

Distance Energy of Certain Mesh Derived Networks and Milovanović

Bounds

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Abstract – In this paper, we compute the distance energy of grid, cylinder, torus, extended grid networks by using matlab. Also we obtained Milovanović bounds for distance energy of a graph.

Keywords – Distance matrix, Distance energy, Grid, Cylinder, Torus, Extended grid, Milovanović bounds.

Introduction

The concept of energy of a graph was introduced by I. Gutman [4] in the year 1978. Let G be a graph with n vertices $\{v_1, v_2, \dots, v_n\}$ and m edges. Let $A = (a_{ij})$ be the adjacency matrix of the graph. The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of A, assumed in non increasing order are the eigenvalues of the graph G. As A is real symmetric, the eigenvalues of G are real with sum equal to zero. The energy E(G) of G is defined to be the sum of the absolute values of the eigenvalues of G.

$$\text{i.e., } E(G) = \sum_{i=1}^n |\lambda_i|$$

For details on the mathematical aspects of the theory of graph energy see the papers [2, 3, 5, 6] and the references cited there in.

1. DISTANCE ENERGY

On addressing problem for loop switching, R. L. Graham, H. O. Pollak [7] defined distance matrix of a graph. The Concept of distance energy was defined by Prof. G. Indulal et.al [10] in the year 2008. Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set E. Let d_{ij} be the distance between the vertices v_i and v_j then the $n \times n$ matrix $D(G) = (d_{ij})$ is called the distance matrix of G. The characteristic polynomial of D(G) is denoted by $f(G; \mu) = |\mu I - D(G)|$, where I is the unit matrix of order n. The roots $\mu_1, \mu_2, \dots, \mu_n$ assumed in non increasing order are called the distance eigenvalues of G. The

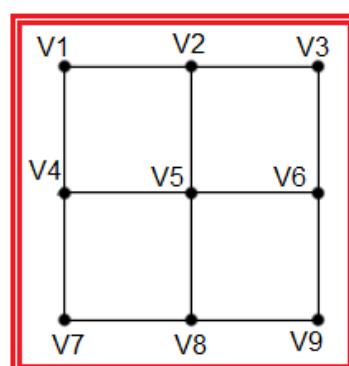
distance energy of a graph G is defined as $E_D(G) = \sum_{i=1}^n |\mu_i|$. Since $D(G)$ is a real symmetric matrix with zero trace, these distance eigenvalues are real with sum equal to zero.

2. DISTANCE ENERGY OF SOME STANDARD GRAPH NETWORKS

In the year 2012, Bharati Rajan [1] computed energy of certain mesh derived networks of order $n \times n$ by using matlab. In this paper we compute distance energy of grid, cylinder, torus and extended grid of order $m \times n$ by using matlab.

2.1. GRID G (m, n)

The topological structure of a grid network, denoted by $G(m, n)$, is defined as the cartesian product $P_m \times P_n$ of undirected paths P_m and P_n . The spectrum of the graph does not depend on the numbering of the vertices. However here we adopt a particular numbering such that the distance matrix has a pattern which is common for any dimension. We follow the sequential numbering from left to right as shown in the diagram.



Example2.1. Distance matrix of grid G (3, 3) is

$$\begin{pmatrix} 0 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 1 & 2 & 3 & 2 & 3 \\ 2 & 1 & 0 & 3 & 2 & 1 & 4 & 3 & 2 \\ 1 & 2 & 3 & 0 & 1 & 2 & 1 & 2 & 3 \\ 2 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 2 \\ 3 & 2 & 1 & 2 & 1 & 0 & 3 & 2 & 1 \\ 2 & 3 & 4 & 1 & 2 & 3 & 0 & 1 & 2 \\ 3 & 2 & 3 & 2 & 1 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 3 & 2 & 1 & 2 & 1 & 0 \end{pmatrix}$$

Distance eigenvalues are -6.0000, -6.0000, -2.2195, -2, 0.0000, 0.0000, 0.0000, 0.0000, 16. 2195.

Therefore distance energy of a grid G (3, 3) is 32.4391.

2.2. The following MATLAB programme generates the distance energy of a Grid G (m, n).

```

clc;
fprintf('DISTANCE ENERGY OF A GRID
mXn\n');
m=input('Enter the value of m: ');
n=input('Enter the value of n: ');
if(m>1 && n>1)
    A=zeros(m*n);
    c=1;
    for i=1:m
        for j=1:n
            v(i,j)=c;
            c=c+1;
        end
    end
    for i=1:m
        for j=1:n
            if(m>2 && n>2)
                if((v(i,j)+1 > 0) && (v(i,j)-1 > 0)
                && (v(i,j)-n > 0) && (v(i,j)+n <= m*n) &&
                (j > 1) && (j < n))
                    A(v(i,j),v(i,j)-1)=1;
                    A(v(i,j)-1,v(i,j))=1;
                    A(v(i,j),v(i,j)+1)=1;
                    A(v(i,j)+1,v(i,j))=1;
                    A(v(i,j),v(i,j)-n)=1;
                    A(v(i,j)-n,v(i,j))=1;
                    A(v(i,j),v(i,j)+n)=1;
                    A(v(i,j)+n,v(i,j))=1;
                else
                    if(i>1 && i<m)
                        A(v(i,j),v(i,j)-n)=1;
                        A(v(i,j)-n,v(i,j))=1;
                        A(v(i,j),v(i,j)+n)=1;
                    end
                end
            end
        end
    end
    DM=graphallshortestpaths(sparse(A));
    fprintf('Distance matrix: \n');
    disp(DM);
    eigenvaluesofgrid=eig(DM);
    fprintf('Co-efficients of characteristic polynomial
are\n');

```

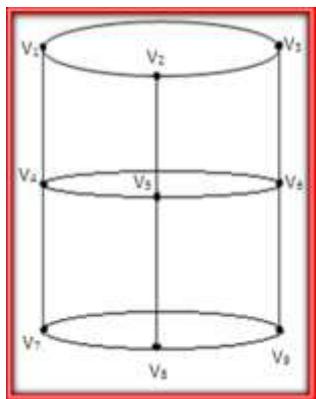
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fprintf('%4.4f\t',poly(DM));
fprintf('\n');
fprintf('Eigenvalues are\n');
fprintf('%4.4f\t',eigenvaluesofgrid);
fprintf('\n');
energy=sum(abs(eigenvaluesofgrid));
fprintf('distance energy of a grid
is %4.4f\n',energy);
else
    fprintf('Not a grid. m and n values must be
greater than 1\n');
end

```

2.3. CYLINDER C(m, n)

The topological structure of a cylinder network is denoted by $C(m, n)$ and is defined as the Cartesian product $P_m \times C_n$ of undirected path P_m and an undirected cycle C_n .



The numbering of vertices adopted for cylinder is same as that of a grid.

Example2.2. Distance matrix of Cylinder C (3, 3) is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 1 & 0 & 1 & 2 & 1 & 2 & 3 & 2 & 3 \\ 1 & 1 & 0 & 2 & 2 & 1 & 3 & 3 & 2 \\ 1 & 2 & 2 & 0 & 1 & 1 & 1 & 2 & 2 \\ 2 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 2 \\ 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & 1 \\ 2 & 3 & 3 & 1 & 2 & 2 & 0 & 1 & 1 \\ 3 & 2 & 3 & 2 & 1 & 2 & 1 & 0 & 1 \\ 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Distance eigenvalues are -6.0000, -3.0000, -3.0000,

-2.1240, 0.0000, 0.0000, 0.0000, 0.0000, 14.1240

Therefore distance energy of a cylinder C (3, 3) is 28.2481.

2.4. The following MATLAB program generates the distance energy of cylinder C(m, n)

```

clc;
fprintf('DISTANCE ENERGY OF A CYLINDER
mXn\n');
m=input('Enter the value of m: ');
n=input('Enter the value of n: ');
if(m>1 && n>1)
    A=zeros(m*n);
    c=1;
    for i=1:m
        for j=1:n
            v(i,j)=c;
            c=c+1;
        end
    end
    for i=1:m
        for j=1:n
            if(j == 1)
                A(v(i,j),v(i,j)-1+n)=1;
                A(v(i,j)-1+n,v(i,j))=1;
            end
            if(m>2 && n>2)
                if((v(i,j)+1 > 0) && (v(i,j)-1 > 0) &&
(v(i,j)-n > 0) && (v(i,j)+n <= m*n))
                    if(j>1 && j<n)
                        A(v(i,j),v(i,j)-1)=1;
                        A(v(i,j)-1,v(i,j))=1;
                        A(v(i,j),v(i,j)+1)=1;
                        A(v(i,j)+1,v(i,j))=1;
                    end
                    A(v(i,j),v(i,j)-n)=1;
                    A(v(i,j)-n,v(i,j))=1;
                    A(v(i,j),v(i,j)+n)=1;
                    A(v(i,j)+n,v(i,j))=1;
                if(j == 1)
                    A(v(i,j),v(i,j)-1+n)=1;
                    A(v(i,j)-1+n,v(i,j))=1;
                end
            else
                if(i>1 && i<m)
                    A(v(i,j),v(i,j)-n)=1;
                    A(v(i,j)-n,v(i,j))=1;
                    A(v(i,j),v(i,j)+n)=1;
                    A(v(i,j)+n,v(i,j))=1;
                end
                if(j>1 && j<n)
                    A(v(i,j),v(i,j)-1)=1;
                    A(v(i,j)-1,v(i,j))=1;
                    A(v(i,j),v(i,j)+1)=1;
                    A(v(i,j)+1,v(i,j))=1;
                end
            end
        end
    end
    if(m>2)
        if(i > 1) && (i < m))
            A(v(i,j),v(i,j)-n)=1;
            A(v(i,j)-n,v(i,j))=1;
            A(v(i,j),v(i,j)+n)=1;
    end

```

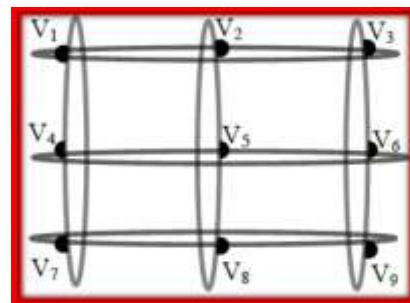
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A(v(i,j)+n,v(i,j))=1;
if(j==1)
    A(v(i,j),v(i,j)+1)=1;
    A(v(i,j)+1,v(i,j))=1;
end
else
    if(i < m && j <= n)
        A(v(i,j),v(i,j)+n)=1;
        A(v(i,j)+n,v(i,j))=1;
    end
    if(mod(j,n) ~ = 0)
        A(v(i,j), v(i,j)+1)=1;
        A(v(i,j)+1, v(i,j))=1;
    end
end
else
    if((j > 1) && (j < n))
        A(v(i,j),v(i,j)-1)=1;
        A(v(i,j)-1,v(i,j))=1;
        A(v(i,j),v(i,j)+1)=1;
        A(v(i,j)+1,v(i,j))=1;
        if(i < m)
            A(v(i,j),v(i,j)+n)=1;
            A(v(i,j)+n,v(i,j))=1;
        end
    else
        if(i < m && j <= n)
            A(v(i,j),v(i,j)+n)=1;
            A(v(i,j)+n,v(i,j))=1;
        end
        if(mod(j,n) ~ = 0)
            A(v(i,j), v(i,j)+1)=1;
            A(v(i,j)+1, v(i,j))=1;
        end
    end
end
end
end
DM=graphallshortestpaths(sparse(A));
fprintf('Distance matrix: \n');
disp(DM);
eigenvaluesofgrid=eig(DM);
fprintf('Co-efficients of characteristic polynomial
are\n');
fprintf('%4.4f\t',poly(DM));
fprintf('\n');
fprintf('Eigenvalues are\n');
fprintf('%4.4f\t',eigenvaluesofgrid);
fprintf('\n');
energy=sum(abs(eigenvaluesofgrid));
fprintf('distance energy of a cylinder
is %4.4f\n',energy);
else
    fprintf('not a cylinder. m and n values must be
greater than 2\n');
end

```

2.5. TORUS T(m, n)

The topological structure of a torus network is denoted by $T(m, n)$ and is defined as the Cartesian product $C_m \times C_n$ where C_m and C_n are undirected cycles. The numbering adopted for torus is same as that of a grid.



Example2.3. Distance matrix of Torus $T(3, 3)$ is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 2 & 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 & 1 & 2 & 2 & 1 & 2 \\ 1 & 1 & 0 & 2 & 2 & 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 0 & 1 & 1 & 1 & 2 & 2 \\ 2 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 2 \\ 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 & 2 & 0 & 1 & 1 \\ 2 & 1 & 2 & 2 & 1 & 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 2 & 2 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Distance eigenvalues are $-3.0000, -3.0000, -3.0000, -3.0000, 0.0000, 0.0000, 0.0000, 0.0000, 12.0000$.

Therefore distance energy of a torus is 24.0000.

2.6. The following MATLAB program generates the distance energy of Torus $T(m, n)$

```

clc;
fprintf('DISTANCE ENERGY OF A TORUS
mXn\n');
m=input('Enter the value of m: ');
n=input('Enter the value of n: ');
if(m>1 && n> 1)
    A=zeros (m*n);
    c=1;
    for i=1:m
        for j=1:n
            v(i,j)=c;
            c=c+1;
        end
    end
    DM=graphallshortestpaths(sparse(A));
    fprintf('Distance matrix: \n');
    disp(DM);
    eigenvaluesofgrid=eig(DM);
    fprintf('Co-efficients of characteristic polynomial
are\n');
    fprintf('%4.4f\t',poly(DM));
    fprintf('\n');
    fprintf('Eigenvalues are\n');
    fprintf('%4.4f\t',eigenvaluesofgrid);
    fprintf('\n');
    energy=sum(abs(eigenvaluesofgrid));
    fprintf('distance energy of a cylinder
is %4.4f\n',energy);
else
    fprintf('not a cylinder. m and n values must be
greater than 2\n');
end

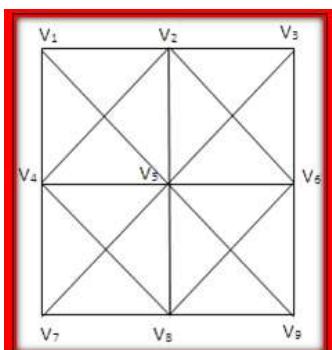
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    end
end
for i=1:m
    for j=1:n
        if(j == 1)
            A(v(i,j),v(i,j)-1+n)=1;
            A(v(i,j)-1+n,v(i,j))=1;
        end
        if(i == 1)
            A(v(i,j),v(i,j)+((m-1)*n))=1;
            A(v(i,j)+((m-1)*n),v(i,j))=1;
        end
        if(j < n)
            A(v(i,j), v(i,j)+1)=1;
            A(v(i,j)+1, v(i,j))=1;
        end
        if(i < m)
            A(v(i,j),v(i,j)+n)=1;
            A(v(i,j)+n,v(i,j))=1;
        end
    end
end
DM=graphallshortestpaths(sparse(A));
fprintf('Distance matrix: \n');
disp(DM);
eigenvaluesofgrid=eig(DM);
fprintf('Co-efficients of characteristic polynomial
are\n');
fprintf('%4.4f\t',poly(DM));
fprintf('\n');
fprintf('Eigenvalues are\n');
fprintf('%4.4f\t',eigenvaluesofgrid);
fprintf('\n');
energy=sum(abs(eigenvaluesofgrid));
fprintf('distance energy of a torus
is %4.4f\n',energy);
else
    fprintf('not a torus. m and n values must be
greater than 1\n');
end

```

2.7. EXTENDED GRID EX(m, n)



By making each 4-cycle in a $m \times n$ mesh into a complete graph we obtain an architecture called an extended mesh denoted by EX (m, n). The number of vertices in EX (m, n) is mn and the number of edges is

$4mn - 3m - 3n + 2$. We follow the sequential numbering from left to right.

Example2.4. Distance matrix of Extended grid EX (3, 3) is

$$\begin{pmatrix} 0 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 1 & 0 & 2 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 0 & 1 & 2 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 & 0 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 0 \end{pmatrix}$$

Distance eigenvalues are $-3.4142, -3.4142, -2.0000, -1.1842, -0.5858, -0.5858, -0.5745, 0.0000, 11.7587$.

Therefore distance energy of a extended grid is 23.5174.

2.8. The following MATLAB program generates the distance energy of Extended Grid EX(m, n)

```

clc;
fprintf('DISTANCE      ENERGY      OF      AN
EXTENDED GRID mXn\n');
m=input('Enter the value of m: ');
n=input('Enter the value of n: ');
if(m>1 && n>1)
    A=zeros(m*n);
    c=1;
    for i=1:m
        for j=1:n
            v(i,j)=c;
            c=c+1;
        end
    end
    for i=1:m
        for j=1:n
            if(m>2 && n>2)
                if((v(i,j)+1 > 0) && (v(i,j)-1 > 0) &&
                (v(i,j)-n > 0) && (v(i,j)+n <= m*n) && (j > 1) && (j
                < n))
                    A(v(i,j),v(i,j)-1)=1;
                    A(v(i,j)-1,v(i,j))=1;
                    A(v(i,j),v(i,j)+1)=1;
                    A(v(i,j)+1,v(i,j))=1;
                    A(v(i,j),v(i,j)-n)=1;
                    A(v(i,j)-n,v(i,j))=1;
                    A(v(i,j),v(i,j)+n)=1;
                end
            end
        end
    end

```

```

A(v(i,j)+n,v(i,j))=1;
A(v(i,j), v(i,j)+n+1)=1;
A(v(i,j)+n+1, v(i,j))=1;
A(v(i,j), v(i,j)+n - 1)=1;
A(v(i,j)+n - 1, v(i,j))=1;
A(v(i,j), v(i,j)-n+1)=1;
A(v(i,j), v(i,j)-n+1)=1;
A(v(i,j)-n+1, v(i,j))=1;
A(v(i,j), v(i,j)-n - 1)=1;
A(v(i,j)-n - 1, v(i,j))=1;
else
    if(i>1 && i<m)
        A(v(i,j),v(i,j)-n)=1;
        A(v(i,j)-n,v(i,j))=1;
        A(v(i,j),v(i,j)+n)=1;
        A(v(i,j)+n,v(i,j))=1;
        if(j<n)
            A(v(i,j),v(i,j)+n+1)=1;
            A(v(i,j)+n+1,v(i,j))=1;
        end
        if(j>1)
            A(v(i,j),v(i,j)+n-1)=1;
            A(v(i,j)+n-1,v(i,j))=1;
        end
    end
    if(j>1 && j<n)
        A(v(i,j),v(i,j)-1)=1;
        A(v(i,j)-1,v(i,j))=1;
        A(v(i,j),v(i,j)+1)=1;
        A(v(i,j)+1,v(i,j))=1;
        if(i<m)
            A(v(i,j), v(i,j)+n-1)=1;
            A(v(i,j)+ n-1, v(i,j))=1;
            A(v(i,j),v(i,j)+ n +1)=1;
            A(v(i,j)+ n + 1,v(i,j))=1;
        end
    end
end
if(m>2)
    if(i>1 && i<m)
        A(v(i,j),v(i,j)+1)=1;
        A(v(i,j)+1,v(i,j))=1;
        A(v(i,j),v(i,j)-1)=1;
        A(v(i,j)-1,v(i,j))=1;
        A(v(i,j),v(i,j)+n)=1;
        A(v(i,j)+n,v(i,j))=1;
        A(v(i,j),v(i,j)-n)=1;
        A(v(i,j)-n,v(i,j))=1;
    if(j > 1)
        A(v(i,j),v(i,j)- n - 1)=1;
        A(v(i,j)- n - 1,v(i,j))=1;
        A(v(i,j),v(i,j)- n + 1)=1;
        A(v(i,j)- n + 1,v(i,j))=1;
    end
    if(j<n)
        A(v(i,j),v(i,j)+ n - 1)=1;
        A(v(i,j)+ n - 1,v(i,j))=1;
        A(v(i,j),v(i,j)+ n + 1)=1;
        A(v(i,j)+ n + 1,v(i,j))=1;
    end
else
    if(j<n)
        A(v(i,j),v(i,j)+1)=1;
        A(v(i,j)+1,v(i,j))=1;
        if(j>1)
            A(v(i,j),v(i,j)-1)=1;
            A(v(i,j)-1,v(i,j))=1;
        end
        if(i>1)
            A(v(i,j),v(i,j)-n)=1;
            A(v(i,j)-n,v(i,j))=1;
        end
    end
    if(i< m)
        A(v(i,j),v(i,j)+n)=1;
        A(v(i,j)+n,v(i,j))=1;
        if(j == 1)
            A(v(i,j),v(i,j)+n + 1)=1;
            A(v(i,j)+n + 1,v(i,j))=1;
        else
            A(v(i,j),v(i,j)+n - 1)=1;
            A(v(i,j)+n - 1,v(i,j))=1;
        end
    end
end

```

```

        end
    end
end
end
DM=graphallshortestpaths(sparse(A));
fprintf('Distance matrix: \n');
disp(DM);
eigenvaluesofgrid=eig(DM);
fprintf('Co-efficients of characteristic polynomial
are\n');
fprintf('%4.4f\t',poly(DM));
fprintf('\n');
fprintf('Eigenvalues are\n');
fprintf('%4.4f\t',eigenvaluesofgrid);
fprintf('\n');
energy=sum(abs(eigenvaluesofgrid));
fprintf('distance energy of a extended grid
is %4.4f\n',energy);
else
    fprintf('not an extended grid. m and n values
must be greater than 2\n');
end

```

3. PROPERTIES OF DISTANCE EIGENVALUES

H. S. Ramane et al. [10] proved the following

Lemma3.1. Let G be a simple graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set E. If $\mu_1, \mu_2, \dots, \mu_n$ are the eigenvalues of distance matrix D(G) then i)

$$\sum_{i=1}^n |\mu_i| = 0.$$

$$\text{ii)} \sum_{i=1}^n \mu_i^2 = 2 \sum_{i \leq j} d(v_i, v_j)^2.$$

4. BOUNDS FOR DISTANCE ENERGY

Similar to bounds for energy of a graph, bounds for distance energy $E_D(G)$ are discussed in the following section. H. S. Ramane et al. [10] proved the following lemma.

Lemma4.1. Let G be a simple (n, m) graph then

$$E_D(G) \leq \sqrt{n \left(2 \sum_{i \leq j} d(v_i, v_j)^2 \right)}.$$

Recently Milovanović et al.[9] gave a sharper lower bounds for energy of a graph. In this paper similar bounds for distance energy of a graph are established.

Theorem 4.1. Let G be a graph with n vertices and m edges. Let $|\mu_1| \geq |\mu_2| \geq \dots \geq |\mu_n|$ be a non-increasing order of eigenvalues of D(G) then

$$E_D(G) \geq \sqrt{n \left(2 \sum_{i \leq j} d(v_i, v_j)^2 \right) - \alpha(n)(|\mu_1| - |\mu_n|)^2}$$

$$\text{where } \alpha(n) = n \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) \text{ and } [x]$$

denotes

the integral part of a real number.

Proof. Let $a, a_1, a_2, \dots, a_n, A$ and $b, b_1, b_2, \dots, b_n, B$ be real numbers such that $a \leq a_i \leq A$ and $b \leq b_i \leq B \forall i = 1, 2, \dots, n$ then the following inequality is valid.

$$\left| n \sum_{i=1}^n a_i b_i - \sum_{i=1}^n a_i \sum_{i=1}^n b_i \right| \leq \alpha(n)(A-a)(B-b)$$

where

$$\alpha(n) = n \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) \text{ and equality holds if and}$$

only if $a_1 = a_2 = \dots = a_n$ and $b_1 = b_2 = \dots = b_n$.

If $a_i = |\mu_i|, b_i = |\mu_i|, a = b = |\mu_n|$ and $A = B = |\mu_1|$ then

$$\left| n \sum_{i=1}^n |\mu_i|^2 - \left(\sum_{i=1}^n |\mu_i| \right)^2 \right| \leq \alpha(n)(|\mu_1| - |\mu_n|)^2$$

$$\text{But } \sum_{i=1}^n |\mu_i|^2 = 2 \sum_{i \leq j} d(v_i, v_j)^2 \text{ and}$$

$$E_D(G) \leq \sqrt{n \left(2 \sum_{i \leq j} d(v_i, v_j)^2 \right)} \text{ then the above inequality becomes}$$

$$n \left(2 \sum_{i \leq j} d(v_i, v_j)^2 \right) - (E_D(G))^2 \leq \alpha(n)(|\mu_1| - |\mu_n|)^2$$

i.e.,

$$E_D(G) \geq \sqrt{n \left(2 \sum_{i \leq j} d(v_i, v_j)^2 \right) - \alpha(n)(|\mu_1| - |\mu_n|)^2}$$

Theorem 4.2. Let G be a graph with n vertices and m edges. Let $|\mu_1| \geq |\mu_2| \geq \dots \geq |\mu_n|$ be a non-increasing order of eigenvalues of D(G) then

$$E_D(G) \geq \frac{2 \sum_{i \leq j} d(v_i, v_j)^2 + n |\mu_1| |\mu_n|}{(|\mu_1| + |\mu_n|)}$$

Proof. Let $a_i \neq 0$, b_i , r and R be real numbers satisfying $ra_i \leq b_i \leq Ra_i$, then the following inequality holds. [Theorem 2, [9]]

$$\sum_{i=1}^n b_i^2 + rR \sum_{i=1}^n a_i \leq (r+R) \sum_{i=1}^n a_i b_i$$

Put $b_i = |\mu_i|$, $a_i = 1$, $r = |\mu_n|$ and $R = |\mu_1|$ then

$$\sum_{i=1}^n |\mu_i|^2 + |\mu_1| |\mu_n| \sum_{i=1}^n 1 \leq (|\mu_1| + |\mu_n|) \sum_{i=1}^n |\mu_i|$$

$$2 \sum_{i \leq j} d(v_i, v_j)^2 + n |\mu_1| |\mu_n| \leq (|\mu_1| + |\mu_n|) E_D(G)$$

$$\therefore E_D(G) \geq \frac{2 \sum_{i \leq j} d(v_i, v_j)^2 + n |\mu_1| |\mu_n|}{(|\mu_1| + |\mu_n|)}$$

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