On The Properties of Fibonacci-Like Sequence

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Abstract - *The Fibonacci sequences are well known examples of second order recurrences. In this paper, we introduce and study Fibonacci-Like sequence that is defined by the recurrence relation as* $T_n = T_{n-1} + T_{n-2}$, $n \geq 2$, $T_0 = m, T_1 = m$, *where m being a fixed positive integer. In this paper we present identities of Fibonacci-Like sequence in addition to this we shall define Binet's formula and generating function of Fibonacci-Like sequence and almost all of the identities are proved by Binet's formula.*

Keywords: *Fibonacci sequence, Fibonacci-Like sequence, Binet's formula.*

1. Introduction

The Fibonacci sequences are well known examples of second order recurrences. Fibonacci sequence is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, $F_0 = 0$, $F_1 = 1$. Most of the authors defined Fibonacci pattern based sequences in many ways which are known as Fibonacci-Like sequences. Some authors have maintained the recurrence relation and changed the first two terms of the sequence. While others have maintained the first two terms of the sequence and changed the recurrence relation little bitty. As illustrated in the tome by Koshy [6], the Fibonacci sequence is a source of many nice and interesting identities in all of mathematics.

The sequence of Fibonacci numbers [6] is a sequence of numbers starting with integer 0 and 1, where each next term of the sequence weighed as the sum of the previous two. i.e.,

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2,\ F_0 = 0,\ F_1 = 1 \tag{1.1}$$

The Binet's formula for Fibonacci sequence is given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} \tag{1.2}$$

Where $\alpha = \frac{1+\sqrt{5}}{2} \approx 1.618$ and

$$\beta = \frac{1-\sqrt{5}}{2} \approx -0.618$$

Horadam in [4] and Jaiswal in [5] have generalized the Fibonacci sequence by preserving the recurrence and altering the first two terms of the sequence.

Fibonacci-Like sequence by Singh et.al in [8] is defined by the recurrence relation

$$S_n = S_{n-1} + S_{n-2},\ n \geq 2,\ S_0 = 2,\ S_1 = 2 \tag{1.3}$$

The Fibonacci-Like sequences are also defined in [1, 3].

The main motive of this paper, to generalize the Fibonacci sequence to obtain a sequence which is called Fibonacci-Like sequence and to present some basic properties of Fibonacci-Like sequence which is defined by

$$T_n = T_{n-1} + T_{n-2},\ n \geq 2,\ T_0 = m,\ T_1 = m \tag{1.4}$$

The few terms of the sequence T_n are

m, m, 2m, 3m, 5m, 8m,13m, … where m being a fixed positive integer.

2. Binet's formula of Fibonacci-Like Sequence

The recurrence relation (1.4) has the characteristic equation $x^2 - x - 1$ which produces two roots as

$$\alpha = \frac{1+\sqrt{5}}{2} \approx 1.618 \text{ and}$$

$$\beta = \frac{1-\sqrt{5}}{2} \approx -0.618$$

Also

$$\alpha + \beta = 1,\ \alpha\beta = -1,$$
$$\alpha^2 - 1 = \alpha \ and\ \beta^2 - 1 = \beta,$$
$$1+\alpha^2 = \sqrt{5}\alpha \ and\ 1+\beta^2 = -\sqrt{5}\beta$$
$$\alpha^4 - 1 = \sqrt{5}\alpha^2 \ and\ \beta^4 - 1 = -\sqrt{5}\beta^2$$

Since $T_n = mF_{n+1}$ Then

The Binet's formula of Fibonacci-Like Sequence is given by

$$T_n = m\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \frac{m}{\alpha - \beta}\left(\alpha^{n+1} - \beta^{n+1}\right)$$

And the generating functions of the Fibonacci-Like sequence is given by

$$\sum_{n=0}^{\infty} T_n x^n = \frac{m}{1 - x - x^2} \text{ and}$$

$$\sum_{n=0}^{\infty} \frac{T_n}{n!} x^n = \frac{m}{\alpha - \beta}\left(\alpha e^{\alpha x} - \beta e^{\beta x}\right)$$

3. Identities for Fibonacci-Like Sequence.

Fibonacci-Like sequence has many $\{T_n\}$ captivating identities [2, 7, 8, 9, 10] here we shall prove almost all of the identities by Binet's formula instead of induction or by any other method.

Sums of Fibonacci –Like terms:

Theorem 3.1: Sum of first n terms of the of the Fibonacci-Like sequence is defined by

$$T_1 + T_2 + \ldots + T_n = \sum_{k=1}^{n} T_k = T_{n+2} - 2m \tag{3.1}$$

Proof. By the Binet's formula of Fibonacci-Like sequence, we have

$$\sum_{k=1}^{n} T_k = \frac{m}{\alpha - \beta}\begin{pmatrix} \alpha^2 - \beta^2 + \alpha^3 - \beta^3 \\ + \ldots + \alpha^{n+1} - \beta^{n+1} \end{pmatrix}$$

$$\sum_{k=1}^{n} T_k = \frac{m}{\alpha - \beta}\begin{bmatrix} \alpha^2 + \alpha^3 + \ldots + \alpha^{n+1} \\ -\left(\beta^2 + \beta^3 + \ldots + \beta^{n+1}\right) \end{bmatrix}$$

$$\sum_{k=1}^{n} T_k = \frac{m}{\alpha - \beta}\left(\frac{\alpha^{n+2} - \alpha^2}{\alpha - 1} - \frac{\beta^{n+2} - \beta^2}{\beta - 1}\right)$$

$$\sum_{k=1}^{n} T_k = \frac{m}{\alpha - \beta}\left(\frac{\alpha^{n+2} - \alpha^2}{-\beta} - \frac{\beta^{n+2} - \beta^2}{-\alpha}\right)$$

$$\sum_{k=1}^{n} T_k = \frac{m}{\alpha - \beta}\left(\alpha^{n+3} - \alpha^3 - \beta^{n+3} + \alpha^3\right)$$

$$\sum_{k=1}^{n} T_k = m\left(\frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta}\right) - m\left(\frac{\alpha^3 - \beta^3}{\alpha - \beta}\right)$$

$$\sum_{k=1}^{n} T_k = T_{n+2} - T_2 = T_{n+2} - 2m$$

Theorem 3.2: Sum of first n terms with odd indices is defined by

$$T_1 + T_3 + T_5 + \ldots + T_{2n-1} = \sum_{k=1}^{n} T_{2k-1} = T_{2n} - m \tag{3.2}$$

Proof.

$$\sum_{k=1}^{n} T_{2k-1} = \frac{m}{\alpha - \beta}\begin{pmatrix} \alpha^2 - \beta^2 + \alpha^4 - \beta^4 \\ + \ldots + \alpha^{2n} - \beta^{2n} \end{pmatrix}$$

$$\sum_{k=1}^{n} T_{2k-1} = \frac{m}{\alpha - \beta}\begin{bmatrix} \alpha^2 + \alpha^4 + \ldots + \alpha^{2n} \\ -\left(\beta^2 + \beta^4 + \ldots + \beta^{2n}\right) \end{bmatrix}$$

$$\sum_{k=1}^{n} T_{2k-1} = \frac{m}{\alpha-\beta}\left(\frac{\alpha^{2n+2}-\alpha^2}{\alpha^2-1} - \frac{\beta^{2n+2}-\beta^2}{\beta^2-1}\right)$$

$$\sum_{k=1}^{n} T_{2k-1} = m\left(\frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta}\right) - m\left(\frac{\alpha-\beta}{\alpha-\beta}\right)$$

$$\sum_{k=1}^{n} T_{2k-1} = T_{2n} - T_0 = T_{2n} - m$$

Theorem 3.3: Sum of first n terms with even indices is defined by

$$T_2 + T_4 + T_6 + \ldots + T_{2n} = \sum_{k=1}^{n} T_{2k} = T_{2n+1} - m \tag{3.3}$$

Proof.

$$\sum_{k=1}^{n} T_{2k} = \frac{m}{\alpha-\beta}\begin{pmatrix}\alpha^3-\beta^3+\alpha^5-\beta^5 \\ +\ldots+\alpha^{2n+1}-\beta^{2n+1}\end{pmatrix}$$

$$\sum_{k=1}^{n} T_{2k} = \frac{m}{\alpha-\beta}\begin{bmatrix}\alpha^3+\alpha^5+\ldots+\alpha^{2n+1} \\ -\left(\beta^3+\beta^5+\ldots+\beta^{2n+1}\right)\end{bmatrix}$$

$$\sum_{k=1}^{n} T_{2k} = \frac{m}{\alpha-\beta}\left(\frac{\alpha^{2n+3}-\alpha^3}{\alpha^2-1} - \frac{\beta^{2n+3}-\beta^3}{\beta^2-1}\right)$$

$$\sum_{k=1}^{n} T_{2k} = T_{2n+1} - T_1$$

$$\sum_{k=1}^{n} T_{2k} = T_{2n+1} - m$$

Theorem 3.4: The alternating sum of first n terms is defined by

$$T_1 - T_2 + T_3 + \ldots + (-1)^{n+1} T_n = (-1)^{n+1} T_{n-1} \tag{3.4}$$

Proof.

$$T_1 - T_2 + T_3 + \ldots + (-1)^{n+1} T_n$$

$$= \frac{m}{\alpha-\beta}\begin{bmatrix}\left(\alpha^2-\beta^2\right)-\left(\alpha^3-\beta^3\right)+\left(\alpha^4-\beta^4\right) \\ +\ldots+\left(\alpha^{n+1}-\beta^{n+1}\right)\end{bmatrix}$$

$$= \frac{m}{\alpha-\beta}\begin{bmatrix}\left(\alpha^2-\alpha^3+\alpha^4+\ldots+\alpha^{n+1}\right) \\ -\left(\beta^2-\beta^3+\beta^4+\ldots+\beta^{n+1}\right)\end{bmatrix}$$

$$= \frac{m}{\alpha-\beta}\left[\frac{(-\alpha)^n\alpha^2-\alpha^2}{-\alpha-1} - \frac{(-\beta)^n\beta^2-\beta^2}{-\beta-1}\right]$$

$$= \frac{m}{\alpha-\beta}\left[\frac{(-1)^n\alpha^{n+2}-\alpha^2}{-\alpha^2} - \frac{(-1)^n\beta^{n+2}-\beta^2}{-\beta^2}\right]$$

$$= \frac{m}{\alpha-\beta}\left[\frac{(-1)^{n+1}\alpha^{n+2}+\alpha^2}{\alpha^2} - \frac{(-1)^{n+1}\beta^{n+2}+\beta^2}{\beta^2}\right]$$

$$= \frac{m}{\alpha-\beta}\left[(-1)^{n+1}\alpha^n+1-(-1)^{n+1}\beta^n-1\right]$$

$$= (-1)^{n+1}\frac{m}{\alpha-\beta}\left(\alpha^n-\beta^n\right)$$

$$T_1 - T_2 + T_3 + \ldots + (-1)^{n+1} T_n = (-1)^{n+1} T_{n-1}$$

Theorem 3.5: For positive integer n

$$T_{-n} = (-1)^n T_{n-2}, \; n > 1 \tag{3.5}$$

Proof. To prove this we use Binet's formula then

$$T_{-n} = \frac{m}{\alpha-\beta}\left(\alpha^{-n+1} - B^{-n+1}\right)$$

$$T_{-n} = \frac{m}{\alpha-\beta}\left(\alpha^{-(n-1)} - \beta^{-(n-1)}\right)$$

$$T_{-n} = \frac{m}{\alpha-\beta}\left[\left(\frac{1}{\alpha}\right)^{n-1} - \left(\frac{1}{\beta}\right)^{n-1}\right]$$

Since $\left(\frac{1}{\alpha}\right)^{n-1} = (-1)^{n-1}\beta^{n-1}$ and $\left(\frac{1}{\beta}\right)^{n-1} = (-1)^{n-1}\alpha^{n-1}$ then

$$T_{-n} = \frac{m}{\alpha-\beta}\left[(-1)^{n-1}\beta^{n-1} - (-1)^{n-1}\alpha^{n-1}\right]$$

$$T_{-n} = \frac{m}{\alpha-\beta}(-1)^n\left(\alpha^{n-1}-\beta^{n-1}\right)$$

$$T_{-n} = (-1)^n T_{n-2}$$

Theorem 3.6: For the whole number n

$$mT_{2n+2} = T_n^2 + T_{n+1}^2 \tag{3.6}$$

Proof.

$$mT_{2n+2} = \frac{m^2}{\sqrt{5}}\left(\alpha^{2n+3} - \beta^{2n+3}\right)$$

$$mT_{2n+2} = \frac{m^2}{5}\left(\sqrt{5}\alpha^{2n+3} - \sqrt{5}\beta^{2n+3}\right)$$

$$mT_{2n+2} = \frac{m^2}{5}\left[\alpha^{2n+2}\left(1+\alpha^2\right) + \beta^{2n+2}\left(1+\beta^2\right)\right]$$

$$mT_{2n+2} = \frac{m^2}{5}\begin{pmatrix}\alpha^{2n+2} + \alpha^{2n+4} + \beta^{2n+2} + \beta^{2n+4} \\ -2\alpha^{2n+1}\beta^{2n+1} - 2\alpha^{2n+2}\beta^{2n+2}\end{pmatrix}$$

$$mT_{2n+2} = \frac{m^2}{5}\left[\left(\alpha^{n+1} - \beta^{n+1}\right)^2 + \left(\alpha^{n+2} - \beta^{n+2}\right)^2\right]$$

$$mT_{2n+2} = T_n^2 + T_{n+1}^2$$

Theorem 3.7: For positive integer n

$$T_3 + T_6 + T_9 + ... + T_{3n} = \frac{T_{3n+2} - T_2}{2} = \frac{T_{3n+2}}{2} - m \tag{3.7}$$

Proof.

$$T_3 + T_6 + T_9 + ... + T_{3n}$$

$$= \frac{m}{\sqrt{5}}\begin{bmatrix}\left(\alpha^4 - \beta^4\right) + \left(\alpha^7 - \beta^7\right) + \left(\alpha^{10} - \beta^{10}\right) \\ + ... + \left(\alpha^{3n+1} - \beta^{3n+1}\right)\end{bmatrix}$$

$$= \frac{m}{\sqrt{5}}\begin{bmatrix}\left(\alpha^4 + \alpha^7 + \alpha^{10} + ... + \alpha^{3n+1}\right) \\ -\left(\beta^4 + \beta^7 + \beta^{10} + ... + \beta^{3n+1}\right)\end{bmatrix}$$

$$= \frac{m}{\sqrt{5}}\left[\left(\frac{\alpha^{3n+4} - \alpha^4}{\alpha^3 - 1}\right) - \left(\frac{\beta^{3n+4} - \beta^4}{\beta^3 - 1}\right)\right]$$

$$= \frac{m}{2\sqrt{5}}\left[\left(\alpha^{3n+3} - \alpha^3\right) - \left(\beta^{3n+3} - \beta^3\right)\right]$$

$$T_3 + T_6 + T_9 + ... + T_{3n} = \frac{1}{2}\left(T_{3n+2} - T_2\right)$$

$$T_3 + T_6 + T_9 + ... + T_{3n} = \frac{T_{3n+2}}{2} - m$$

Theorem 3.8: For the whole number n

$$T_n^3 = \frac{m^2}{5}\left[T_{3n+2} - 3(-1)^{n+1}T_n\right] \tag{3.8}$$

Proof.

$$T_n^3 = \frac{m^3}{5\sqrt{5}}\left(\alpha^{n+1} - \beta^{n+1}\right)^3$$

$$T_n^3 = \frac{m^3}{5\sqrt{5}}\left[\alpha^{3n+3} - \beta^{3n+3} - 3(-1)^{n+1}\left(\alpha^{n+1} - \beta^{n+1}\right)\right]$$

$$T_n^3 = \frac{m^2}{5}\left[T_{3n+2} - 3(-1)^{n+1}T_n\right]$$

Theorem 3.9: For positive integer n,

$$T_4 + T_7 + T_{10} + ... + T_{3n+1} = \frac{T_{3n+3} - 3m}{2} \tag{3.9}$$

Proof.

$$T_4 + T_7 + T_{10} + ... + T_{3n+1}$$

$$= \frac{m}{\sqrt{5}}\begin{bmatrix}\left(\alpha^5 - \beta^5\right) + \left(\alpha^8 - \beta^8\right) + \left(\alpha^{11} - \beta^{11}\right) \\ + ... + \left(\alpha^{3n+2} - \beta^{3n+2}\right)\end{bmatrix}$$

$$= \frac{m}{\sqrt{5}}\begin{bmatrix}\left(\alpha^5 + \alpha^8 + \alpha^{11} + ... + \alpha^{3n+2}\right) \\ -\left(\beta^5 + \beta^8 + \beta^{11} + ... + \beta^{3n+2}\right)\end{bmatrix}$$

$$= \frac{m}{\sqrt{5}}\left[\left(\frac{\alpha^{3n+5} - \alpha^5}{\alpha^3 - 1}\right) - \left(\frac{\beta^{3n+5} - \beta^5}{\beta^3 - 1}\right)\right]$$

$$= \frac{m}{2\sqrt{5}}\left[\left(\frac{\alpha^{3n+5} - \alpha^5}{\alpha}\right) - \left(\frac{\beta^{3n+5} - \beta^5}{\beta}\right)\right]$$

$$= \frac{m}{2\sqrt{5}}\left[\left(\alpha^{3n+4} - \alpha^4\right) - \left(\beta^{3n+4} - \beta^4\right)\right]$$

$$= \frac{m}{2\sqrt{5}}\left[\left(\alpha^{3n+4} - \beta^{3n+4}\right) - \left(\alpha^4 - \beta^4\right)\right]$$

$$T_4 + T_7 + T_{10} + ... + T_{3n+1} = \frac{T_{3n+3} - 3m}{2}$$

Theorem 3.10: For positive integer n

$$T_5 + T_8 + T_{11} + \ldots + T_{3n+2} = \frac{T_{3n+4} - 5m}{2} \tag{3.10}$$

Proof.

$$T_5 + T_8 + T_{11} + \ldots + T_{3n+2}$$
$$= \frac{m}{\sqrt{5}}\left[\begin{matrix}(\alpha^6 - \beta^6) + (\alpha^9 - \beta^9) + (\alpha^{12} - \beta^{12}) \\ + \ldots + (\alpha^{3n+3} - \beta^{3n+3})\end{matrix}\right]$$
$$= \frac{m}{\sqrt{5}}\left[\begin{matrix}(\alpha^6 + \alpha^9 + \alpha^{12} + \ldots + \alpha^{3n+3}) \\ -(\beta^6 + \beta^9 + \beta^{12} + \ldots + \beta^{3n+3})\end{matrix}\right]$$
$$= \frac{m}{\sqrt{5}}\left[\left(\frac{\alpha^{3n+6} - \alpha^6}{\alpha^3 - 1}\right) - \left(\frac{\beta^{3n+6} - \beta^6}{\beta^3 - 1}\right)\right]$$
$$= \frac{m}{2\sqrt{5}}\left[\left(\frac{\alpha^{3n+6} - \alpha^6}{\alpha}\right) - \left(\frac{\beta^{3n+6} - \beta^6}{\beta}\right)\right]$$
$$= \frac{m}{2\sqrt{5}}\left[(\alpha^{3n+5} - \alpha^5) - (\beta^{3n+5} - \beta^5)\right]$$
$$= \frac{m}{2\sqrt{5}}\left[(\alpha^{3n+5} - \beta^{3n+5}) - (\alpha^5 - \beta^5)\right]$$
$$T_5 + T_8 + T_{11} + \ldots + T_{3n+2} = \frac{T_{3n+4} - T_4}{2}$$
$$T_5 + T_8 + T_{11} + \ldots + T_{3n+2} = \frac{T_{3n+4} - 5m}{2}$$

Theorem 3.11: For positive integer n

$$S_1^3 + S_2^3 + S_3^3 + \ldots + S_n^3 = \frac{m^2}{10}\left[T_{3n+4} - 5m - 6(-1)^{n+1} T_{n-1}\right] \tag{3.11}$$

Proof. By theorem (3.8), we have

$$S_1^3 + S_2^3 + S_3^3 + \ldots + S_n^3$$
$$= \frac{m^2}{5}\left[\begin{matrix}T_5 - 3(-1)^2 T_1 + T_8 - 3(-1)^3 T_2 + \\ T_{11} - 3(-1)^4 T_3 \ldots + T_{3n+2} - 3(-1)^{n+1} T_n\end{matrix}\right]$$
$$= \frac{m^2}{5}\left[\begin{matrix}T_5 - 3T_1 + T_8 + 3T_2 + T_{11} - \\ 3T_3 + \ldots + T_{3n+2} - 3(-1)^{n+1} T_n\end{matrix}\right]$$
$$= \frac{m^2}{5}\left[\begin{matrix}(T_5 + T_8 + T_{11} + \ldots + T_{3n+2}) - \\ 3(T_1 - T_2 + T_3 + \ldots + (-1)^{n+1} T_n)\end{matrix}\right]$$

Use theorems (3.10) and (3.4), we have

$$S_1^3 + S_2^3 + S_3^3 + \ldots + S_n^3 = \frac{m^2}{5}\left[\begin{matrix}\left(\frac{T_{3n+4} - 5m}{2}\right) - \\ 3(-1)^{n+1} T_{n-1}\end{matrix}\right]$$
$$= \frac{m^2}{10}\left[T_{3n+4} - 5m - 6(-1)^{n+1} T_{n-1}\right]$$
$$S_1^3 + S_2^3 + S_3^3 + \ldots + S_n^3 = \frac{m^2}{10}\left[T_{3n+4} - 5m - 6(-1)^{n+1} T_{n-1}\right]$$

Theorem 3.12: For positive integer n

$$mT_{2n+1} = T_{n+1}^2 - T_{n+1}^2 \tag{3.12}$$

Proof.

$$mT_{2n+1} = \frac{m^2}{\sqrt{5}}\left(\alpha^{2n+2} - \beta^{2n+2}\right)$$
$$mT_{2n+2} = \frac{m^2}{5}\left(\sqrt{5}\alpha^{2n+2} - \sqrt{5}\beta^{2n+2}\right)$$
$$mT_{2n+2} = \frac{m^2}{5}\left[\alpha^{2n}(\alpha^4 - 1) + \beta^{2n}(\beta^4 - 1)\right]$$
$$mT_{2n+2} = \frac{m^2}{5}\left(\begin{matrix}\alpha^{2n+4} - \alpha^{2n} + \beta^{2n+4} - \beta^{2n} - \\ 2\alpha^{n+2}\beta^{n+2} - 2\alpha^n\beta^n\end{matrix}\right)$$
$$mT_{2n+1} = \frac{m^2}{5}\left[(\alpha^{n+2} - \beta^{n+2})^2 - (\alpha^n - \beta^n)^2\right]$$
$$mT_{2n+1} = T_{n+1}^2 - T_{n-1}^2$$

Theorem 3.13: For positive integer n

$$T_{2n} = T_{2n+1} - T_{2n-1}, \; n \geq 1 \tag{3.13}$$

Proof.

$$T_{2n} = \frac{m}{\sqrt{5}}\left(\alpha^{2n+1} - \beta^{2n+1}\right)$$

$$T_{2n} = \frac{m}{\sqrt{5}}\left[\alpha^{2n}\left(\alpha^2 - 1\right) - \beta^{2n}\left(\beta^2 - 1\right)\right]$$

$$T_{2n} = \frac{m}{\sqrt{5}}\left[\alpha^{2n+2} - \beta^{2n+2} - \left(\alpha^{2n} - \beta^{2n}\right)\right]$$

$$T_{2n} = T_{2n+1} - T_{2n-1}$$

Theorem 3.14: For the positive integer n

$$T_{n-1}T_{n+1} - T_n^2 = (-1)^{n+1}m^2,\ n \geq 1 \tag{3.14}$$

Proof.

$$T_{n-1}T_{n+1} - T_n^2$$

$$= \frac{m^2}{5}\left[-(-1)^n\beta^2 - (-1)^n\alpha^2 + 2(-1)^{n+1}\right]$$

$$= \frac{m^2}{5}\left[(-1)^{n+1}\beta^2 + (-1)^{n+1}\alpha^2 + 2(-1)^{n+1}\right]$$

$$= \frac{m^2}{5}\left[(-1)^{n+1}\left(\alpha^2 + \beta^2\right) + 2(-1)^{n+1}\right]$$

$$= \frac{m^2}{5}\left[(-1)^{n+1}(3) + 2(-1)^{n+1}\right]$$

$$= \frac{m^2}{5}\left[(-1)^{n+1}(5-2) + 2(-1)^{n+1}\right]$$

$$= \frac{m^2}{5}\left[(-1)^{n+1}5 - 2(-1)^{n+1} + 2(-1)^{n+1}\right]$$

$$T_{n-1}T_{n+1} - T_n^2 = (-1)^{n+1}m^2$$

Theorem 3.15: For the whole number n

$$T_n = m\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k} \tag{3.15}$$

Proof. By generating function of Fibonacci-Like Sequence, we have

$$\sum_{n=0}^{\infty} T_n x^n = m\left(1 - x - x^2\right)$$

$$\sum_{n=0}^{\infty} T_n x^n = m\sum_{n=0}^{\infty}(1+x)^n x^n$$

$$\sum_{n=0}^{\infty} T_n x^n = m\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}x^k x^n$$

$$\sum_{n=0}^{\infty} T_n x^n = m\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n+k}{k}x^{n+2k}$$

$$\sum_{n=0}^{\infty} T_n x^n = m\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}x^{n+k}$$

$$\sum_{n=0}^{\infty} T_n x^n = m\sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}x^n$$

Equating the coefficient of x^n on both sides, to get

$$T_n = m\sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n-k}{k}$$

Theorem 3.16: For the positive integer n

$$T_{2n} = \sum_{k=0}^{n}\binom{n}{k}T_{n-k} \tag{3.16}$$

Proof.

$$T_{2n} = \frac{m}{\sqrt{5}}\left(\alpha^{2n+1} - \beta^{2n+1}\right)$$

$$T_{2n} = \frac{m}{\sqrt{5}}\left[(1+\alpha)^n\alpha - (1+\beta)^n\beta\right]$$

Since $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}$ then , we have

$$T_{2n} = \frac{m}{\sqrt{5}}\left[\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\alpha - \sum_{k=0}^{n}\binom{n}{k}\beta^{n-k}\beta\right]$$

$$T_{2n} = \frac{m}{\sqrt{5}}\left[\sum_{k=0}^{n}\binom{n}{k}\left(\alpha^{n-k+1} - \beta^{n-k+1}\right)\right]$$

$$T_{2n} = \sum_{k=0}^{n}\binom{n}{k}T_{n-k}$$

Theorem 3.17: For whole number n

$$\lim_{n\to\infty}\frac{T_{n+1}}{T_n}=\alpha \tag{3.17}$$

Proof.

$$\lim_{n\to\infty}\frac{T_{n+1}}{T_n}=\lim_{n\to\infty}\left(\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha^{n+1}-\beta^{n+1}}\right)$$

$$\lim_{n\to\infty}\frac{T_{n+1}}{T_n}=\lim_{n\to\infty}\frac{\left(1-\frac{\beta^{n+2}}{\alpha^{n+2}}\right)}{\left(\frac{\alpha^{n+1}}{\alpha^{n+2}}-\frac{\beta^{n+1}}{\alpha^{n+2}}\right)}$$

$$\lim_{n\to\infty}\frac{T_{n+1}}{T_n}=\lim_{n\to\infty}\frac{1-\left(\frac{\beta}{\alpha}\right)^n\left(\frac{\beta}{\alpha}\right)^2}{\frac{1}{\alpha}-\left(\frac{\beta}{\alpha}\right)^n\frac{\beta}{\alpha^2}}$$

Since $\left|\frac{\beta}{\alpha}\right|<1$ then $\lim_{n\to\infty}\left(\frac{\beta}{\alpha}\right)^n=0$, we have

$$\lim_{n\to\infty}\frac{T_{n+1}}{T_n}=\lim_{n\to\infty}\frac{1}{1/\alpha}=\alpha$$

$$\lim_{n\to\infty}\frac{T_{n+1}}{T_n}=\alpha$$

5. Conclusion

In this paper, we introduced Fibonacci-Like sequence and presented some basic identities about it. The main thing in this paper is that almost all of the identities are proved by Binet's formula in spite of the induction method or any other method.

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References

[1]. Badshah V. H., Teeth M. S., Dar M. M., "Generalised Fibonacci-Like sequence and its Properties" Int. J. Contemp. Math. Sciences, Vol. 7, No. 24 (2012) 1155-1164.

[2]. Cartilz L., "A Note on Fibonacci Numbers", The Fibonacci Quarterly., 2 (1964) 15-28

[3]. Harnay S., Singh B., Pal S., "Generalised Fibonacci-Like sequence and Fibonacci-Sequence" Int. J. Contemp. Math. Sciences, Vol. 9, No. 5 (2012) 235-241.

[4]. Horadam, A. F., "Generalized Fibonacci Sequences", The American Mathematical Monthly, Vol. 68, No. 5 (1961) 455-459.

[5]. Jaiswal, D. V., "On a Generalized Fibonacci Sequence", Labdev J. Sci. Tech. Part A, Vol.7 (1969) 67-71.

[6]. Koshy T., "Fibonacci and Lucas Numbers with Applications" Wiley, 2001

[7]. LEE J. Z. & LEE J. S., "Some Properties of the Generalizations of the Fibonacci Sequence", The Fibonacci Quarterly Vol. 25, No. 2 (1987) 111-117.

[8]. Singh B., Sikhwal O., and Bhatnagar S., "Fibonacci-Like Sequence and its Properties", Int. J. Contemp. Math. Sciences, Vol. 5, No. 18 (2010) 859-868.

[9]. Harris V.C., "On Identities Involving Fibonacci Numbers"., The Fibonacci Quarterly., Vol.3 (1970) 61-73

[10]. N. N., Vorobyoy., "The Fibonacci Numbers". D. C. Health and company Boston 1963.