

Ricci Flow Equations on Quasi-C-Reducible Finsler Space

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Abstract — In the present paper, we deal with one of the special Finsler space such that quasi- $\mathcal{C}$ -reducible space and find out the un-normal and normal Ricci flow equations on quasi- $\mathcal{C}$ -reducible space with (α, β) -metric.

Keywords— Ricci flow, (α, β) -metric, Quasi- $\mathcal{C}$ -reducible space.

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I. Introduction

Geometric flows are important in many sections of mathematics and physics. A geometric flow is an evolution of geometric structure under a different equation related to a fundamental on a manifold, usually associated with some curvature. The well-known geometric flows in mathematics are the heat flow, the Ricci flow and the mean curvature flow. They are all related to dynamical systems in the infinite-dimensional space of all metrics on a given manifold. Geometry flow equations are completely difficult to be solved time existence of solutions is obtained by the parabolic or hyperbolic nature of the equation.

For the first time in 1982, Hamilton introduced the Ricci flow a Riemannian manifold M with a Riemannian metric g_0 and the family g_{ij} of Riemannian metrics on M as satisfying:

$$\frac{d}{dt} g_{ij} = -2Ric_{ij}, \quad g(0) = g_0,$$

where, Ric_{ij} is the Ricci tensor of g_{ij} and is known as the un-normalised Ricci-flow in Riemannian geometry [3].

Hamilton et al showed that there is a unique solution to this equation for an arbitrary smooth metric on a closed manifold over a sufficiently short time. Hamilton also showed that Ricci flow preserved positivity on the Ricci curvature tensor in three dimensions and the curvature operator in four dimensions. It is not an easy way to define Ricci flow of mutually compatible fundamental geometric structures on Finsler manifolds. the problem of constructing the Finsler-Ricci flow theory contains a number of new conceptual and fundamental issues on compatibility of geometrical and physical objects and

their optimal configurations. The same equations can be used in the Finsler setting. Since both the fundamental tensor g_{ij} and Ricci tensor Ric_{ij} have been generalized to that broader framework, albeit gaining a y dependence in the process. However, there are some reasons why we shall refrain from doing so : (i) Not every symmetric covariant 2-tensor $g_{ij}(x, y)$ arises from a Finsler metric $L(x, y)$ and (ii) There is more than one geometrical context in which g_{ij} makes sense.

The main results on Ricci flow evolution were proved originally for (pseudo) Riemannian and Kähler geometries. Thus the Ricci flow theory became a very powerful method in understanding the geometry and topology of Riemannian and Kählerian manifolds. Vacaru ([12]-[15]) studied on nonholonomic Ricci flows, evolution equations and dynamics, exact solutions in gravity, symmetric and non-symmetric metrics, the entropy of lagrange-Finsler spaces and Ricci flows, spectral functionals, nonholonomic Dirac operators and non-commutative Ricci flows, fractional nonholonomic Ricci flows, Nonholonomic Ricci flows and parametric deformations of the solitonic pp-waves and schwarzschild solutions. Narasimhamurthy et al and Thayebi et al ([5] [11]) studied on Ricci flow equations

on special Finsler space with (α, β) -metrics.

The aim of this paper, we study one of the special Finsler spaces such as quasi- $\mathcal{C}$ -reducible space and find out the un-normal Ricci flow and normal Ricci flow equations on quasi- $\mathcal{C}$ -reducible space with (α, β) -metric.

II. Preliminaries

We call the indicatrix in $x \in M$, the hypersurface S_x in $T_x M$ defined by the equation $L(x, y) = 1$ and denote by SM the fiber bundle of unitary tangent vectors to M . We obtain a symmetric tensor $\mathcal{C}$, Cartan tensor, on $\pi^* TM$ defined by,

$$\mathcal{C}(u, v, w) = C_{ijk}(y)u_i v_j w_k,$$

where $u = u_i \frac{\partial}{\partial x^i}$, $v = v_i \frac{\partial}{\partial x^i}$, $w = w_i \frac{\partial}{\partial x^i}$ and $C_{ijk} = \frac{1}{4} [L^2]_{y_i y_j y_k}$. It is well known that $C = 0$ if and only if L is Riemannian.

Some authors proposed special form of C_{ijk} ,

$$C_{ijk} = A_{ij} B_k + A_{jk} B_i + A_{ki} B_j,$$

where, A_{ij} is a symmetric tensor and B_i a covariant vector. The equations $A_{i0} = 0$ and $B_0 = 0$ were shown. The angular metric tensor has these properties of A_{ij} and it was also shown that $A_{ij} = h_{ij}$ implies $B_i = C_i / (n + 1) h_{ij}$. Thus we are led to the special form,

$$(n + 1) C_{ijk} = h_{ij} C_k + h_{jk} C_i + h_{ki} C_j.$$

A non-Riemannian $L^n (n \geq 3)$ with C_{ijk} of the above form is called C -reducible. On the other hand, in the special case B_i is equal to the torsion vector C_i ,

$$C_{ijk} = A_{ij} C_k + A_{jk} C_i + A_{ki} C_j,$$

and non-Riemannian $L^n (n \geq 3)$ of the above was called quasi- C -reducible. It was shown that any non-Riemannian $L^n (n \geq 3)$ with the so called (α, β) -metric is quasi- C -reducible.

For a Finsler metric $L = L(x, y)$ on a manifold M , the spray $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ is a vector field on TM , where $G^i = G^i(x, y)$ are defined by,

$$G^i = \frac{g^{il}}{4} \{ [L^2]_{x^k y^l y^k} - [L^2]_{x^l} \}.$$

Let $x \in M$ and $L_x = L|_{T_x M}$. To measure the non-Euclidean feature of L_x , define $C_y: T_x M \otimes T_x M \rightarrow \mathbb{R}$ by,

$$C_y(u, v, w) = \frac{1}{2} \frac{d}{dt} [g_y + tw(u, v)]|_{t=0}, u, v, w \in T_x M.$$

The family $C = C_y, y \in T_0 M$ is called the Cartan torsion. It is well known that $C = 0$ if and only if L is Riemannian. For $y \in T_x M$, define mean Cartan torsion I_y by $I_y(u) = I_i(y) u^i$, where $I_i = g^{jk} C_{ijk}$, $C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}$ and $u = u \frac{\partial}{\partial x^i} |_x$.

By Deicke's Theorem, L is Riemannian if and only if $I_y = 0$.

A deformation of Finsler metrics means a 1-parameter family of metrics $g_{ij}(x, y, t)$, such that $t \in [-\epsilon, \epsilon]$ and $\epsilon > 0$ is sufficiently small. For such a metric $w = u_i dx^i$, the volume element as well as the connections attached to it depend on t . The same equation can be used in the Finsler setting. Another Ricci flow equation can also be used instead

of this tensor evolution equation [2]. By contracting $\frac{d}{dt} g_{ij} = -2Ric_{ij}$ with y^i and y^j gives, via Euler's theorem, we get

$$\frac{\partial L^2}{\partial t} = -2L^2 R,$$

where, $R = \frac{1}{L^2} Ric$. That is,

$$d \log L = -R, L(t = 0) = L_0.$$

This scalar equation directly addresses the evolution of the Finsler metric L and makes geometrical sense on both the manifold of nonzero tangent vectors TM_0 and the sphere bundle SM . One of the advantages of above equation is its independence to choice of Cartan, Berwald or Chern connections. It is therefore suitable as an unnormalized Ricci flow for Finsler geometry.

By using the elegance work of Akbar-Zadeh in [1], Bao [2] proposed the following normalised Ricci flow equation for Finsler metrics,

$$\frac{d}{dt} \log L = -R + \frac{1}{Vol(SM)} \int_{SM} R dv, L(t = 0) = L_0,$$

where the underlying manifold M is compact [2].

The Cartan tensor of an (α, β) -metric on n -dimensional manifold M is given by,

$$C_{ijk} = A_{ij} I_k + A_{jk} I_i + A_{ki} I_j,$$

where, A_{ij} is the symmetric tensor and angular metric tensor h_{ij} have same properties of A_{ij} . So $A_{ij} = h_{ij}$.

III. Un-normal Ricci flow equation on Quasi- C -reducible Finsler space with (α, β) -metrics.

In this section, we study (α, β) -metrics satisfying un-normal Ricci flow equation. First, we prove the following results:

Lemma 3.1. Let L_t be a deformation of an (α, β) -metric L , which is quasi- C -reducible, on a manifold M of dimension $n \geq 3$. Then the variation of Cartan tensor is given by the following,

$$C'_{ijk} I^i I^j I^k = -6R \|I\|^4 - \frac{1}{2} L^2 R_{i,j,k} I^i I^j I^k - 3 \|I\|^2 I^m R_{m,i,j,k} I^i I^j I^k, \quad (3.1)$$

where, $\|I\|^2 = I_m I^m$.

Proof: First, assume that L_t be a deformation of a Finsler metric on a two-dimensional manifold M satisfies Ricci flow equation, that is, $\frac{d}{dt} g_{ij} = g'_{ij} = -2Ric_{ij}$, $d \log L = \frac{L'}{L} = -R$, (3.2)

Where $R = \frac{1}{L^2} Ric$. By definition of Ricci tensor, we have

$$Ric_{ij} = \frac{1}{2} [RL^2]_{y^i y^j},$$

$$= Rg_{ij} + \frac{1}{2} L^2 R_{,i,j} + R_{,i} y_j + R_{,i} y_i \quad (3.3)$$

where $R_{,i} = \frac{\partial R}{\partial y^i}$ and $R_{,i,j} = \frac{\partial^2 R}{\partial y^i \partial y^j}$. Taking a vertical derivative of (3.3) and using $y_{ij} = g_{ij}$ and $LL_k = y_k$ yields

$$Ric_{ij,k} = 2RC_{ijk} + \frac{1}{2} L^2 R_{,i,j,k} + \{g_{jk} R_{,i} + g_{ij} R_{,k} + g_{ki} R_{,j}\} + \{R_{,j,k} y_k + R_{,i,j} y_k + R_{,k,i} y_j\}.$$

Contracting (3.4) with $I^i I^j I^k$ and using $y_i I^i = y^i I_i = 0$ implies that

$$Ric_{ij,k} I^i I^j I^k = 2RC_{ijk} I^i I^j I^k + \frac{1}{2} L^2 R_{,i,j,k} I^i I^j I^k + 3\|I\|^2 I^m R_{,m} \quad (3.5)$$

The Cartan tensor of an (α, β) -metric on n -dimensional manifold M is given by,

$$C_{ijk} = A_{ij} I_k + A_{jk} I_i + A_{ki} I_j. \quad (3.6)$$

Multiplying (3.6) with $I^i I^j I^k$ yields,

$$C_{ijk} I^i I^j I^k = 3\|I\|^2. \quad (3.7)$$

Then by (3.5) and (3.7), we get

$$Ric_{ij,k} I^i I^j I^k = 6R\|I\|^4 + \frac{1}{2} L^2 R_{,i,j,k} I^i I^j I^k + 3\|I\|^2 I^m R_{,m} \quad (3.8)$$

On the other L_t satisfies Ricci flow equation, then

$$C'_{ijk} = \frac{1}{2} \frac{\partial g'_{ij}}{\partial y^k},$$

$$= \frac{1}{2} \frac{\partial (-2Ric_{ij})}{\partial y^k},$$

$$= -Ric_{ij,k} \quad (3.9)$$

By (3.8) and (3.9), we get (3.1).

Lemma 3.2. Let L_t be a deformation of an (α, β) -metric L , which is quasi- C -reducible, on a manifold M of dimension $n \geq 3$. Then $C'_{ijk} I^i I^j I^k$ is a factor of $\|I\|^2$.

Proof: Since $g^{ij} g_{jk} = \delta^i_k$, we have

$$(g^{ij} g_{jk})' = 0,$$

$$\Rightarrow g'^{ij} g_{jk} + g^{ij} g'_{jk} = 0,$$

$$\Rightarrow g'^{ij} g_{jk} + g^{ij} (-2Ric_{jk}) = 0,$$

$$\Rightarrow g'^{ij} g_{jk} - 2g^{ij} Ric_{jk} = 0 \quad (3.10)$$

or equivalently, $(g^{ij})' g_{jk} = 2g^{ij} Ric_{jk}$.

Contracting with g_{lk} gives,

$$(g^{il})' = 2Ric^{il} \quad (3.11)$$

Then, we have

$$I'_i = (g^{jk} c_{ijk})'$$

$$= (g^{jk})' c_{ijk} + g^{jk} (c_{ijk})'$$

$$= 2Ric^{jk} c_{ijk} + g^{jk} (-Ric_{ij,k})$$

$$= 2Ric^{jk} c_{ijk} + g^{jk} (-Ric_{ij,k})$$

$$= Ric^{jk} \frac{\partial g_{ij}}{\partial y^k} - (g^{jk} Ric_{jk})_{,i} + g^{jk} Ric_{jk}$$

(3.12)

Since,

$$-g^{jk} Ric_{ij,k} = -(g^{jk} Ric_{jk})_{,i} + g^{jk} Ric_{jk}, \quad (3.13)$$

we have,

$$I'_i = Ric^{jk} g_{jk,i} - (g^{jk} Ric_{jk})_{,i} + g^{jk} Ric_{jk},$$

$$= -(g^{jk} Ric_{jk})_{,i}$$

$$= -\rho_i, \quad (3.14)$$

where $\rho = g^{jk} Ric_{jk}$ and $\rho_i = \frac{\partial \rho}{\partial y^i}$. Thus

$$I'^i = (g^{ij} I_j)'$$

$$= (g^{ij})' I_j + g^{ij} I'_j,$$

$$= 2Ric^{ij} I_j + g^{ij} (-\rho_j),$$

$$= 2Ric^{ij} I_j - \rho^i \quad (3.15)$$

The variation of $y_i = LL_{y^i}$ with respect to t is given by,

$$y'_i = 2Ric_{im} y^m \quad (3.16)$$

Therefore, we can compute the variation of angular metric h_{ij} as follows

$$h'_{ij} = (g_{ij} - L^{-2} y_i y_j)',$$

$$= (g_{ij})' - (L^{-2} y_i y_j)',$$

$$= -2Ric_{ij} - \{L^{-2} [y'_i y_j + y_i y'_j] + y_i y_j (L^{-2})'\}$$

$$= -2Ric_{ij}$$

$$- L^{-2} [-2Ric_{im} y^m y_j - 2Ric_{jm} y^m y_i]$$

$$- 2L^{-2} R y_i y_j,$$

$$= 2Ric_{ij} + 2(h_{ij} - g_{ij})R$$

$$+ 2(Ric_{im} L^{-1} y_j) L^{-1} y^m$$

$$+ 2(Ric_{jm} L^{-1} y_i) L^{-1} y^m,$$

$$= -2Ric_{ij} + 2(h_{ij} - g_{ij})R$$

$$+2(Ric_{im}I_j + Ric_{jm}l_i)l^m, \quad (3.17)$$

where $l_i = L^{-1}y_i$ and $l^m = L^{-1}y^m$.

Thus, we consider the variation of Cartan tensor

$$\begin{aligned} C'_{ijk} &= \{A_{ij}I_k + A_{jk}I_i + A_{ki}I_j\}', \\ C'_{ijk} &= -\rho_k A_{ij} + \rho_i A_{jk} + \rho_j A_{ki}, \\ &\quad -2(Ric_{ij}I_k + Ric_{jk}I_i + Ric_{ki}I_j) \\ &\quad +2R(A_{ij}I_k + A_{jk}I_i + A_{ki}I_j) \\ &\quad -2R(g_{ij}I_k + g_{jk}I_i + g_{ki}I_j) \\ &\quad +2(\Delta_{ij}I_k + \Delta_{jk}I_i + \Delta_{ki}I_j) \end{aligned} \quad (3.18)$$

where $\Delta_{ij} = (Ric_{im}l_j + Ric_{jm}l_i)l^m$. Multiplying

(3.18) with $I^i I^j I^k$ gives

$$\begin{aligned} C'_{ijk} &= \{A_{ij}I_k + A_{jk}I_i + A_{ki}I_j\}', \\ &= -3[\rho_m I^m + 2Ric_{pq} I^p I^q] \|I\|^2 \end{aligned} \quad (3.19)$$

which implies $C'_{ijk} I^i I^j I^k$ is a factor of $\|I\|^2$. This completes the proof.

Next, we prove the following main theorem.

Theorem 3.1. Suppose that L is an (α, β) -metric on M , which is quasi- C -reducible, then every deformation L_t of the metric L satisfying un-normal Ricci flow equation is an Einstein metric.

Proof: By virtue of Lemma 3.1 and Lemma 3.2, $R_{i,j,k} I^i I^j I^k$ is a factor of $\|I\|^2$. Since $R_{i,j,k} I^i I^j I^k$ is a factor of $\|I\|^2$, multiplying it with y^k or y^j implies $R_{,i} = 0$. It means that $R = R(x)$ and then L_t is an Einstein metric.

IV. Normal Ricci flow equation on Quasi- C -reducible space with (α, β) -metrics

If M is a compact manifold, then $S(M)$ is compact and we can normalize the Ricci flow equation by requiring that the flow keeps the volume of SM constant. Recalling the Hilbert form $w = L_{,i} dx^i$, that volume is

$$Vol_{SM} = \int_{SM} \frac{(-1)^{n(n-1)/2}}{(n-1)!} w \wedge (dw)^{n-1} = \int_{SM} dV_{SM}.$$

During the evolution, L, w and consequently the volume form dV_{SM} and the volume Vol_{SM} , all depend on t . On the other hand, the domain of integration SM , being the quotient space of TM_0 under the equivalence relation $z \sim y, z = \lambda y$ for some $\lambda > 0$, is totally independent of any Finsler metric and hence does not depend on t . We have

$$\frac{d}{dt}(dV_{SM}) = \left[g_{ij} \frac{d}{dt} g_{ij} - n \frac{d}{dt} \log L \right] dV_{SM}.$$

A normalized Ricci flow for Finsler metrics is proposed by Bao [2] as follows

$$\begin{aligned} \frac{d}{dt} \log L &= -R + \frac{1}{Vol(SM)} \int_{SM} R dv, \\ L(t=0) &= L_0, \end{aligned} \quad (4.1)$$

where the underlying manifold M is compact. Now, we let $Vol(SM) = 1$. Then all of Ricci constant metrics are exactly the fixed points of the above flow. Let

$$Ric_{ij} = \frac{1}{2}(L^2 R)_{,y^i y^j},$$

and differentiating (4.1) with respect to y^i and y^j , the following normal Ricci flow tensor evaluation equation is concluded.

$$\begin{aligned} \frac{d}{dt} g_{ij} &= -2Ric_{ij} + \frac{2}{Vol(SM)} \int_{SM} R dv g_{ij}, \\ g(t=0) &= g_0. \end{aligned} \quad (4.2)$$

Starting with any familiar metric on M as the initial data L_0 , we may deform it using the proposed normalized Ricci flow, in the hope of arriving at a Ricci constant metric.

Theorem 4.2. Suppose that L is an (α, β) -metric on M , which is quasi- C -reducible, then every deformation L_t of the metric L satisfying normal Ricci flow equation is an Einstein metric.

Proof: Consider Finsler surfaces which satisfy the normal Ricci flow equation. Then

$$\begin{aligned} \frac{dg_{ij}}{dt} &= -2Ric_{ij} + 2 \int_{SM} R dv g_{ij}, \\ d \log L &= \frac{L'}{L} = -R + \int_{SM} R dv. \end{aligned} \quad (4.3)$$

By the same argument in the un-normal Ricci flow case, we can calculate the variation of mean Cartan tensor as follows

$$\begin{aligned} I'_i &= (g^{ik} C_{ijk})', \\ &= (g^{jk})' C_{ijk} + g^{jk} (C_{ijk})', \\ &= \left[2Ric^{jk} - 2 \int_{SM} R dv g_{ij} \right] C_{ijk} \\ &\quad + g^{jk} \left[Ric_{jk,i} + 2 \int_{SM} R dv C_{ijk} \right], \\ &= \rho_i. \end{aligned} \quad (4.4)$$

Then we have,

$$I'^i = (g^{ij} I_j)' + g^{ij} I'_j,$$

$$= \left[2Ric^{ij} - 2 \int_{SM} R dv g_{ij} \right] I_j - g^{ij} \rho_j. \quad (4.5)$$

As the similar way that we used in un-normal Ricci flow, it follows that

$$C'_{ijk} = \left[\frac{1}{\|I\|^2} I_i I_j I_k \right]',$$

$$= - \frac{(I'^m I_m + I'^m I'_m) C_{ijk}}{\|I\|^2} - \frac{(\rho_i I_j I_k + \rho_j I_i I_k + \rho_k I_i I_j)}{\|I\|^2}. \quad (4.6)$$

Contracting it with $I^i I^j I^k$, we can say $C'_{ijk} I^i I^j I^k$ is a factor of $\|I\|^2$. By Lemma 3.2, we deduce that $R_{i,j,k} I^i I^j I^k$ is a factor of $I^i I^j I^k$. By the same argument, it results that every deformation L_t of the metric L satisfying normal Ricci flow equation is an Einstein metric.

V. Conclusion

The Ricci flow theory became a very powerful method in understanding the geometry and topology of Riemannian manifolds. There were proposed a number important innovations in modern physics and mechanics. The purpose of this paper, we found the Ricci flow equations in Finsler geometry. Further studied the Ricci flow equations with special Finsler space that is, quasi- C -reducible space with (α, β) -metric.

IV. References

- [1] H. Akbar-Zadeh, Generalized Einstein manifolds, J. Geom. Phys., 17 (1995), 342-380.
- [2] A. H. M. Hamza and H. A. Hussein, Construction new type of matrices, IJMTT, 24(2), (2015).
- [3] D. Bao, On two curvature-driven problems in Riemann-Finsler geometry, Advanced Studies in Pure Mathematics, (2007).
- [4] R. S. Hamilton, Three manifolds with positive Ricci curvature, J. Diff. Geom., 17 (1982), 255-306.
- [5] Henri Poincare, Cinquieme complement 'a lanalysis situs. In OEuvres. Tome VI, Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics] pages v+541, Editions Jacques Gabay, Sceaux, Reprint of the 1953 edition, (1996).
- [6] S. K. Narasimhamurthy, D.M. Vasantha, Ricci flow equation on special Finsler space, Journal of Tensor Soc., 8 (2014) 121-130.
- [7] Narasimhamurthy, S.K. and Vasantha, D.M., Projective change between two Finsler spaces with (α, β) - metric, KYUNGPOOK Math. J., 52 (1) (2012), 81-89.
- [8] Nagesh S, G, Nanjundan, Suresh P, S. Pasha, A characterization of zero-inflated geometric model, IJMTT, 23(1), (2015).
- [9] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv: math.DG/ 0211159.
- [10] G. Perelman, Ricci flow with surgery on three manifolds, arXiv: math.DG/ 03109.
- [11] G. Perelman, Finite extinction time for the solutions to the Ricci flow on certain three manifolds, arXiv: math.DG/ 0307245.
- [12] Prasad, B.N., Gupta, B.N. and Singh, D.D., Conformal transformation in Finsler spaces with (α, β) - metric, Indian J. Pure and Appl. Math., 18 (4) (1961), 290-301.
- [13] A. Thayebi, E. Peyghan and B. Najafi, Ricci flow equation on (α, β) - metrics, arXiv:1108.0134v1.

[14] S. Vacaru, Nonholonomic Ricci flows: II. evolution equations and dynamics, J. Math. Phys., 49 (2008), 043504.

[15] S. Vacaru, Nonholonomic Ricci flows, exact solutions in gravity, and Symmetric and Non-symmetric Metrics, Int. J. Theor. Phys., 48 (2009), 579-606.

[16] S. Vacaru, The Entropy of Lagrange-Finsler spaces and Ricci flows, Rep. Math. Phys., 63 (2009), 95-110.

[17] S. Vacaru, Spectral functionals, nonholonomic Dirac operators, and non commutative Ricci flows, J. Math. Phys., 50 (2009).