# On the Analysis of the Fibonacci Numbers: Critique to Kepler

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***ABSTRACT:*** *What has the theoretical breeding of rabbits got to do with geometry? The connection of the Fibonacci sequence to the Golden Mean was made by Johannes Kepler in the 17th century CE, which sequence was introduced by Leonardo Pissano, better known as Fibonacci, in the Liber Abaci in the 13th century CE. We judiciously ignore the historical background of the Fibonacci sequence, and proceed to analyse it mindful of the fact that the so-called Fibonacci numbers were known well before 1202. Herein the author, after presenting the 'Kepler Paradox' proposes and pursues a more precise analysis of the Fibonacci sequence with regard the Golden Mean using an existing geometric tool, and henceforth shows how similar sequences can be assembled, among which is the Lucas sequence.*

**KEYWORDS***: Fibonacci, Kepler, Kepler Paradox, Lucas, Cassini, Golden Mean, Divine Proportion, Phi, Turing Sunflower Project, Phyllotaxis, Fibonacci Group Theory, Systems of Fibonacci Sequences, Fibonacci Family of Spirals.*

## I. Introduction

Given the sequence of numbers:

$F_n$= 1,2,3,5,8,13 …

two adjacent numbers are found to be in Divine Proportion to each other, according to Kepler, see [1], that is, 'as 13 is to 8, so is 8 to 5, approximately'. In essence, Kepler's analysis states that the ratio of adjacent terms in the Fibonacci sequence approaches phi (=[(1 + √5)/2)]) as the sequence tends to infinity. We herein call this analysis the golden line analogy, or simply, linear analogy. This might be unfair on Kepler's part, for he might have been referring also to the golden rectangle, but for purposes of this article, we shall term his analysis as such, and the justification for this can be inferred from the subsequent presentation.

The motivation to revisit the work of Kepler springs firstly from an observation that any sequence created from two arbitrary terms and employing the same recurrence formula as that of the Fibonacci sequence will have the ratio of adjacent terms approach phi as the sequence goes into infinity. We duly say that this is not a property unique to the Fibonacci sequence. This approximation to phi is rather an oscillation about it. This oscillation phenomenon, coupled with the fact that the series will only start to *reasonably* approximate phi at an advanced stage, is sufficient pretext for an investigation. What then makes the Fibonacci numbers special, i.e. observable in nature, e.g. in pine cones, sunflower seeds arrangements, flower petals, e.t.c, as reported in literature such as [2]? Secondly, the fact that Kepler's analysis does not yield the exact value of phi, and gives different values at different stages in the sequence, naturally warrants an investigation into either the legitimacy of linking the Fibonacci numbers to the Divine Proportion, or the adequacy and suitability of the method of analysis, or both such legitimacy of concept and suitability of method, so to speak.

Whereas it is noted that the breeding of rabbits has no direct relationship with geometry, we proceed to geometrically analyse the sequence by ignoring the rabbit problem and rather bring to the knowledge of the critic the fact that the numbers were used (therefore known) elsewhere even centuries before Fibonacci, e.g. in Egypt around 3000 BCE [3].

## II. The Kepler Paradox

The problem with Kepler's analysis is twofold: arithmetical and geometrical. The latter is more pertinent than the former, since it is the one that connects the sequence to the Divine Proportion. As a result, we begin by examining the latter.

### A. *Geometric*

We conceive Kepler's analysis as assuming that the $n^{th}$ term in the Fibonacci sequence is the Golden Cut of a line whose length is equal to the $(n + 1)^{th}$ term; in principle.

The geometrical problem is presented thus: since the Divine Proportion only begins to be approximated at an advanced stage in the sequence, it cannot have been used in the construction of the sequence from the beginning. This seems to be a mechanical conception of the matter at hand, but it suffices to note that seeking the Divine Proportion from the sequence implies that it was used (consciously or subconsciously) in the construction of the sequence. Here it is implied that if the Divine Proportion cannot be found from the beginning of the sequence, then its emergence higher up in the sequence becomes a matter of coincidence, more so because any series created with the Fibonacci recurrence formula exhibit the same property. One would therefore be forgiven for concluding that the Fibonacci sequence has nothing to do with the Divine Proportion and only a cult of mystics revering Fibonacci numbers from elsewhere can *believe* in that. Kepler's linear analogy also falls short of the overall sequence mechanics: it cannot be used in the construction of the sequence and will have to invoke arithmetic, a different concept altogether. Linear analogy therefore is limited to the analysis of a given set of numbers, and cannot be used for sequence construction. This leads us to the other facet of the Kepler problem.

### B. *Arithmetic*

Arithmetically the sequence is assembled from this recurrence relation:

$$F_n = F_{n-1} + F_{n-2}. \quad (1.1)$$

Firstly let's assume phi is rational.

**Rule 1**: $F_n$ is the length of a 'golden' line if and only if the ratios $F_n/ F_{n-1}$ and $F_{n-1}/ F_{n-2}$ equal phi.

In other words, only when $F_{n-1}$ is the Golden Cut of a line $F_n$ units long, then shall we find the Divine Proportion. Let us consider the next term. It is given by:

$$F_{n+1} = F_n + F_{n-1} \quad (1.2)$$

$$= 2F_{n-1} + F_{n-2}. \quad (1.3)$$

It follows from relation (1.3) that

$$(F_{n+1})/(2F_{n-1})$$

will equal phi if $F_n$ is the Golden Cut of a line $F_{n+1}$ units long. It can be deduced that:

$$F_{n+1}/ F_n = 1 + [F_{n-1}/(F_{n-1} + F_{n-2})]. \quad (1.4)$$

Consistency with the definition requires that the value of

$$F_{n-1}/F_{n-2}$$

be equal to the value obtained from equation (1.4). Visual inspection of the expression

$$F_{n-1}/F_{n-2}$$

and equation (1.4) shows that these ratios will never be equal, i.e. consecutive ratios will never equal; the reason for the oscillation phenomenon. This holds for any sequence created from relation (1.1), and further, this ratio can never equal phi. This is the arithmetic component of the Kepler paradox. Phi is not a rational number, i.e. cannot be expressed as a ratio of two integers, and when faced with the fact that any sequence of numbers constructed from the recurrence relation (1.1) will have the ratio of any two adjacent terms tend to approximate phi as the sequence goes into infinity, Kepler's analysis does not distinguish the Fibonacci sequence from any such other two-term recurrence. Here it is meant that this particular arithmetic property is not unique to the Fibonacci sequence, and the sequence will therefore remain mathematically undistinguished from other sequences assembled with the same recurrence formula, the main point made in [4]. This will therefore render the whole process a matter of coincidence, and one would be 'lucky' to stumble into a meaningful series (observable in nature) like the Lucas sequence which was constructed from interchanging the position of the first two terms of the Fibonacci sequence, e.g. see [5], a purely arithmetical exercise which has nothing to do with geometry, so to speak. We cannot honestly dispute the existence of the Fibonacci numbers in nature, e.g. in phyllotaxis, as we cannot also honestly use Kepler's analysis to convincingly attribute the phenomenon to phi, at least from a mathematical point of view..

## III. PROPOSED METHOD OF ANALYZING THE FIBONACCI NUMBERS

Having discussed the shortcomings of Kepler's analysis which is purely arithmetical, and having noted that it is limited to only analyzing existing sequences (now also including the Lucas sequence), and falls short of overall sequence mechanics, let us use an existing geometric tool to analyze the sequence.

It is assumed that the reader is conversant with golden rectangle construction, and the reader shall see literature like [6] for a treatment of the subject. In the Fibonacci sequence, $F_{n+1}$ is the length of the golden rectangle created from a square of sides $F_n$. This can be represented mathematically in the form:

$$F_{n+1} = 0.5F_n + \sqrt{[(0.5F_n)^2 + F_n^2]} \text{ ; for } n \geq 1 \quad (1.5)$$

Rounding off to the nearest whole number is needed. For example, when $F_n$ is 5, $F_{n+1}$ from equation (1.5) is given as 8.09… and rounded off to 8. This 8 is then used to 'geometrically extrapolate' the next term, which is given as 12.94… and rounded off to 13. This procedure can be performed at any point in the series. For example, 233 will give 377.00…, also rounded off to 377 for the purposes of computing the next member and representation by rational numbers. The immediate question that arises is why employing equation (1.5) when the recurrence relation (1.1) can be used? This is the main purpose of this paper, to communicate the importance of equation (1.5), which is derived from geometry. Firstly, it guards against the random selection of numbers, i.e. the first two terms of a sequence. It is meant that given the number 1 for example, one must not apply a rule of thumb to know that the second term is 2, but shall use equation (1.5). This will prove an invaluable tool in the construction of other sequences as meaningful as the Fibonacci sequence. Secondly, through this equation only are we able to appreciate The Divine Proportion in a sequence. Phi is computed before rounding off. For example, from a square of sides 2 units, using equation (1.5) we find a golden rectangle of length 3.236067977. Before rounding this off to 3, we divide it by 2 to get 1.618033989. This procedure can be repeated for every member in the Fibonacci sequence and the Divine Proportion be found to remain constant throughout the series. The following rules may guide the assemblage of Fibonacci sequences:

a) Equation (1.5) gives the $(n+1)^{th}$ term; rounding off to the nearest whole number is warranted,
b) Recurrence relation (1.1) may be used after the second term in the sequence for quick assemblage,
c) The lowest number missing in a series begins the next series,
d) The Divine Proportion is found from dividing $F_{n+1}$ as given by equation (1.5) (before rounding off) by $F_n$, and
e) One number is therefore a member of one sequence only.

As a simple proof of point (d), note that relation (1.5) is reducible to

$$F_{n+1} = [(1 + \sqrt{5})/2]F_n$$
$$=\varphi F_n$$

It follows therefore that $\varphi = F_{n+1}/F_n$.
But $\varphi$ is irrational, so $F_{n+1}/F_n$ is irrational. Since $F_n$ is rational, we say $F_{n+1}$ is irrational. So only before rounding off (*rationalizing*) $F_{n+1}$ shall we divide it by $F_n$ to get $\varphi$. This also becomes an extended proof to point (e) because $F_n$ will give a unique $F_{n+1}$. The geometrical interpretation is that there exists only one golden cut of a line and in the definition adopted for purposes of this article, one square only gives rise to one golden rectangle. This means that sequences created according to the Divine Proportion have unique terms, i.e. one number only participates in one sequence.

These rules are especially important as they lead us to some notable results. The Fibonacci sequence is given as:

1,2,3,5,8,13 …

The lowest missing number is 4. Now starting from 4 and using equation (1.5) we get 6. Thereafter we can choose to continue employing equation (1.5) or use (1.1) for quick assemblage of the sequence. We thus get the series:

4,6,10,16,26,42,68, …

Interestingly, the Turing Sunflower Project e.g. see [7] reports the existence of ‘double Fibonacci numbers’ in sunflower spirals.

The next lowest missing number is 7. We therefore construct the series:

7,11,18,29,47 …

This series is identifiable with the Lucas series. For this reason we call it the modLucas series. The next lowest missing number is 9. We construct the series:

9,15,24,39, …

This procedure is repeated to infinity, to give an infinite number of sequences in the Fibonacci family of spirals.

## IV. INTRODUCING SYSTEMS OF FIBONACCI SEQUENCES AND FIBONACCI GROUP THEORY

From the previous section, it can be deduced that, for example, the series:

4,6,10,16,26,42, …

can be written as

2{2,3,5,8,13,21, …}.

And also the series:

20,32,52,84 …

can be reduced to

4{5,8,13,21 …}.

**Lemma 1**: *Any sequence on which division by an integer n reduces it to a sequence beginning at some point in the 1202 Fibonacci sequence is in group* $F_A$.

We can further subdivide group $F_A$ into $F_{A1}$ and $F_{A2}$ groups.

**Lemma 2**: *From the* $n^{th}$ *term in the Fibonacci sequence, multiplication can be performed by an integer n to yield a sequence in the* $F_{A1}$ *group.*

$F_{A1}$ is composed of these four sequences only:

1,2,3,5,8,13,21 …

4,6,10,16,26,42, …

9,15,24,39, …

20,32,52,84, …

Note that for instance from the fourth term in the Fibonacci sequence, multiplication by 4 gives the sequence:

20,32,52, …

It has to be noted that multiplication by 5 from the fifth term produces

40,65,105,…

This is deemed 'counterfeit' since it comes from

25,40,65,…

We therefore restrict the $F_{A1}$ group to the four sequences given above, because of the 'discontinuity' that is introduced by the violation of Lemma 2 when n = 5. Series such as

25,40,65, …

on which division by an integer n will also yield a series starting at some point in

1,2,3,5,8 …

are in the $F_{A2}$ group.

We now move to our next group: $F_B$. This group is composed of those series resembling the Lucas sequence. Subdivisions in this group are given thus:

**Lemma 3**: *Any sequence on which division by an integer n reduces it to some sequence beginning at some point in the modLucas sequence is in group $F_B$.*

Again we further subdivide this group into $F_{B1}$ and $F_{B2}$.

**Lemma 4**: *From the $n^{th}$ term in the modLucas sequence, multiplication can be performed by an integer n to yield a sequence in the $F_{B1}$ group.*

a) $F_{B1}$: this group is composed of three sequences, viz:

7,11,18,29, …
22,36,58,94, …

54,87,141,228, …

Here, the discontinuity is at n = 4; i.e., when n=4, we get the sequence

116,188,304, …

This is another 'counterfeit' since it comes from

72,116,188,304, …

b) $F_{B2}$: all series on which division by an integer n can be performed to yield a sequence beginning at some point in
7,11,18,29, …

but violate Lemma 3. This group is composed of sequences like

72,116,188,304, …

which can be written as

4{18,29,47 …}.

Let us proceed to the next and last group: the $F_C$ group. This consists of such series as

12,19,31,50 …

and

35,57,92, …

e.t.c upon which factorization by an integer cannot be performed for reduction to either the 1202 Fibonacci sequence or the modLucas sequence. It is not here implied that the sequences in the $F_C$ group do not have their 'copies'. For example, the sequence:

93,150,243,393, …

can be written as

3{31,50,81,131, …}

and can be found to come from this sequence:

12,19,31,50,81,131, …

Subdivisions and trends in the $F_C$ group are very interesting (even complicated) but they fall outside the scope of this paper. We restrict ourselves to three groups only because we deem the 1202 Fibonacci sequence and the modLucas sequence to be the main sequences, when we consider parent numbers from 1 to 9. We here define a parent number as one which starts a sequence, e.g. 1, 4, 7, 9, 12 e.t.c. The geometrical interpretation is that no square of rational sides (in this case 'rational'means integer) can give rise to a golden rectangle whose length can be approximated by a parent number. To illustrate, take the number 7 for example. None of the integers 1 to 6 can be used as the size of a square that will give rise to a golden rectangle whose length can be approximated by the number 7. The number 4 therefore can only precede 6 in any sequence created using the principle of the Divine Proportion presented herein, itself being preceded by none.

Before we leave this section, we shall state the following Lemma.

**Lemma 5:** *The difference between any two successive sequences is either the 1202 Fibonacci sequence or the Lucas series.*

For example,

9,15,24,39,…
-[7,11,18,29,…]
=2,4,6,10, …
=2{1,2,3,5,…}

and

4,6,10,16,…
-[1,2,3,5,8,…]
=3,4,7,11,18,…

Note however that in the sequence

3,4,7,11,…

The first three terms are not in Divine Proportion to each other. In further group analysis beyond the scope of this presentation, we take the point where the difference between two successive sequence is given as

3,4,7,11,…

as the point of discontinuity.

## V. CASSINI FORMS FOR SYSTEMS OF FIBONACCI SEQUENCES

We shall call the above groups systems of Fibonacci sequences. The series

1,2,3,5, …

we shall call it the group sequence for the $F_A$ group, and the series

7,11,18, ..

we shall call it the group sequence for the $F_B$ group. For example, the Cassini rule for the series

4,6,10,16, …

is given as

$$F_{n+1}F_{n-1} – F_n^2 = 4(-1)^n;\ \text{for } n \geq 2. \quad (1.6)$$

For the modLucas sequence, i.e.:

7,11,18,29, …

the Cassini rule is

$$L_{n+1}L_{n-1} – L_n^2 = 5(-1)^n;\ \text{for } n \geq 2. \quad (1.7)$$

For the purposes of group analysis, we present the general Cassini formula as:

$$F_{n+1}F_{n-1} – F_n^2 = ab(-1)^n;\text{for } n \geq 2 \quad (1.8)$$

where

*a* is the group Cassini constant and

*b* is the square of the factor reducing or scaling down a particular sequence to the group sequence.

The group Cassini constant is the (positive) Cassini value for the group sequence. The (positive) Cassini value for

1,2,3,5 …

is 1 and for the series

7,11,18, …

is 5. It follows that for $F_A$ sequences the general Cassini formula would be given as:

$$F_{n+1}F_{n-1} – F_n^2 = b(-1)^{n-1};\ \text{for } n \geq 2 \quad (1.9)$$

and for $F_B$ series it is given as:

$$L_{n+1}L_{n-1} – L_n^2 = 5b(-1)^n;\ \text{for } n \geq 2 \quad (1.10)$$

The scaling factor for the sequence

20,32,52,84, …

is 4; i.e. 4 can be factored out from this sequence to yield a sequence starting at some point in the 1202 Fibonacci series. This means b is $4^2 = 16$. The group Cassini constant is 1 and therefore ab = 16. So for this series the Cassini rule is:

$$F_{n+1}F_{n-1} - F_n^2 = 16(-1)^n; \text{ for } n \geq 2 \quad (1.11)$$

Similar results can be obtained for members in the group, and a similar procedure can be followed in the $F_B$ group, being guided by equation (1.10). The same can be done in the many subgroups in the $F_C$ group, which exercise is beyond the scope of this presentation.

It suffices at this point to note that the 'Cassini power' alternates between *n-1* and *n* in the $F_A$ and $F_B$ groups and at the points of discontinuity described in group formation, there is no change of power. This can be illustrated simply by taking for example, $F_A$ group. The point of discontinuity is seen when we multiply by 5 from the 5th term in the Fibonacci sequence. Now for the first four series which also make up $F_{A1}$ group, the Cassini values from the first three terms are: -1,4,-9,16. For the next series which begins the $F_{A2}$ group, the first Cassini value is 25 (positive). This violates the pattern -+-+ that we see in -1,4,-9,16. This shows there is a double alternation, i.e. within the sequence and within the group. The group alternation is discontinued and started afresh exactly at the point of discontinuity as defined earlier.

## VI. CONCLUSION AND RECOMMENDATION

The method of analysis presented herein provides self-proof as evidenced by, for example, the reduction of certain sequences to the well-known Fibonacci and Lucas series. This is taken as proof of the adequacy of the method of analysis and/or assemblage of the series. Physical proof can be found from observations in nature. As a simple example, the very existence of say a flower with 14 petals is proof that the number 14 has also been employed in Creation, albeit it is not found in the 1202 Fibonacci sequence. Other interesting (even controversial) examples can be found in '*applications*' in some works in fields like music. In [8], J.F. Putz analysed Mozart's piano sonatas. Sonata 1 in C Major was divided into 38 (Exposition) and 62 (Development and Recapitulation) measures. This is a novel 'application' of the sequence

62,100,162, …

The Golden Ratio is not therefore on 38 and 62, though from this sequence:

38,61,99,160 …

we are almost there. Instead, the Divine Proportion in this particular example is on 62 and 100. As a sidenote, from the discussion in this paper, the sequence

62,100,162, …

can be conceived to spring from

12,19,31,50, …

since we can write it as

2{31,50,81, …}

In this case we would view the sonata as 50 measures divided into 19 and 31, with a scale factor of 2; but we do not here intend to reverse-engineer Mozart's work. Whether Mozart was conscious (or subconscious) of the Divine Proportion in it is immaterial mathematically since we are interested in the mathematical accuracy of the analysis. It is meant that for the mathematical study of the Golden Section this and other works of composers like Dufay, see [9], who implemented the sequence

271,438,709, …

with no 'apparent' knowledge of the ratio are good examples, and the student can be asked to assume that the composer consciously applied the golden section, and this assumption (or lack thereof) will not affect the mathematics in any way. It is interesting however to note that 7 out of the 29 sonatas presented in [8] strictly reveal the Divine Proportion, if we follow the principle presented herein.

This example from Mozart's music illustrates the main point being driven home in this paper. Pursuant to the definition adopted herein, the number 100 cannot be said to be divisible into two segments, that is to say we restrict the analysis to two numbers because we deem the larger one to be the length of a golden rectangle and the smaller one the size of the square giving rise to such rectangle. In this particular example, we need to note that 62 is a parent number, and therefore is preceded by no other number in the sequence 62,100,162,... This means in Mozart's sonata, 38 becomes correct by default, but is not in Divine Proportion with 62. This concept will ensure departure from linear analogy which is easily confused with arithmetic, and hinders construction of other sequences according to the Divine Proportion.

It is recommended that the concept of the Divine Proportion be shifted from a linear or *'one-dimensional'* to a rectangular or *'two-dimensional'* conception with regard to Fibonacci numbers. One-dimensional analysis relies on arithmetic for sequence construction, but two-dimensional analysis is purely geometrical and does not rely on arithmetic. The fact that a geometrically created sequence will arithmetically become a two-term recurrence must be viewed from an aesthetic rather than a mechanical viewpoint, and must be interpreted carefully geometrically. In other words, the fact that a geometrically sound sequence becomes a two-term recurrence does not mean any two-term recurrence is geometrically sound. This will distinguish the Fibonacci sequences from the generalized Fibonacci sequences. This important point will therefore guide the assemblage of sequences according to the Divine Proportion, at least according to this presentation.

We finally point out that Kepler's analysis was purely arithmetical and non-geometric. This means his analysis holds arithmetically but is not geometrically sound. Believing that we have presented a geometric analysis of the Fibonacci sequence, we here conclude that all numbers can be found in the Universe. It is not prudent to marvel at a flower with 13 petals and find nothing special in one with 12 petals, for example. In terms of the Fibonacci sequences, the fact that the number 12 is a parent number, i.e. starts its own series, might even make it more special than the number 13 which is not a parent number. The same can also be said of the number 7 or the number 20, but this is a different subject altogether. Many avenues of research can therefore be pursued in light of the proposed method of creating the sequences as presented herein, including Botany, Finance, Religion, and Astronomy. Fibonacci Group Theory introduced in this paper can also be applied in the study of The Platonic Solids and can find 'extensive' applications in Computer Science, to name a few.

## ACKNOWLEDGMENT

*I said, 'Age should speak, and multitude of years should teach wisdom'. But there is a spirit in man, and the breath of the Almighty gives him understanding. – Job 32:7-8.*

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## ADDENDUM: THE RABBIT PROBLEM

We wish to address the question why all two term recurrences created from general formula:

$$C_{n+2} = C_{n+1} + C_n;\ n \geq 1 \qquad \text{(A.1)}$$
$$C_1 = a;\ C_2 = b$$

have the ratio of successive terms tend to approximate phi as the sequence tends into infinity.

Having arbitrary a and b, we assemble the sequence:

$$a,b,a+b,a+2b,2a+3b,3a+5b,5a+8b,... \qquad \text{(A.2)}$$

The two conditions a=b=1 and a=0;b=1 yields the sequence:

1,1,2,3,5,8,13,...

This sequence, being the solution to the rabbit problem presented and solved in the Liber Abaci in 1202 CE by Leonardo Pissano, we add to the extensive work by Scott and Marketos [10] another conjecture, namely that it is probable that Fibonacci arrived at the rabbit problem (it solution in essence) from purely algebraic means, i.e. he needed not have any knowledge in geometry or biological breeding models, only an experiment in the algebraic laboratory sufficed. We find it ridiculous to claim that he followed the above procedure verbatim, on the contrary we say that he was not unaware of the recurrence, formulated/conceived in whatever way. The point being made here is that it was possible for him to assemble the sequence from purely algebraic thought. This is left to interested researchers for further handling.

Now let's take a = 1 and b =5 (note that a does not necessarily need to be lower than b). From relation (A.1) we assemble:

1,5,6,11,17,28,...

From the relationship (A.2) we write this as:

1,5,1+5,1+2(5),2(1)+3(5),3(1)+5(5),...

In terms of the Divine Proportion according to the concept introduced in this article, this sequence is incorrect, albeit the ratio of successive terms will approach phi as the sequence goes into infinity. It shall be noted that as the sequence goes higher, the coefficients of a and b continue to increase while a and b remain constant. As the sequence tends to infinity, a and b will become sufficiently small as compared to their coefficients such that in the ratio of any two successive terms, a and b can be equated to unity with negligible margin of error. We illustrate this by using the 39th and 40th terms in 7,11,18,29,...

$$\frac{L40}{L39} = \frac{969323029}{599074578} = \frac{39088169(7) + 63245986(11)}{24157817(7) + 39088169(11)} \approx \varphi.$$

Now 7 and 11 are too small as compared to their coefficients. We can equate a=b=1 and write

$$\frac{L40}{L39} = \frac{39088169(1) + 63245986(1)}{24157817(1) + 39088169(1)} \approx \varphi.$$

Note that this value equals $F_{40}/F_{39}$ in

1,1,2,3,5,….

i.e. $\frac{102334155}{63245986}$.

We therefore say that as the coefficients of a and b tend to infinity, a and b tend to unity in the ratio of any two adjacent terms. This means that the phi revealed in any two term recurrence when subjected to Kepler's analysis is due to the

1,1,2,3,5,..

sequence (the DNA of any two term recurrence) which 'dominates' as the sequence goes higher. We mean that it is not possible to seperate any two term recurrence from

1,1,2,3,5,..

We hold that since the terms in this sequence are also terms of a sequence created according to the Divine Proportion, as the sequence goes higher the error of representing phi by rational numbers becomes too small such that it will start to be approximated, though never exactly since phi is irrational.

Finally, we point out that for algebraic purposes, we write the Fibonacci sequence as

1,1,2,3,5,…

and for geometric purposes we write it as

1,2,3,5,8,…

In light of the analysis carried out in this paper, 1 cannot follow another 1 geometrically, otherwise a condition will never be met to change from 1 to 2. We mean that a square of 1 unit will give a golden rectangle with a side equal to phi and this is rounded off to 2 for computation of the next rectangle. Since Fibonacci represented his sequence as

1,1,2,3,5,…

which is geometrically wrong at least according to this article, we insist on the algebraic origins of the rabbit problem.