^2_2$. Then $0 \leq b_1 - 1 \leq 2;\ 0 \leq b_2 - 1 \leq 2$ and $0 \leq a_1 + 1 < 2;\ 0 \leq a_2 + 1 < 2$. Notice now that for each integer $t \geq 4$, $C^1_1 \cup C^1_2$ can be written as a sum of *t* real orthogonal bimatrices.

In the second case, set $C^1_1 \cup C^1_2 \equiv C^0_1 - I^{II}_1 \cup C^0_2 - I^{II}_2$. Then we have $0 \leq a_1 - 1 \leq b_1 - 1 \leq 2;\ 0 \leq a_2 - 1 \leq b_2 - 1 \leq 2$. Theorem 3.4 guarantees that for each integer $t \geq 4$, $C^1_1 \cup C^1_2$ can be written as a sum of t real orthogonal bimatrices. Hence, for each integer $t \geq 5$, $C^0_1 \cup C^0_2$ can be written as a sum of t real orthogonal bimatrices.

We now use induction to show that if $k \geq 2$ is an integer satisfying $b_1 \leq k;\ b_2 \leq k$, then for each integer $t \geq k + 2$, $C^0_1 \cup C^0_2$ can be written as a sum of t real orthogonal bimatrices.

Suppose that the claim is true for some integer $k \geq 3$. We show that the claim is true when $0 < b_1 \leq k + 1;\ 0 < b_2 \leq k + 1$. if $0 \leq b_1 \leq k;\ 0 \leq b_2 \leq k$, then our inductive hypothesis guarantees that for each integer $t \geq k + 2$, $C^0_1 \cup C^0_2$ can be written as a sum of *t* and hence, also of $t \geq k + 3$ real orthogonal bimatrices.

If $k < b_1 \leq k + 1;\ k < b_2 \leq k + 1$, we take a look at two cases:

(i) $1 \leq a_1 \leq k + 1;\ 1 \leq a_2 \leq k + 1$ And

(ii) $0 \leq a_1 \leq 1;\ 0 \leq a_2 \leq 1$;

In case (i), set $C_1^1 \cup C_2^1 \equiv C_1^0 \cup C_2^0 - I_1^{II} \cup I_2^{II}$; and in case (ii), set $C_1^1 \cup C_2^1 \equiv C_1^0 \cup C_2^0 - K_1^{II} \cup K_2^{II}$.

Lemma 3.6

Let $C_1 \cup C_2 \in M_2$ be given suppose that $k \geq 2$ is an integer such that $\sigma_1^1\ C_1 \leq k$ and $\sigma_2^1\ C_2 \leq k$. Then for each integer $t \geq k+2,\ C_1 \cup C_2$ can be written as a sum of t matrices from u_2 .

Let $A_1 \cup A_2 \in {}_{2n}$ be given, and suppose that the bi singular values of $A_1 \cup A_2$ are $\sigma_1^1 \geq \ldots \geq \sigma_1^{2n} \geq 0; \sigma_2^1 \geq \ldots \geq \sigma_2^{2n} \geq 0.$

Set $D_1 \cup D_2 \equiv \left[diag\ \sigma_1^1,\ldots,\sigma_1^{2n} \cup diag\ \sigma_2^1,\ldots,\sigma_2^{2n} \right]$

Write $D_1 \cup D_2 \equiv \left[diag\ \sigma_1^1,\ldots,\sigma_1^2 \oplus \ldots \oplus diag\ \sigma_1^{2n-1},\sigma_1^{2n} \right]$

$$\cup\ diag\ \sigma_2^1,\ldots,\sigma_2^2 \oplus \ldots \oplus diag\ \sigma_2^{2n-1},\sigma_2^{2n} \Big].$$

Let $k \geq 2$ be an integer such that $\sigma_1^1\ A \leq k; \sigma_2^1\ A_2 \leq k$. Then Lemma 3.6 guarantees that for each integer $t \geq k+2,$ and for each $j = 1,\ldots,n, diag\ \sigma_1^{2j-1},\sigma_1^j \cup diag\ \sigma_2^{2j-1},\sigma_2^j$, can be written as a sum of *t* real orthogonal bimatrices. We conclude that for each integer $t \geq k+2,\ A_1 \cup A_2$ can be written as a sum of t real orthogonal bimatrices.

Theorem 3.7

Let *n* be a positive integer, and let $A_1 \cup A_2 \in {}_{2n}$ be given. Suppose that $k \geq 2$ is an integer such that $\sigma_1^1\ A_1 \leq k;\ \sigma_2^1\ A_2 \leq k$. then for each integer $t \geq k+2,\ A_1 \cup A_2$ can be written as a sum of t matrices in u_{2n} .

Proof

Let $A_1 \cup A_2 \in {}_{3\times 3}$ be given. Suppose that $A_1 \cup A_2 = P_1 \cup P_2\ \Sigma_1 \cup \Sigma_2\ Q_1 \cup Q_2$, with $P_1 \cup P_2$, $Q_1 \cup Q_2 \in O_3$ and $\Sigma_1 \cup \Sigma_2 = \left[diag\ a_1,b_1,c_1 \cup diag\ a_2,b_2,c_2 \right]$ with $0 \leq c_1 \leq b_1 \leq a_1 \leq 2; 0 \leq c_2 \leq b_2 \leq a_2 \leq 2$.

If $a_1 = a_2 = 2,$ then notice that $diag\ b_1,c_1 \cup diag\ b_2,c_2$ can be written as a sum of four orthogonal bimatrices. One checks that $\Sigma_1 \cup \Sigma_2$ can be written as a sum of four real orthogonal bimatrices.

Suppose $a_1 < 2;\ a_2 < 2.$ if $c_1 = c_2 = 0,$ then $\Sigma_1 \cup \Sigma_2$ can be written as a sum of four orthogonal bimatrices. If $c_1 = c_2 = 2,$ then $A_1 \cup A_2$ is a sum of two orthogonal bimatrices. If $0 \neq c_1 < 2;\ 0 \neq c_2 < 2,$ then, choose θ_1,θ_2 that $2\ Cos\ \theta_1 = c_1;\ 2\ Cos\ \theta_2 = c_2$.

Notice that $\left[A_1 \; \theta_1 \; + A_1 \; -\theta_1 \;\; \cup \; A_2 \; \theta_2 \; + A_2 \; -\theta_2 \;\; \right] = 2\left[Cos \; \theta_1 I_1^{II} \cup Cos \; \theta_2 I_2^{II} \right]$

Set $U_1^I \cup U_2^I \;=\; 1 \oplus A_1 \; \theta_1 \;\; \cup \;\; 1 \oplus A_2 \; \theta_2$ and

set $U_1^{II} \cup U_2^{II} \;=\; -1 \oplus A_1 \; -\theta_1 \;\; \cup \;\; -1 \oplus A_2 \; -\theta_2$.

Then $\Sigma_1 \cup \Sigma_2 \; - \; U_1^I \cup U_1^{II} \; + \; U_2^I \cup U_2^{II} \;=\; diag \; a_1, b_1 - c_1, 0 \; \cup diag \; a_2, b_2 - c_2, 0$, which can be written as a sum of four real orthogonal bimatrices. Hence, $A_1 \cup A_2$ can be written as a sum of six real orthogonal bimatrices.

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