Differentiable Riemannian Geometry

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Abstract *In this paper uniform upper and lower continuous function $f \in (M)$ on manifolds spaces with curvature bounds on (M) as surfaces and applications compact Riemannian boundary $(f, f^{-1}) \in M^{\pm} \in R$ is complete with Riemannian and we prove is integration on differential on* R

Keywords *Basic differential geometry – differentiable manifolds charts – integration smooth manifolds.*

I. Introduction

The Riemannian geometry with boundary, in the Euclidean domain the interior geometry is given ,flat and trivial, and the interesting phenomena come from the shape of the boundary ,Riemannian manifolds have no boundary, and the geometric phenomena are those of the interior is called differential geometry .

II. A BASIC NOTIONS ON DIFFERENTIAL GEOMETRY

In this section is review of basic notions on differential geometry:

2.1 First principles

Hausdrff 2.1.1

A topological space M is called (Hausdorff) if for all $x, y \in M$ there exist open sets such that $x \in U$ and $y \in V$ and $U \cap V = \phi$

Second countable 2.1.2

A topological space M is second countable if there exists a countable basis for the topology on M .

Locally Euclidean of dimension (N) 2.1.3

A topological space M is locally Euclidean of dimension n if for every point $x \in M$ there exists on open set $U \in M$ and open set $w \subset R^n$ so that U and W are (homeomorphic).

Definition 2.1.3

A topological manifold of dimension n is a topological space that is Hausdorff, second countable and locally Euclidean of dimension n .

Definition 2.1.4

A smooth atlas A of a topological space M is given by : (i) An open covering $\{U\}_{i\in I}$ where $U_i \subset M$

Open and $M = \cup_{i\in I} U_i$.

(ii) A family $\{\phi_i : U_i \to W_i\}_{i\in I}$ of homeomorphism ϕ_i onto open subsets $W_i \subset R^n$ so that if $U_i \cap U_j \neq \phi$ then the map $\phi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is (a diffoemorphism)

Definition 2.1.5

If $(U_i \cap U_j) \neq \phi$ then the diffeomorphism $\phi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is known as the (transitition map).

Definition 2.1.6

A smooth structure on a Hausdorff topological space is an equivalence class of atlases, with two atlases A and B being equivalent if for $(U_i, \varphi_i) \in A$ and $(V_j, \Psi_j) \in B$ with $U_i \cap V_j \neq \varphi$ then the transition $\phi_i(U_i \cap V_j) \to \Psi_j(U_i \cap V_j)$ map is a diffeomorphism (as a map between open sets of R^n).

Definition 2.1.7

A smooth manifold M of dimension n is a topological manifold of dimension n together with a smooth structure .

Definition 2.1.8

Let M and N be two manifolds of dimension m, n respectively a map $F : M \to N$ is called smooth at $p \in M$ if there exist charts $(U, \phi), (V, \Psi)$ with $p \in U \subset M$ and $F(p) \in V \subset N$ with $F(U) \subset V$ and the composition $\Psi \circ F \circ \phi^{-1} : \phi(U) \to \Psi(V)$ is a smooth (as map between open sets of R^n is called smooth if it smooth at every $p \in M$.

Definition 2.1.9

A map $F : M \to N$ is called a diffeomorphism if it is smooth bijective and inverse $F^{-1} : N \to M$ is also smooth.

Definition 2.1.10

A map F is called an embedding if F is an immersion and (homeomorphic) onto its image .

Definition 2.1.11

If $F : M \to N$ is an embedding then $F(M)$ is an immersed (submanifolds) of N .

2.2 Tangent space and vector fields

Let $C^\infty(M, N)$ be smooth maps from M and N and let $C^\infty(M)$ smooth functions on M is given a point $p \in M$ denote, $C^\infty(p)$ is functions defined on some open neighbourhood of p and smooth at p .

Definition 2.2.1

(i) The tangent vector X to the curve $c : (-\varepsilon, \varepsilon) \to M$ at $t = 0$ is the map $c(0) : C^\infty(c(0)) \to R$ given by the formula .

(1)

$$X(f) = c(0)(f) = \left(\frac{d(f \circ c)}{dt}\right)_{t=0} \quad \forall f \in C^\infty c(0)$$

(ii) A tangent vector X at $p \in M$ is the tangent vector at $t=0$ of some curve $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$ this is $X = \alpha'(0) : C^{\infty}(p) \to R$.

Remark 2.2.2

A tangent vector at p is known as a liner function defined on $C^{\infty}(p)$ which satisfies the (Leibniz property)

(2)

$X(f\,g) = X(f)g + f\,X(g)\ \ , \forall f, g \in C^{\infty}(p)$

.

2.3 Differential Geometrics

Given $F \in C^{\infty}(M, N)$ and $p \in M$ and $X \in T_p M$ choose a curve $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$ and $\alpha'(0) = X$ this is possible due to the theorem about existence of solutions of liner first order ODEs , then consider the map $F_{*p} : T_p M \to T_{F(p)} N$ mapping $X \to F_{*p}(X) = (F \circ \alpha)'(0)$, this is liner map between two vector spaces and it is independent of the choice of α .

Definition 2.3.1

The liner map F_{*p} defined above is called the derivative or differential of F at p while the image $F_{*p}(X)$ is called the push forward X at $p \in M$.

Definition 2.3.2

Given a smooth manifold M a vector field V is a map $V : M \to TM$ mapping $p \to V(p) \equiv V_p$ and V is called smooth if it is smooth as a map from M to TM .

(Not) $X(M)$ is an R vector space for $Y, Z \in X(M)$, $p \in M$ and $a, b \in R$, $(aY + bZ)_p = aV_p + bZ_p$ and for $f \in C^{\infty}(M)$, $Y \in X(M)$ define $fY : M \to TM$ mapping $p \to (fY)_p = f(p)Y_p$

2.4 Cotangent space and Vector Bundles and Tensor Fields

Let M be a smooth n-manifolds and $p \in M$ .We define cotangent space at p denoted by $T_p^* M$ to be the dual space of the tangent space at $p : T_p^*(M) = \{ f : T_p M \to R \}$, f smooth Element of $T_p^* M$ are called cotangent vectors or tangent convectors at p .(i) For $f : M \to R$ smooth the composition $T_p^* M \to T_{f(p)} R \cong R$ is called df_p and referred to the differential of f .Not that $df_p \in T_p^* M$ so it is a cotangent vector at p (ii) For a chart (U, ϕ, x^i) of M and $p \in U$ then $\{dx^i\}_{i=1}^{n}$ is a basis of $T_p^* M$ in fact $\{dx^i\}$ is the dual basis of $\left\{ \frac{d}{dx^i} \right\}_{i=1}^{n}$.

Definition 2.4.1

The elements in the tensor product $V_s^r = V \otimes .... \otimes V \otimes V^* \otimes ... \otimes V^*$ are called (r, s) tensors or r-contravariant , s- contravariant tensor .

Remark 2.4.2

The Tensor product is bilinear and associative however it is in general not commutative that is $(T_1 \otimes T_2) \neq (T_2 \otimes T_1)$ in general .

Definition 2.4.3

$T \in V_s^r$ is called reducible if it can be written in the form $T = V_1 \otimes ... \otimes V_r \otimes L^1 \otimes ... \otimes L^s$ for.

(3) $V_i \otimes V_r , L^j \in V^*$ for $1 \le i \le r , 1 \le j \le s$.

Definition 2.4.4

Choose two indices (i, j) where $1 \le i \le r , 1 \le j \le s$ for any reducible tensor $T = V_1 \otimes ... \otimes V_r \otimes L^1 \otimes .... \otimes L^2$ let $C_i^r(T) \in V_{s-1}^{r-1}$ We extend this linearly to get a linear map $C_i^j : V_s^r \to V_{s-1}^{r-1}$ which is called tensor-contraction.

Remark 2.4.4

An ant symmetric (or alternating $(0, k)$ tensor) $T \in V_k^0$ is called a k-form on V and the space of all k-forms on V is denoted $\wedge^k V^* = \{T \in V_k^0 : T\}$ alternating .

Definition 2.4.5

A smooth real vector boundle of rank k denoted (E, M, π) is a smooth manifold E of dimension $n+1$ the total space a smooth manifold M of dimension n the manifold dimension $n+k$ and a smooth subjective map $\pi : E \to M$ (projection map) with the following properties :

(i) There exists an open cover $\{V_\alpha\}_{\alpha \in I}$ of M and diffeomorphisms $\Psi_\alpha : \pi^{-1}(V_\alpha) \to V_\alpha \times R^k$.

(ii) For any point $p \in M$, $\Psi_\alpha(\pi^{-1}(p)) = \{p\} \times R^k \cong R^k$ and we get a commutative diagram (in this case $\pi_1 : V_\alpha \times R^k \to V_\alpha$ is projection onto the first component .

(iii) whenever $V_\alpha \cap V_\beta \neq \phi$ the diffeomorphism.

(4)

$\psi_\alpha \circ \Psi_\beta^{-1} : (V_\alpha \cap V_\beta) \times R^k \to (V_\alpha \cap V_\beta) \times R^k$

takes the form

$\Psi_\alpha \circ \Psi_\beta^{-1}(p, a) = (p, A_{\alpha\beta}(p)(a)), a \in R^k$ where $A_{\alpha\beta} : V_\alpha \cap V_\beta \to GL(k, R)$ is called transition maps.

2.5 Bundle Maps and isomorphisms

Suppose (E, M, π) and $(\tilde{E}, \tilde{M}, \tilde{\pi})$ are two vector bundles a smooth map $F : E \to \tilde{E}$ is called a smooth bundle map from (E, M, π) to $(\tilde{E}, \tilde{M}, \tilde{\pi})$.

(i) There exists a smooth map $f : M \to \tilde{M}$ such that the following diagram commutes that $\pi(F(q)) = f(\pi(q))$ for all $p \in M$

(ii) F induces a linear map from E_p to $\tilde{E}_{f(p)}$ for any $p \in M$.

Definition 2.5.1 Dual Bundle

Take a vector bundle (E, M, π) where $E : \cup_{p \in M} E_p$ replace E_p with its dual $E^*{}_p$ and consider $E^* : \cup_{p \in M} E^*{}_p$. Let $V_\alpha, \psi_\alpha, A_{\alpha\beta}$ by an in the transition maps for the dial bundle E^* are denoted $(A^{dual})_{\alpha\beta} = (A^{-1}_{\alpha\beta})^T$ observe that $(A^{dual})_{\alpha\beta} = (A^{dual})_{\beta\gamma}$.

Definition 2.5.2 Tensor product of vector Bundles

Suppose (E, M, π) is vector bundle of rank k and $(\tilde{E}, \tilde{M}, \tilde{\pi})$ is vector bundle of rank l over the same base manifold M then define $E \otimes \tilde{E} = \cup_{p \in M} E_p \otimes \tilde{E}_p$, this is well defined because E_p and $\tilde{E}_p$ are vector spaces . Let be an open cover of $M, \Psi_\alpha, \tilde{\Psi}_\alpha, A_{\alpha\beta}, \tilde{A}_{\alpha\beta}$ be the local trivializations and transition maps to E and $\tilde{E}$ respectively then the transudation maps and local trivializations for $E \otimes \tilde{E}$ are given .

(5)

$$a \otimes \tilde{a} \to A_{\alpha\beta} a \otimes \tilde{A}_{\alpha\beta} \tilde{a} \in R^k \otimes R^l \cong R^{k+1},$$
$$\forall a \in R^k, \tilde{a} \in R^l$$

Definition 2.5.3

Let $F : M \to N$ be a smooth map between two smooth manifolds and $w \in \Gamma(T^0_k N)$ be a k covariant tensor field we define a k covariant tensor field $F^* w$ over M by .

(6)

$$(F^* w)_p (v_1, \ldots, v_k) = w_{F(p)} (F_{*p}(v_1), \ldots, F_{*p}(v_k))$$
$$, \forall v_1, \ldots, v_k \in T_p M$$

In this case $F^* w$ is called the pullback of w by F.

Proposition 2.5.4

Suppose $F : M \to N$ is a smooth map and $G : N \to Q$ a smooth map for M, N, Q smooth manifolds and $w \in T(T^0_k N), \eta \in T(T^0_l N)$ and $f \in C^\infty(N)$ then .

(i) $(G \circ F)^* = F^* \circ G^*$.

(ii) $F^*(w \otimes \eta) Fw \otimes F^* w \otimes F^* \eta$ in particular , $F^*(f \circ w) = (f \circ F) F^* w$.

(iii) $F(df) = d(f \circ F)$ (iv) if $p \in M$ and (y^i) are local coordinates in a chart containing the point $F(p) \in N$ then

(7) .

$$F^*(w_{j1,\ldots,jk} dy^{i1} \otimes \ldots \otimes dy^{ik}) =$$
$$(w_{j1,\ldots jk}, \circ F) d(y^{j1} \circ F) \otimes \ldots \otimes d(y^{ik} \circ F).$$

2.6 Exterior derivative

The exterior derivative is a map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ which is R linear such that $d \circ d = 0$ and if f is a k vector field on k then $(df)(X) = Xf$.

2.7 Integration of differential forms

$\int_M w$ is well defined only if M is orient able $\dim(M) = n$ and has a partition of unity and w has compact support and is a differential n-form on M.

2.8 Riemannian Manifolds

An inner product (or scalar product) on a vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to R$ that is :

(i)symmetric $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$ (ii)Bilinear $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$ $\langle u, av + bw \rangle = a\langle u, v \rangle + b\langle u, w \rangle$ for all $a, b \in R$ and $u, v, w, \in V$.

(iii) positive definite $\langle u, v \rangle \succ 0$ for all $u \neq 0$.

Definition 2.8.1

A pair (M, g) of a manifold M equipped with a Riemannian metric g is called a Riemannian manifold.

2.9 Length and Angle between tangent vectors

Suppose (M, g) is a Riemannian manifold and $p \in M$ we define the length (or norm) of a tangent vector $v \in T_p M$ to be $|v| = \sqrt{\langle v, v \rangle_p}$ Recall $g(\cdot,\cdot) = \langle \cdot,\cdot \rangle$ and the angle v, w between $v, w \in T_p M$ $(v \neq 0 \neq w)$ by $\cos(v, w) = \dfrac{\langle v, w \rangle_p}{|v||w|}$.

Examples of Riemannian metrics 2.9.1

1. Euclidean metric (canonical metric) g_{Eucl} on R^n.

(8)

$$g_{Eucl} = \delta_{ij} dx^i \otimes dx^j = dx^1 \otimes dx^1 + \ldots + dx^n \otimes dx^n$$
$$= dx^1 dx^1 + \ldots + dx^n dx^n$$

2. Induced metric

Let (M, g) be a Riemannian manifold and $f : N \to (M, g)$ an immersion where N is a smooth manifold (that is f is a smooth map and f is injective) then induced metric on N is defined .

(9)

$$(f\, g)_p (v, w) = g_{f(p)}(f_*(v)), (f_*(w))$$
$$, \forall v, w \in T_p N, p \in N$$

3. Induced metric $i^* g_{Eucl}$ on S^n

The induced metric S^n sometimes denoted $g_{Eucl}|_{S^n}$ from the Euclidean space R^{n+1} and g_{Eucl} by the inclusion $i : S^2 \to R^{n+1}$ is called the standard (or round) metric on S^n clearly i is an immersion .Consider stereographic projection $S^2 \to R^3$ and denote the inverse map

$u : R^2 \rightarrow S^2$ then $u^* g_{Eucl}$ Given the Riemannian metric for R^2.

4. Product metric

If (M_1, g_1), (M_2, g_2) are two Riemannian manifolds then the product $M_1 \times M_2$ admits a Riemannian metric $g = g_1 \oplus g_2$ is called the product metric defined by .

$$g(u_1 \oplus u_2, v_1 \oplus v_2) = g_1(u_1, v_1) \oplus g_2(u_2, v_2)$$

Where $u_i, v_i \in T_{pi} M_i$ for $i = 1,2,....$ we use the fact that $T_{p_1,p_2}(M_1 \times M_2) \cong T_{p_1} M_1 \oplus T_{p_1} M_2$.

5. Warped product

Suppose (M_1, g_1), (M_2, g_2) are two Riemannian manifolds then $(M_1 \times M_2, g_1 \oplus f^2 g_2)$ is the warped product of g_1, g_2 or denoted $(M_1, g_1) \times_f (M_2, g_2)$ where $f : M_1 \rightarrow R$ is a smooth positive function.

(10) $$\begin{aligned}&(g_1 \oplus f^2 g_2)_{p_1 p_2}(u_2 \oplus u_2, v_1 \oplus v_2)\\ &= g_{1_{p_1}}(u_1, v_1) \oplus f(p_1) g_{2_{p_1}}(v_2, w_2)\end{aligned}$$

2.10 Conformal map and Isometric

A smooth map $f : (M, g) \rightarrow (N, h)$ between two Riemannian manifolds is called a conformal map with conformal factor $\lambda : M \rightarrow R^+$ if $(f^* h) = \lambda^2 g$. (Not)A conformal map preserves angles that is $(v, w) = (f_*(v), f_*(w))$ for all $u, v \in T_p M$ and $p \in M$.

Example 2.10.1

$S^2 \subset R^3$ we consider stereographi projection $S^2 / p_n \rightarrow R^2$. As stereographic projection is a (diffeomorphism) its inverse $u : R \rightarrow S / p_n$ is a conformal map . It follows from an exercise sheet that u is a conformal map with conformal factor $\rho(x, y) = 2 / (1 + x^2 + y^2)$.

Definition 2.10.2

A Riemannian manifold (M , g) is locally flat if for every point $p \in M$ there exist a conformal (diffeomorphism) $f : U \rightarrow V$ between an open neighbourhoods U of p and $V \subset R^n$ of $f(p)$.

Definition 2.10.3

Given two Riemannian manifold (M , g) and (N, h) they are called isometric of there is a diffeomorphism $f : M \rightarrow N$ such that $f^* h = g$ such that a differomorphism f is called an (isometric).

Remark 2.10.4

In particular an isometrics $f : (M, g) \rightarrow (M, g)$ is called an isometric of (M, g) . All isometrics on a Riemannian manifold from a group.

Definition 2.10.5

$(M, g), (N, h)$ are called locally isometric if for every point $p \in M$ there is an isometric $f : U \rightarrow V$ from an open neighbourhood U of p in M and an open neighbourhood V of $f(p)$ in N .

Definition 2.10.6

Suppose $f : (M, g) \rightarrow (N, h)$ is an immersion then f is isometric if $f^* h = g$.

Definition 2.10.7

Let (M, g) be an oriented Riemannian n-manifold with its Riemannian volume from dV_g if f is a compactly supported smooth function on M then $f\, dV_g$ is a new n-form which is compactly supported we can define the integral of f over M as .

(11) $$\int_M f = \int_M f\, dV_g$$

Recall the integration of n-forms over n-manifolds.

2.11 Bundle metrics

The recall from liner algebra on a vector space V a bilinear from $B : V \times V \rightarrow R$ can be considered as an element $B \in E^* \otimes E^*$ given a vector bundle (E, M, π) a bundle metric is a map that assigns each fiber E_p an inner product $\langle \cdot, \cdot \rangle_p$ which depends smoothly on $p \in M$.

Definition 2.11.1

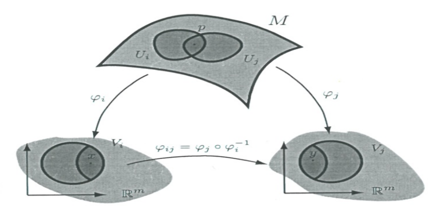
A bundle metric h on the vector bundle (E, M, π) is an element of $\Gamma(E^* \otimes E^*)$ which is stmmetric and positive definite.

Remark 2.11.2

Given a vector bundle (E, M, π) with a bundle metric h we can define an isomorphism $E \rightarrow E^*$ we can extend h to any (r, s) tensor products of E and E^* .

III.DIFFERENTIABLE MANIFOLDS CHARTS

In this section, the basically an m-dimensional topological manifold is a topological space M which is locally homeomorphic to R^m , definition is a topological space M is called an m-dimensional (topological manifold) if the following conditions hold. (i) M is a hausdorff space. (ii) for any $p \in M$ there exists a neighborhood U of P which is homeomorphic to an open subset $V \subset R^m$. (iii) M has a countable basis of open sets coordinate charts (U, φ) Axiom (ii) is equivalent to saying that $p \in M$ has a open neighborhood $U \in P$ homeomorphic to open disc D^m in R^m , axiom (iii) says that M can covered by countable many of such neighborhoods , the coordinate chart (U, φ) where U are coordinate neighborhoods or charts and φ are coordinate . A homeomorphisms , transitions between different choices of coordinates are called transitions maps $\varphi_{ij} = \varphi_j \circ \varphi_i$, which are again homeomorphisms by definition , we usually write $p = \varphi^{-1}(x), \varphi : U \to V \subset R^n$ as coordinates for U , see Figure (1) , and $p = \varphi^{-1}(x), \varphi^{-1} : V \to U \subset M$ as coordinates for U , the coordinate charts (U, φ) are coordinate neighborhoods, or charts , and φ are coordinate homeomorphisms , transitions between different choices of coordinates are called transitions maps $\varphi_{ij} = \varphi_j \circ \varphi_i$ which are again homeomorphisms by definition , we usually $x = \varphi(p), \varphi : U \to V \subset R^n$ as a parameterization U a collection $A = \{(\varphi_i, U_i)\}_{i \in I}$ of coordinate chart with $M = \cup_i U_i$ is called atlas for M .



Figurer (1) : $\varphi_{ij} : \varphi_j \circ \varphi_i^{-1}$ the transition maps

The transition maps φ_{ij} Figurer (1) a topological space M is called (hausdorff) if for any pair $p, q \in M$, there exist open neighborhoods $p \in U$ and $q \in U'$ such that $U \cap U' \neq \phi$ for a topological space M with topology $\tau \in U$ can be written as union of sets in β , a basis is called a countable basis β is a countable set .

Definition 3.1.1

A topological space M is called an m-dimensional topological manifold with boundary $\partial M \subset M$ if the following conditions .

(i) M is hausdorff space.

(ii) for any point $p \in M$ there exists a neighborhood U of p which is homeomorphism to an open subset $V \subset H^m$.

(iii) M has a countable basis of open sets , can be rephrased as follows any point $p \in U$ is contained in neighborhood U to $D^m \cap H^m$ the set M is a locally homeomorphism to R^m or H^m the boundary $\partial M \subset M$ is subset of M which consists of points p .

Definition 3.1.2

A function $f : X \to Y$ between two topological spaces is said to be continuous if for every open set U of Y the pre-image $f^{-1}(U)$ is open in X .

Definition 3.1.3

Let X and Y be topological spaces we say that X and Y are homeomorphic if there exist continuous function such that $f \circ g = id_y$ and $g \circ f = id_x$ we write $X \cong Y$ and say that f and g are homeomorphisms between X and Y , by the definition a function $f : X \to Y$ is a homeomorphisms if and only if .(i) f is a bijective .(ii) f is continuous (iii) f^{-1} is also continuous.

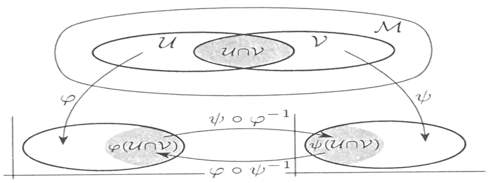
3.2 Differentiable manifolds

A differentiable manifolds is necessary for extending the methods of differential calculus to spaces more general R^n a subset $S \subset R^3$ is regular surface if for every point $p \in S$ the a neighborhood V of P is R^3 and mapping $x : u \subset R^2 \to V \cap S$ open set $U \subset R^2$ such that.

(i) x is differentiable homomorphism.

(ii) the differentiable $(dx)_q : R^2 \to R^3$, the mapping x is called a parametnzation of S at P the important consequence of differentiable of regular surface is the fact that the transition also example below if $x_\alpha : U_\alpha \to S^1$ and $x_\beta : U_\beta \to S^1$ are $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = w \neq \phi$, the maps $x_\beta^{-1} \circ x_\alpha : x^{-1}(w) \to R^2$ and $x_\alpha^{-1} \circ x_\beta = x_\beta^{-1}(w) \to R$

Are differentiable structure on a set M induces a natural topology on M it suffices to $A \subset M$ to be an open set in M if and only if $x_\alpha^{-1}(A \cap x_\alpha(U_\alpha))$ is an open set in R^n for all α it is easy to verify that M and the empty set are open sets that a union of open sets is again set and that the finite intersection of open sets remains an open set. Manifold is necessary for the methods of differential calculus to spaces more general than de R^n , a differential structure on a manifolds M induces a differential structure on every open subset of M , in particular writing the entries of an $n \times k$ matrix in succession identifies the set of all matrices with $R^{n,k}$, an $n \times k$ matrix of rank k can be viewed as a k-frame that is set of k linearly independent vectors in R^n , $V_{n,k} K \leq n$ is called the steels manifold ,the general linear group $GL(n)$ by the foregoing $V_{n,k}$ is differential structure on the group n of orthogonal matrices, we define the smooth maps function $f : M \to N$ where M, N are differential manifolds we will say that f is smooth if there are atlases (U_α, h_α) on M , (V_B, g_B) on N , such that the maps $g_B f h_\alpha^{-1}$ are smooth wherever they are defined f is a homeomorphism if is smooth and a smooth inverse. A differentiable structures is topological is a manifold it an open covering U_α where each set U_α is homeoomorphic, via some homeomorphism h_α to an open subset of Euclidean space R^n , let M be a topological space , a chart in M consists of an open subset $U \subset M$ and a homeomorphism h of U onto an open subset of R^m, a C^r atlas on M is a collection (U_α, h_α) of charts such that the U_α cover M and h_B, h_α^{-1} the differentiable .



Figurer (2) :

$(\varphi \circ \psi^{-1}) = (\psi^{-1} \circ \varphi)$

Definition 3.2.1

Let M be a metric space we now define what is meant by the statement that M is an n-dimensional C^∞ manifold.

(i) A chart on M is a pair (U, φ) with U an open subset of M and φ a homeomorphism a (1-1) onto, continuous function with continuous inverse from U to an open subset of R^n, think of φ as assigning coordinates to each point of U .

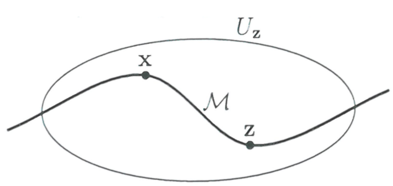
(ii) Two charts (U, φ) and (V, ψ) are said to be compatible if the transition functions . see Fig (2)

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \subset R^n \to \psi(U \cap V) \subset R^n$$

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \subset R^n \to \varphi(U \cap V) \subset R^n$$

Are C^∞ that is all partial derivatives of all orders of $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ exist and are continuous.

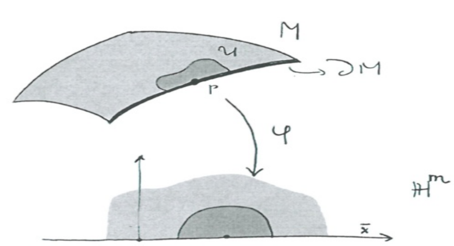
(iii) An atlas for M is a family $A = \{ (U_i, \varphi_i) : i \in I \}$ of charts on M such that $\{ U_i \}_{i \in I}$ is an open cover of M and such that every pair of charts in A are compatible . The index set I is completely arbitrary . It could consist of just a single index. It could consist of uncountable many indices . An atlas A is called maximal if every chart (U, φ) on M that is compatible with every chat of A .



Figyer (3)

Example 3.2.2 (Surfaces)

Any smooth n-dimensional R^{n+1} is an n-dimensional manifold. Roughly speaking a subset of R^{n+m} a an n-dimensional surface if , locally m of the $m+n$ coordinates of points on the surface are determined by the other n coordinates in a C^∞ way , For example , the unit circle S^1 is a one dimensional surface in R^2 . Near (0.1) a point $(x, y) \in R^2$ is on S^1 if and only if $y = \sqrt{1 - x^2}$ and near (-1.0) , (x, y) is on S^1 if and only if $y = -\sqrt{1 - x^2}$. The precise definition is that M is an n-dimensional surface in R^{n+m} if M is a subset of R^{n+m} with the property that for each $z = (z_1, ..., z_{n+m}) \in M$ there are a neighborhood U_z of z in R^{n+m} , and n integers . $1 \le J_1 \le j_2 \le ... \le j_{n+m}$ C^∞ function $f_k(x_{j1}, ..., x_{jn})$, $k \in \{1, ..., n+m\} / \{j_1, ..., j_n\}$ such that the point $x = (x_1, ..., x_{n+m}) \in U_z$. That is we may express the part of M that is near z as $x_{i1} = f_{i1}(x_{j1}, x_{j2}, ..., x_{jn})$, $x_{i2} = f_{i2}(x_{j1}, x_{j2}, ..., x_{jn})$, $x_{im} = f_{im}(x_{j1}, x_{j2}, ..., x_{jn})$. Where there for some C^∞ function $f_1, ..., f_m$. We many use $x_{j1}, x_{j2}, ..., x_{jn}$ as coordinates for R^2 in $M \cap U_z$ .Of course an atlas is with $\varphi_z(x) = (x_{j1}, ..., x_{jn})$ Equivalently, M is an n-dimensional surface in R^{n+m} if for each $z \in M$, there are a neighborhood U_z of z in R^{n+m} , and $m\, C^\infty$ functions $g_k : U_z \to R$ with the vector $\{ \nabla_{gz}(z) |, 1 \le k \le m \}$ linearly independent such that the point $x \in U_z$ is in M if and only if $g_k(x) = 0$ for all $1 \le k \le m$ .To get from the implicit equations for M given by the g_k to the explicit equations for M given by the f_k one need only invoke (possible after renumbering of x) .



Figurer (4) : coordinate maps for boundary points

A topological space M is called an m-dimensional topological manifold with boundary $\partial M \subset M$ if the following conditions.

(i) M is hausdorff space.

(ii) for any point $p \in M$ there exists a neighborhood U of p which is homeomorphic to an open subset $V \subset H^m$

(iii) M has a countable basis of open sets , Figure (4) can be rephrased as follows any point $p \in U$ is contained in neighborhood U to $D^m \cap H^m$ the set M is a locally homeomorphic to R^m or H^m the boundary $\partial M \subset M$ is subset of M which consists of points p .

Definition 3.2.3

Let X be a set a topology U for X is collection of X satisfying:

(i) ϕ and X are in U .

(ii) the intersection of two members of U is in U .

(iii) the union of any number of members U is in U . The set X with U is called a topological space the members $U \in u$ are called the open sets . let X be a topological space a subset $N \subseteq X$ with $x \in N$ is called a neighborhood of X if there is an open set U with $x \in U \subseteq N$, for example if X a metric space then the closed ball $D_{\varepsilon}(x)$ and the open ball $D_{\varepsilon}(x)$ are neighborhoods of X a subset C is said to closed if $X \setminus C$ is open

Definition 3.2.4

A function $f : X \to Y$ between two topological spaces is said to be continuous if for every open set U of Y the pre-image $f^{-1}(U)$ is open in X .

Definition 3.2.5

Let X and Y be topological spaces we say that X and Y are homeomorphic if there exist continuous function $f : X \to Y$, $g : Y \to X$ such that $f \circ g = id_y$ and $g \circ f = id_X$ we write $X \cong Y$ and say that f and g are homeomorphisms between X and Y , by the definition a function $f : X \to Y$ is a homeomorphisms if and only if (i) f is a bijective (ii) f is continuous (iii) f^{-1} is also continuous.

3 .3 Differentiable manifolds

A differentiable manifolds is necessary for extending the methods of differential calculus to spaces more general R^n a subset $S \subset R^3$ is regular surface if for every point $p \in S$ the a neighborhood V of P is R^3 and mapping $x : u \subset R^2 \to V \cap S$ open set $U \subset R^2$ such that (i) x is differentiable homomorphism (ii) the differentiable $(dx)_q : R^2 \to R^3$, the mapping x is called aparametnzation of S at P the important consequence of differentiable of regular surface is the fact that the transition also example below if $x_\alpha : U_\alpha \to S^1$ and $x_\beta : U_\beta \to S^1$ are $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = w \neq \phi$ the mappings $x_\beta^{-1} \circ x_\alpha : x^{-1}(w) \to R^2$ and $x_\alpha^{-1} \circ x_\beta = x_\beta^{-1}(w) \to R$

Are differentiable

A differentiable structure on a set M induces a natural topology on M it suffices to $A \subset M$ to be an open set in M if and only if $x_\alpha^{-1}(A \cap x_\alpha(U_\alpha))$ is an open set in R^n for all α it is easy to verify that M and the empty set are open sets that a union of open sets is again set and that the finite intersection of open sets remains an open set manifold is necessary for the methods of differential calculus to spaces more general than de R^n , a differential structure on a manifolds M induces a differential structure on every open subset of M , in particular writing the entries of an $n \times k$ matrix in succession identifies the set of all matrices with $R^{n,k}$, an $n \times k$ matrix of rank k can be viewed as a k-frame that is set of k linearly independent vectors in R^n , $V_{n,k} K \le n$ is called the steels manifold ,the general linear group $GL(n)$ by the foregoing $V_{n,k}$ is differential structure on the group n of orthogonal matrices, we define the smooth maps function $f : M \to N$ where M , N are differential manifolds we will say that f is smooth if there are atlases (U_α, h_α) on M , (V_B, g_B) on N , such that the maps $g_B f h_\alpha^{-1}$ are smooth wherever they are defined f is a homeomorphism if is smooth and a smooth inverse. A differentiable structures is topological is a manifold it an open covering U_α where each set U_α is homeoomorphic, via some homeomorphism h_α to an open subset of Euclidean space R^n , let M be a topological space , a chart in M consists of an open subset $U \subset M$ and homeomorphism h of U onto an open subset of R^m , a C^r atlas on M is a collection (U_α, h_α) of charts such that the U_α cover M and h_B, h_α^{-1} the differentiable vector fields on a differentiable manifold M , let X and Y be a differentiable vector field on a differentiable manifolds M then there exists a unique vector field Z such that such that , for all $f \in D$, $Zf = (XY - YX) f$ if that $p \in M$ and let $x : U \to M$ be a parameterization at p and

$$\left(X = \sum_i a_i \frac{\partial}{\partial x_i} \right) , \left(Y = \sum_j a_j \frac{\partial}{\partial y_j} \right)$$

$$\left(XYf = X (\sum_i b_j \frac{\partial f}{\partial x_i}) \right) , \left(YXf = Y (\sum_j a_i \frac{\partial f}{\partial x_j} \right)$$

Therefore Z is given in the parameterization X by Z .

$$Z f = (X Y f - Y X f) , \sum_{i,j} (a_i \frac{\partial b_j}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j})$$

Are differentiable this a regular surface is intersect from one to other can be made in a differentiable manner the defect of the definition of regular surface is its dependence on R^3 . A differentiable manifold is locally homeomorphic to R^n the fundamental theorem on existence , uniqueness and dependence on initial conditions of ordinary differential equations which is a local theorem extends naturally to differentiable manifolds . For familiar with differential equations can assume the statement below which is all that we need for example X be a differentiable on a differentiable manifold M and $p \in M$ then there exist a neighborhood $p \in M$ and $U_p \subset M$ an inter $(-\delta, \delta)$, $\delta \ge 0$, and a differentiable mapping $\varphi : (-\delta, \delta) \times U \to M$ such that curve $t \to \varphi(t, q)$ and $\varphi(0, q) = q$ a curve

$\alpha:(-\delta,\delta)\to M$ which satisfies the conditions $\alpha^{-1}(t)=X(\alpha(t))$ and $\alpha(0)=q$ is called a trajectory of the field X that passes through q for $t=0$. A differentiable manifold of dimension N is a set M and a family of injective mapping $x_\alpha \subset R^n \to M$ of open sets $u_\alpha \in R^n$ into M such that:

(i) $u_\alpha x_\alpha(u_\alpha)=M$

(ii) for any α,β with $x_\alpha(u_\alpha)\cap x_\beta(u_\beta)$

(iii) the family (u_α, x_α) is maximal relative to conditions (i),(ii) the pair (u_α, x_α) or the mapping x_α with $p\in x_\alpha(u_\alpha)$ is called a parameterization , or system of coordinates of M , $u_\alpha x_\alpha(u_\alpha)=M$ the coordinate charts (U,φ) where U are coordinate neighborhoods or charts , and φ are coordinate homeomorphisms transitions are between different choices of coordinates are called transitions maps

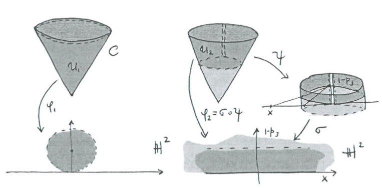
(9) $$\varphi_{i,j}:(\varphi_j\circ\varphi_i^{-1})$$

Which are anise homeomorphisms by definition , we usually write $x=\varphi(p),\varphi:U\to V\subset R^n$ collection U and $p=\varphi^{-1}(x),\varphi^{-1}:V\to U\subset M$ for coordinate charts with is $M=\cup U_i$ called an atlas for M of topological manifolds.

A topological manifold M for which the transition maps $\varphi_{i,j}=(\varphi_j\circ\varphi_i)$ for all pairs φ_i,φ_j in the atlas are homeomorphisms is called a differentiable , or smooth manifold , the transition maps are mapping between open subset of R^m , homeomorphisms between open subsets of R^m are C^∞ maps whose inverses are also C^∞ maps , for two charts U_i and U_j the transitions mapping

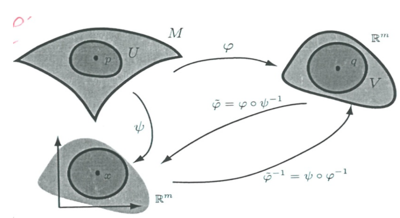
(10)

$$\varphi_{i,j}=(\varphi_j\circ\varphi_i^{-1}):\varphi_i(U_i\cap U_j)\to\varphi_j(U_i\cap U_j)$$



Since $(\psi'\circ\psi^{-1})$ and $(\varphi'\circ\varphi^{-1})$ are homeomorphisms it easily follows that which show that our notion of rank is well defined $(J\,f'')_{x^i}=J(\psi'\circ\psi^{-1})_{y^i}\,J\,f'(\varphi'\circ\varphi^{-1})^{-1}$, if a map has constant rank for all $p\in N$ we simply write $rk(f)$, these are called constant rank mapping.The product two manifolds M_1 and M_2 be two C^k-manifolds of dimension n_1 and n_2 respectively the topological space $M_1\times M_2$ are arbitral unions of sets of the form $U\times V$ where U is open in M_1 and V is open in M_2 , can be given the structure C^k manifolds of dimension n_1,n_2 by defining charts as follows for any charts M_1 on (V_j,ψ_j) on M_2 we declare that $(U_i\times V_j,\ \varphi_i\times\psi_j)$ is chart

on $M_1\times M_2$ where $\varphi_i\times\psi_j:U_i\times V_j\to R^{(n_1+n_2)}$ is defined so that $\varphi_i\times\psi_j(p,q)=(\varphi_i(p),\psi_j(q))$ for all $(p,q)\in U_i\times V_j$. A given a C^k n-atlas, A on M for any other chart (U,φ) we say that (U,φ) is compatible with the atlas A if every map $(\varphi_i\circ\varphi^{-1})$ and $(\varphi\circ\varphi_i^{-1})$ is C^k whenever $U\cap U_i\neq 0$ the two



Figurer (6):coordinate difeomorphisms $\tilde{\varphi}=\varphi\circ\psi^{-1}$ **and** $\tilde{\varphi}^{-1}=\psi\circ\varphi^{-1}$

atlases A and $\tilde{A}$ is compatible if every chart of one is compatible with other atlas see Figure (6).A sub manifolds of others of R^n for instance S^2 is sub manifolds of R^3 it can be obtained as the image of map into R^3 or as the level set of function with domain R^3 we shall examine both methods below first to develop the basic concepts of the theory of Riemannian sub manifolds and then to use these concepts to derive a equantitive interpretation of curvature tensor , some basic definitions and terminology concerning sub manifolds, we define a tensor field called the second fundamental form which measures the way a sub manifold curves with the ambient manifold , for example X be a sub manifold of Y of $\pi:E\to X$ and $g:E_1\to Y$ be two vector brindled and assume that E is compressible , let $f:E\to Y$ and $g:E_1\to Y$ be two tubular neighbourhoods of X in Y then there exists .

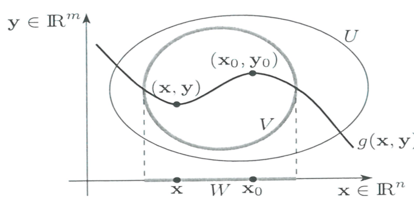
Theorem 3.3.1 (Implicit Function)

Let $m,n\in N$ and let $U\subset R^{n+m}$ be an open set , let $g:U\to R^m$ be C^∞ with $g(x_0,y_0)=0$ for some $x_0\in R^n$, $y_0\in R^m$ with $(x_0,y_0)\in U$. Assume that $\det\left[\frac{\partial g_i}{\partial y_j}(x_0,y_0)\right]_{1\le i,j\le m}\neq 0$ then there exist open sets $V\subset R^{n+m}$ and $W\subset R^n$ with $(x_0,y_0)\in V$ such that , for each $x\in W$ there is a unique $(x\ ,y\ )\in V$ with $g(x\ ,y\ )=0$ if the y above is denoted $f(x_0)=y_0$ and $g\ (x,f(x))=0$ for all $x\in W$ the n-sphere S^n is the n-dimensional surface R^{n+1} given implicitly by equation

$g(x_1, \ldots, x_{n+1}) = x_1^2 + \ldots + x_{n+1}^2 = 0$ in a neighborhood of , for example the northern hemisphere S^n is given explicitly by the equation $x_{n+1} = \sqrt{x_1^2 + \ldots + x_n^2}$ if you think of the set of all 3×3 real matrices as R^9 (because a 3×3 matrix has 9 matrix elements) then .

$$S\,O(3) = \{ 3 \times 3\, real \ \ matrices \ \ R, R^t R = 1, \det R = 1 \}$$

Is a 3-dimensional surface in R^9 , we shall look at it more closely Figurer (7) :



Figurer (7) : 3-dimensional surface in R^9

Example 3.3.2 (A Torus)

The torus T^2 is the two dimensional surface $T^2 = \{(x, y, z) \in R^3, (\sqrt{x^2 + y^2} - 1)^2 + z^2 = 1/4\}$ in R^3 in cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta, z = 0$ the equation of the torus is $(r-1)^2 + z^2 = 1/4$ fix any θ , $say\ \theta_0$. Recall that the set of all points in R^n that have $\theta = \theta_0$ is an open book , it is a hal-plane that starts at the z axis . The intersection of the tours with that half plane is circle of radius 1/2 centered on $r = 1, z = 0$ as φ runs form $0\ to\ 2\pi$, the point $r = 1 + 1/2\cos\varphi$ and $\theta = \theta_0$ runs over that circle. If we now run θ from $0\ to\ 2\pi$ the point $(x, y, z) = ((1 + 1/2\cos\varphi)\cos\theta, (1 + 1/2\sin\varphi)$ Runs over the whole torus . So we may build coordinate patches for T^2 using θ and φ with ranges $(0, 2\pi)$ or $(-\pi, \pi)$ as coordinates)

Definition 3.3.3

(i) A function f from a manifold M to manifold N (it is traditional to omit the atlas from the notation) is said to be C^∞ at $m \in M$ if there exists a chart $\{U, \varphi\}$ for M and chart $\{V, \psi\}$ for N such that $m \in U$, $f(m) \in v$ and $(\psi \circ f \circ \varphi^{-1})$ is C^∞ at $\varphi(m)$.

(ii) Tow manifold M and N are diffeomorphic if there exists a function $f : M \to N$ that is (1-1) and onto with N and f^{-1} on C^∞ everywhere. Then you should think of M and N as the same manifold with m and $f(m)$ being two names for same point , for each $m \in M$.

IV. INEGRATION SMOOTH MANIFOLD

We now onto integration .I shall explicitly define integrals over 0-dimensional .1-dimensional and 2-dimensional regions of a two dimensional manifold and prove a generalization of Stokes theorem . I am restricting to low dimensions purely for pedagogical reason . The same ideas also work for high dimensions . Before getting into the details, here is a little motivational discussion. A curve , i.e a region that can be parameterized by function of real variable, is integral any finite union of , possibly disconnected, curves . We shall call this a 1-chain. We Start off integration of m-forms by considering m-forms R^m , a subset $D \subset R^m$ is called a domain of integration if D is bounded and ∂D has m-dimensional Lrbesgue measure $d\mu = dx_1, \ldots, dx_m$ equal to equal zero . In particular any finite union or intersection of open or closed rectangles is a domain of integration . Any bounded continuous function f on D is integral (i.e) $-\infty \prec \int_D f dx_1, \ldots, dx_m \prec \infty$ since $\Lambda^m(R^m) \cong R$ is a smooth function . For a given (bounded) domain of integration D we define .

$$(11) \qquad \int_D w = \int_D f(x_1, \ldots, x_m) dx_1 \ldots dx_m = \int_D f d\mu = \int_D w_x(e_1 \ldots e_m) d\mu$$

An m-form w is compactly supported if $\mathrm{supp}\,(w) = cl\{x \in R^m : w(x) \neq 0\}$ is a compact set. The set of compactly supported m-form of R^m is denoted by $\Gamma_c^m(R^m)$, and is a linear subspace of $\Gamma_c^m(R^m)$. Similarly for any open set $U \subset R^m$ we can define $w \in \Gamma_c^m(R^m)$. Clearly $\Gamma_c^m(U) \subset \Gamma_c^m(R^m)$, and can be viewed as a linear subspace via zero extension to R^m . For any open set $U \subset R^m$ there exists a domain of integration D such that $D \supset D \supset \sup(w)$. For example let $U, V \in R^m$ be open sets $f : U \to V$ on orientation preserving diffeomorphism, and let $w \in \Gamma_c^m(V)$ then $\int_V w = \int_U f^* w$ if f for the domains D and E .we use coordinates $\{x_i\}$ and $\{y_i\}$ on D and D respectively we start with $w = g(y_1, \ldots, y_m) dy^1 \wedge \ldots \wedge dy^m$. Using the change of variables formula for integrals and the pullback formula , we obtain .

(12)

$$\int_E w = \int_E g(y) dy_1 \ldots dy_m = \int_D (f \circ g)(x) \det(J\,\tilde{f}_x) dx^1 \wedge \ldots \wedge dx^m = \int_D f^* w$$

One has to introduce a-sign in the orientation reversing case .

Theorem 4.1 (Kelvin – Stokes)

$$(13) \qquad \int_D d\alpha = \int_{\partial D} i^* \alpha$$

For every $\alpha \in \Omega^{d-1}(M)$ where $i : \partial D \to M$ denotes the canonical (Moor prosaically , one says that $i^*\alpha$ is the restriction of α to ∂D) the attentive reader should have been worrying both integral above need some orientation to be defined . So we should add that the manifold M is oriented (or at least has a chosen local orientation covering at least D) then the basic ∂D inherits a canonical orientation from that of M , given geometrically by the inner side of D , and analytically by asking that dx_1 (locally) be used to orient the to normal directions to ∂D which will together with only one orientation to ∂D to produce the given orientation of M Figure (8) .

Figure (8) :domains with reasonable singularities

Definition 4.1.1 (0-dimensional Integration)

(i) A 0-form is a function $f : M \to C$.

(ii) A 0-chain is an expression of form $(n_1 P_1 + ... + n_k P_k)$ with $(P_1, ..., P_k)$ distinct points of M and $(n_1, ..., n_k) \in Z$.

(iii) If F is a 0-form and $(n_1 P_1 + ... + n_k P_k)$ is a 0-chain , then we define the integral.

(14) $$\int_{n_1 p_1 + ... + n_k p_k} F = n_1 (P_1) + ... + n_k F(P_k)$$

Definition 4.1.2 (1-dimensional Integration)

(i) A 1-form w is a rule which assigns to each coordinate chart $\{U, \xi = (x, y)\}$ a pair (f, g) of com (f, g) complex valued functions on $\xi(U)$ in a coordinate manner to be defined in $w_{\{U,\xi\}} = f\,dx + g\,dy$ to indicate that w assigns the pair to the chart $\{U, \xi\}$. That w is coordinate invariant means that – If $\{U, \xi\}$ and $\{\tilde{U}, \tilde{\xi}\}$are tow charts with $U \cap \tilde{U} \neq 0$ - If w assigns to $\{U, \xi\}$ the pair of functions (f, g) and assigns to $\{\tilde{U}, \tilde{\xi}\}$ the pair of function $(\tilde{f}, \tilde{g})$. (ii) If the transition function $\{\tilde{\xi}, \tilde{\xi}^{-1}\}$ from $\tilde{\zeta}(U \cap \tilde{U}) \subset R^2$ to $\zeta(U \cap \tilde{U}) \subset R^2$ is $(\tilde{x}(x, y), \tilde{y}(x, y))$ then.

$$f(x, y) = \left(\tilde{f}(\tilde{x}(x, y), \tilde{y}(x, y)) \frac{\partial \tilde{x}}{\partial x}(x, y) \right) + \left(\tilde{g}(\tilde{x}(x, y), \tilde{y}(x, y)) \frac{\partial \tilde{y}}{\partial x}(x, y) \right)$$

$$g(x, y) = \left(\tilde{f}(\tilde{x}(x, y), \tilde{y}(x, y)) \frac{\partial \tilde{x}}{\partial x}(x, y) \right) + \left(\tilde{g}(\tilde{x}(x, y), \tilde{y}(x, y)) \frac{\partial \tilde{y}}{\partial y}(x, y) \right)$$

(iii) If w is a 1-form and $(n_1 C_1 + ... + n_k K_k)$ is a 1-chain then we define integral

(15) $$\int_{n_1 C_1 + ... + n_k C_k} w = n_1 \int_{C_1} w + ... + n_k \int_{C_k} w$$

(iv) Addition of 1-form and multiplication of a 1-form by a function on M are defined as follows , let $\alpha : M \to C$ and let $\{U, \zeta = (x, y)\}$ be a coordinate chart for M . If $w_1|_{\{U,\xi\}} = f_1 dx + g_1 dy$ and $w_2|_{\{U,\xi\}} = f_2 dx + g_2 dy$ then .

(16)

$$w_1 + w_2|_{\{U,\xi\}} = (f_1 + f_2)\,dx + (g_1 + g_2)\,dy$$

$$\alpha w_1|_{\{U,\xi\}} = \left(\alpha \circ \zeta^{-1} f_1\right) dx + \left(\alpha \circ \xi^{-1} g_1\right) dy$$

Definition 4.1.3 (2-dimensional Integrals)

(i) A 2-form Ω is a rule which assigns to each chart $\{U, \xi\}$ a function f on $\xi(U)$ such that $\Omega|_{\{U,\xi\}} = f dx \wedge dy$ is invariant under coordinate transformations . This means that .

(ii) If $\{U, \xi\}$ and $\{\tilde{U}, \tilde{\xi}\}$ are two charts with $U \cap \tilde{U} \neq 0$ If Ω assigns $\{U, \xi\}$the function f and assigns $\{\tilde{U}, \tilde{\xi}\}$ the function $\tilde{f}$ - If the transition function $\xi \circ \zeta^{-1}$ from $\xi(U \cap \tilde{U}) \subset R^2$ to $\tilde{\xi}(U \cap \tilde{U}) \subset R^2$ is $(\tilde{x}(x, y), \tilde{y}(x, y))$ then .

(17)

$$f(x, y) = \tilde{f}(\tilde{x}(x, y), \tilde{y}(x, y))$$

$$= \left[\frac{\partial \tilde{x}}{\partial x}(x, y) + \frac{\partial \tilde{y}}{\partial y}(x, y) - \frac{\partial \tilde{x}}{\partial y}(x, y) \frac{\partial \tilde{y}}{\partial x}(x, y) \right]$$

$Q^2 = \{(x, y) \in R^2, x, y \geq 0, x + y \leq 1\}$ a surface is map $D : Q^2 \to M$ 2-chain is an expression of the from

$(n_1 D_1 + ... + n_k D_k)$ with $(D_1 + ... + D_k)$ surfaces and $(n_1 + ... + n_k)$ surfaces and $(n_1 + ... + n_k) \in Z$.

(iii)Let $\{U, \xi = (x, y)\}$ be a chart and let $\Omega|_{U,\xi} = f(x, y)\,dx \wedge dy$ if $D : Q^2 \to U \subset M$ is a surface with range in U then we define the integral .

$$\int_D \Omega = \iint_{Q^2} f(\xi(D(s,t))) \begin{bmatrix} \frac{\partial}{\partial s} x(D(s,t)) \frac{\partial}{\partial s} y(D(s,t)) \\ - \frac{\partial}{\partial t} x(D(s,t)) \frac{\partial}{\partial s} y(D(s,t)) \end{bmatrix} ds\,dt$$

If D does not have rang in a single chart , split it up into a finite number of pieces, each with range in a single chart. This can always be done , since the range of D is always compact . The answer is independent of chart (s) .

(v) If Ω is a 2-form and $(n_1 D_1 + ... + n_k D_k)$ is a 2-chain , then we define the integral.

(18) $$\int_{n_1 D_1 + ... + n_k D_k} \Omega = n_1 \int_{D_1} \Omega + ... + n_k \int_{D_k} \Omega$$

4.4 Definition (n-dimensional Integrals)

The integrals of n-forms w on M ,we first assume that w is a n-form supported in an orientation compatible coordinate chart $\{\varphi, U, V\}$so that there is a function $f(x^1, ..., x^n)$ supported in U such that $w = f(x^1, ..., x^n)dx^1 \wedge .... \wedge dx^n$ we define $\int_U w = \int_V f(x^1, ..., x^n)dx^1, ..., dx^n$ where the right hand side is the Lebesgue integral on $V \subset R^n$. To integrate a general n-form w on M, we take a locally finite cover $\{U_\alpha\}$ of M that consists of orientation-compatible coordinate charts. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$.Now since each ρ_α is supported in U_α each $\rho_\alpha w$ is supported U_α also .We define

(19) $$\int_M w = \sum_\alpha \int_{U_\alpha} \rho_\alpha w$$

We say that w is integral if the right hand side converges. One need to check that the definition above is independent of choice of orientation compatible coordinate charts , and is independent of choice of partition of unity , so that the integral is well-defined .

Theorem 4.1.4

The expression (6) is independent of choice of U_α and the choice of ρ_α .

Proof :

We first show that

$\int_U w = \int_V f(x^1,\ldots\ldots, x^n)dx^1,\ldots, dx^n$ is well-defined , i.e w is supported in U and if $\{x^i_\alpha\}$ and $\{x^i_\beta\}$ are $w = f_\alpha dx^1_\alpha \wedge \ldots \wedge dx^n_\alpha = f_\beta dx^1_\beta \wedge \ldots. \wedge dx^n_\beta$ then .

$$\int_{V_\alpha} f_\alpha dx^1_\alpha,\ldots., dx^n_\alpha = \int_{V_\beta} f_\beta dx^1_\beta,\ldots, dx^n_\beta$$

then $dx^1_\beta,\ldots., dx^n_\beta = \det(d\varphi_{\alpha\beta})dx^1_\beta,\ldots, dx^n_\beta$ implies that

$f_\alpha = \det(d\varphi_{\alpha\beta})\, f_\beta$ on the other hand side, the change of variable formula in R^n reads

(20)

$$\int_{V_\beta} f\, dx^1_\beta,\ldots., dx^n_\beta = \int_\alpha f \det(d\varphi_{\alpha\beta})dx^1_\alpha,\ldots, dx^n_\alpha$$

So that desired formula follows form the fact $\det(d\varphi_{\alpha\beta}) \succ 0$ since U_α and U_β are orientation compatible . Well-defined ,we suppose U_α and U_β are two locally finite cover of M consisting of orientation-compatible charts, and ρ_α and ρ_β are partitions of unity subordinate to U_α and U_β respectively . We consider a new cover $U_\beta \cap U_\alpha$ with new partition of unity ρ_α, ρ_β it is enough to prove $\sum_\alpha \rho_\alpha w = \int_{U_\beta} (\sum_\beta \rho_\beta)\rho_\alpha w = \sum_\beta \int_{U_0 \cap U_\beta} \rho_\beta . \rho_\alpha w$ obviously the integral defined above is linear $\int_M (aw + b\eta) = a\int_M w + b\int_M \eta$. Now M, N are both oriented manifolds, with volume forms $\eta_1 \eta_2$ respectively .

Definition 4.1.5

A smooth map $f : M \to N$ is said to be orientation-preserving if $f^*\eta_2$ is a volume form on M that defines the same orientation as η_1 does .

Theorem 4.1.6

Let M be compact manifold and α, β $\int_M f^* w = \int_N w$.

Proof :

It is enough to prove this in local charts tow volume forms then there exist a in which case this is merely change of variable formula in R^n .

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The basic notions on differential geometry knowledge of calculus , Encluding $\{E^n\}$ the geometric formulation f of the notion of the differential and the inverse function f^{-1} theorem ∂M . A certain familiarity with the elements of the differential Geometry of surfaces with the basic definition of differentiable manifolds , starting with properties of covering spaces and of the fundamental group and its relation to covering spaces

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