# Generalized Rough sets induced by filters in residuated lattices

K.Reena and I. Arockiarani

*(Department of Mathematics, Nirmala College for women, Coimbatore, India)*

**Abstract:**
The focus of this paper is to study the special properties of the rough sets which can be constructed by means of the congruences determined by filters of residuated lattice. Also the properties of the generalized rough sets with respect to filters of residuated lattice are investigated.

**Keywords:** Rough set, residuated lattice, set-valued mapping, generalized rough set.

## 1. Introduction

Pawlak rough set theory is an extension of the set theory for study and analyze various types of data [29-32,45]. It has been successfully applied such artificial intelligence fields as machine learning, pattern recognition, decision analysis,cognitive sciences, intelligent decision making and process control [7,15–17,36,49,52]. Some of rough set applications are to approximate an arbitrary a universe by two definable subsets called lower and upper approximations, and to reduce the number of the set of attributes in data sets.The notion of attribute reductwas proposed as a minimal subset of attributes that induce the same discernibility relation as the whole set of condition attributes. Nowdays, many types of attribute reductions have been achieved, without any relationships among them [3,39,52–55]. However, equivalence relation, as the indiscernibility tool in Pawlak's rough set theory is still restrictive for many applications such as incomplete information tables can not handled with Pawlak's model ( [28]). So many generalizations of Pawlak's model were proposed [1,30,35–38,49,50,55]. Some researchers introduced approaches to relax the partition to a cover [21,25,44,54]. Pei [32, 33] mainly forced on researching algebraic characterization of rough set extension on two universes. In [49], Yan et al. discussed properties of rough set extension on two universes by introducing character function and relation matrix, proposed algorithms for obtaining lower and upper approximation of rough set extension on two universes and studied Pawlak rough set induced by rough set extension on two universes. The generalization of Pawlak rough set was proposed for two universes on general binary relations. Therefore, congruence relations must be extended to two universes for algebraic sets. From this point of view, Davvaz [10] introduced the concept of set-valued homomorphism for groups. And then, Yamak [44] and Xiao and Li [42] proposed the concepts of (strong) set-valued homomorphism of a ring and of a lattice, respectively. The aim of this paper is to discuss the algebraic properties of rough sets induced by filters in residuated lattices. Also, we study the rough sets which are constructed by congruence relation. Furher, we introduce a special class of set-valued homomorphism with respect to a filter and discuss the properties of the generalized rough set which is an extended notion of the rough set.

## 2. Preliminaries

**Definition 2.1: [41]**

A residuated lattice is an algebraic structure $L = (L, \vee, \wedge, *, \rightarrow, 0, 1)$ satisfying the following axioms:

1. $(L, \vee, \wedge, 0, 1)$ is a bounded lattice
2. $(L, *, 1)$ is a commutative monoid.
3. $(*, 1)$ is an adjoint pair, i.e., for any x, y, z, w$\in$L,
   - i. if $x \leq y$ and $z \leq w$, then $x * z \leq y * w$.
   - ii. if $x \leq y$ and $y \rightarrow z \leq x \rightarrow z$ then $z \rightarrow x \leq z \rightarrow y$.
   - iii. (adjointness condition) $x * y \leq z$ if and only if $x \leq y \rightarrow z$.

In this paper, denote L as residuation lattice unless otherwise specified.

**Theorem 2.2: [41]**

In each residuated lattice L, the following properties hold for all x, y, z ∈ L:

1. $(x * y) \to z = x \to (y \to z)$.
2. $z \le x \to y \Leftrightarrow z * x \le y$.
3. $x \le y \Leftrightarrow z * x \le z * y$.
4. $x \to (y \to z) = y \to (x \to z)$.
5. $x \le y \Rightarrow z \to x \le z \to y$.
6. $x \le y \Rightarrow y \to z \le x \to z, \ y' \le x'$.
7. $y \to z \le (x \to y) \to (x \to z)$.
8. $y \to x \le (x \to z) \to (y \to z)$.
9. $1 \to x = x, x \to x = 1$.
10. $x^m \le x^n$ , m, n ∈ N, m ≥ n.
11. $x \le y \Leftrightarrow x \to y = 1$.
12. $0' = 1, 1' = 0, x' = x^m, x \le x^n$.
13. $x \vee y \to z = (x \to z) \wedge (y \to z)$.
14. $x * x' = 0$.
15. $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

**Definition 2.3: [41]**

A non-empty subset F of a residuated lattice L is called a filter of L if it satisfies

1. $x, y \in F \Rightarrow x * y \in F$.
2. $x \in F, x \le y \Rightarrow y \in F$.

**Theorem 2.4: [41]**

A non-empty subset F of a residuated lattice L is called a filter of L if it satisfies, for any x,y∈ L,

1. $1 \in F$.
2. $x \in F, x \to y \in F \Rightarrow y \in F$.

**Definition 2.5: [28]**

Let U is a non empty set and R an equivalence relation on U. Then the pair (U, R) is called an approximation space.

**Definition 2.6:[28]**

Let (U, R) be an approximation space and X any nonempty subset of U. Then the sets, $\underline{Apr}(X) = \{x \in U / [x]_R \subseteq X\}$ and $\overline{Apr}(X) = \{x \in U / [x]_R \cap X \ne \varphi\}$ are called the lower and upper rough approximations of the set X. Then Apr(X) = $(\underline{Apr}(X), \overline{Apr}(X))$ is called a rough set in (U, R).

**3.Generalized Rough sets induced by filters:**

**Definition 3.1:**

Let L be a distributive residuated Lattice then for any filter F of L we can find a congruence relation C over L defined by $\forall a, b \in L$ aCb if there exist x ∈ F such that a * x = b * x.

**Remark 3.2:**

The Congruence class $[x]_F = \{y \in L / xCy\}$.

**Definition 3.3:**

Let A and B are two non-empty subsets of L. Then we define A * B = {a * b / a ∈ A , b ∈ B} and A → B = {a → b / a ∈ A , b ∈ B}.

**Definition 3.4:**

Let U and W be two non-empty universes. Let $\Sigma$ be a set-valued mapping given by $\Sigma : U \to \rho(W)$. Then the triple (U, W, $\Sigma$) is referred to as a generalized approximation space. For any set A ⊆ W, the lower and upper approximation $\underline{\Sigma}(A)$ and $\overline{\Sigma}(A)$ are defined by $\underline{\Sigma}(A) = \{x \in U / \Sigma(x) \subseteq A\}$ and $\overline{\Sigma}(A) = \{x \in U / \Sigma(x) \cap A \neq \Phi\}$. The pair $(\underline{\Sigma}(A), \overline{\Sigma}(A))$ is referred to as a generalized rough set.

**Theorem 3.5:**

Let U and W be non-empty universes and $\Sigma : U \to \rho^*(W)$ be a set-valued mapping where $\rho^*(W)$ denotes the set of all non-empty subsets of W. If A ⊆ W, then $\underline{\Sigma}(A) \subseteq \overline{\Sigma}(A)$.

**Theorem 3.6:**

Let (U, W, $\Sigma$) be a generalized approximation space. Let $\{A_i\}_{i \in I}$ be an arbitrary family in W. Then 1. $\underline{\Sigma}(\cap_{i \in I} A_i) = \cap_{i \in I} \underline{\Sigma}(A_i)$, $\underline{\Sigma}(\cup_{i \in I} A_i) \supseteq \cup_{i \in I} \underline{\Sigma}(A_i)$     2. $\overline{\Sigma}(\cup_{i \in I} A_I) = \cup_{i \in I} \overline{\Sigma}(A_i)$ , $\overline{\Sigma}(\cap_{i \in I} A_i) \subseteq \cap_{i \in I} \overline{\Sigma}(A_i)$.

**Definition 3.7:**

Let F be a filter of L and A a non-empty subset of L. Then the sets $\underline{Apr}_F(A) = \{x \in L / [x]_F \subseteq A\}$, $\overline{Apr}_F(A) = \{x \in L / [x]_F \cap A \neq \Phi\}$ are called the lower and upper rough approximation of the set A with respect to the filter F of L. The pair $Apr_F(X) = (\underline{Apr}_F(X), \overline{Apr}_F(X))$ is called a rough set in the approximation space (L, F).

**Lemma 3.8:**

Let $F_1$ and $F_2$ be filters of L and $F_1 \subseteq F_2$. If A is a non-empty subset of L, then     1. $\underline{Apr}_{F_2}(A) \subseteq \underline{Apr}_{F_1}(A)$ and $\overline{Apr}_{F_1}(A) \subseteq \overline{Apr}_{F_2}(A)$.

2. $\underline{Apr}_{F_1 \cap F_2}(A) \supseteq \underline{Apr}_{F_1}(A) \cup \underline{Apr}_{F_2}(A)$ and $\overline{Apr}_{F_1 \cap F_2}(A) \subseteq \overline{Apr}_{F_1}(A) \cap \overline{Apr}_{F_2}(A)$

**Proof:** Follows from definitions.

**Lemma 3.9:**

Let $F_1$ and $F_2$ be filters of L and A ⊆ L. Then

1. $Apr_{F_1} \cap Apr_{F_2}$ is a join congruence
2. $(Apr_{F_1} \cap Apr_{F_2})(A) \supseteq \underline{Apr}_{F_1}(A) \cup \underline{Apr}_{F_2}(A)$, $(\overline{Apr}_{F_1} \cap \overline{Apr}_{F_2}) \subseteq \overline{Apr}_{F_1}(A) \cap \overline{Apr}_{F_2}(A)$.

**Proof:**

1. Suppose $x_i, y_i \in L$ such that $x_i \equiv y_i$ (mod $Apr_{F_1} \cap Apr_{F_2}$) (i = 1, 2). Then there exist $d_i \in F_1$ and $e_i \in F_2$ such that $x_i * d_i = y_i * d_i$ and $x_i * e_i = y_i * e_i$ (i = 1, 2). So $(x_1 * x_2) * (d_1 * d_2) = (y_1 * y_2) * (d_1 * d_2)$, $(x_1 * x_2) * (e_1 * e_2) = (y_1 * y_2) * (e_1 * e_2)$. Since $F_1$ and $F_2$ are filters, we have $d_1 * d_2 \in F_1$ and $e_1 * e_2 \in F_2$.     Therefore, $x_1 * x_2 \equiv y_1 * y_2$ (mod $Apr_{F_1} \cap Apr_{F_2}$).
2. It is straightforward.

**Lemma 3.10:**

Let $F_1$ and $F_2$ be filters of L and A $\subseteq$ L. Then

1. $\underline{Apr}_{F_1 \cap F_2}$(A) $\supseteq$ $(\underline{Apr}_{F_1} \cap \underline{Apr}_{F_2})$(A),     $\overline{Apr}_{F_1 \cap F_2}$(A) $\subseteq$ $(\overline{Apr}_{F_1} \cap \overline{Apr}_{F_2})$(A)

**2.** If L is distributive, then $\underline{Apr}_{F_1 \cap F_2}$(A)=$(\underline{Apr}_{F_1} \cap \underline{Apr}_{F_2})$(A),     $\overline{Apr}_{F_1 \cap F_2}$(A) = $(\overline{Apr}_{F_1} \cap \overline{Apr}_{F_2})$(A) .

**Proof:**

1.  Let x $\in$ $(\underline{Apr}_{F_1} \cap \underline{Apr}_{F_2})$(A) and $x' \equiv$ x (mod $Apr_{F_1 \cap F_2}$ ).                    It is clear that $x'$ $\equiv$ x (mod $Apr_{F_1} \cap Apr_{F_2}$ ). So $x' \in$ A which implies that          x $\in$ $\underline{Apr}_{F_1 \cap F_2}$ (A). Let x $\in$ $\overline{Apr}_{F_1 \cap F_2}$(A), there exist $x' \in$ A and d $\in$ $F_1 \cap F_2$ such that      x * d = $x'$ * d. So $x' \equiv$ x (mod $Apr_{F_1} \cap$ $Apr_{F_2}$ ) which implies that          x $\in \overline{Apr}_{F_1} \cap Apr_{F_2})$(A)
2.  Let x $\in \underline{Apr}_{F_1 \cap F_2}$ (A) and $x' \equiv$ x (mod $Apr_{F_1} \cap Apr_{F_2}$ ), there exist d $\in F_1$ and e $\in F_2$ such that $x'$ * d = x * d and $x'$ * e = x * e. So $(x'$ * d) $\to (x'$ * e) = (x * d) $\to$ (x * e). Since L is distributive and $F_1$ , $F_2$ are filters, we have $x'$ * (d $\to$ e) = x * (d $\to$ e) and          d $\to$ e $\in F_1 \cap F_2$. Thus $x' \equiv$ x (mod $Apr_{F_1 \cap F_2}$ )., then $x' \in$ A which implies that          x $\in (\underline{Apr}_{F_1} \cap \underline{Apr}_{F_2})$(A). Hence by statement (1), we have $\underline{Apr}_{F_1 \cap F_2}$ (A) = $(\underline{Apr}_{F_1} \cap \underline{Apr}_{F_2})$(A). Similarly we prove $\overline{Apr}_{F_1 \cap F_2}$(A) = $(\overline{Apr}_{F_1} \cap \overline{Apr}_{F_2})$(A).

**Proposition 3.11:**

Let F be a filter of L, then $\underline{Apr}_F$(F) = F = $\overline{Apr}_F$(F).

**Proof:**

We have $\underline{Apr}_F$(F) $\subseteq$ F $\subseteq \overline{Apr}_F$(F). On the other hand, let x $\in \overline{Apr}_F$(F), we have $[x]_F \cap$ F $\neq \Phi$, then there exist a $\in$ F and d $\in$ F such that x * d = a * d. Since F is a filter, we have x * d $\in$ F, and thus x $\in$ F. This means $\overline{Apr}_F$(F) $\subseteq$ F. Moreover, let x $\in$ F and a $\in [x]_F$, then there exists          d $\in$ F such that a * d = x * d. Then we have a * d $\in$ F, and thus a $\in$ F. So $[x]_F \subseteq$ F which implies that x $\in \underline{Apr}_F$(F). Therefore, F $\subseteq \underline{Apr}_F$(F). From the above, we have $\underline{Apr}_F$(F) = F = $\overline{Apr}_F$(F).

**Lemma 3.12:**

Let $F_1$ and $F_2$ be filters of L, then $F_1 = \overline{Apr}_{F_1}$ $(F_1 \cap F_2$ ).

**Proof:**

Let $F_1$ and $F_2$ be filters of L. Then the following statements are equivalent.

1.  $F_1 \subseteq F_2$
2.  $F_2 = \overline{Apr}_{F_1}$ $(F_2)$
3.  $F_2 = \underline{Apr}_{F_1}(F_2)$.

**Proof:**

(1)$\Rightarrow$ (2): If $F_1 \subseteq F_2$, let x $\in \overline{Apr}_{F_1}$ $(F_2)$, then there exist $x' \in F_2$ and d $\in F_1 \subseteq F_2$ such that x * d = $x'$ * d. Since $F_2$ is a filter, we have $x'$ * d = x * d $\in F_2$, then x $\in F_2$. Therefore $F_2 = \overline{Apr}_{F_1}(F_2)$. (2 ) $\Rightarrow$ (3): If $F_2 = \overline{Apr}_{F_1}$ $(F_2)$. Let

$x \in F_2$ and $x' \equiv x \pmod{Apr_F}$. Assume that $x' \notin F_2$, then $x' \notin \overline{Apr}_{F_1}(F_2)$. Hence $[x]_{F_1} \cap F_2 = [x']_{F_1} \cap F_2$ $= \Phi$ which implies that $x \notin \overline{Apr}_{F_1}(F_2) = F_2$. It contradicts with $x \in F_2$, so $x' \in F_2$. Thus $[x]_{F_1} \subseteq F_2$ this means that $x \in \underline{Apr}_{F_1}(F_2)$. Also $\underline{Apr}_{F_1}(F_2) \subseteq F_2$. Hence we have $F_2 = \underline{Apr}_{F_1}(F_2)$.

(3)$\Rightarrow$(1) Follows from definitions.

**Lemma 3.13:**

Let $F_1$, $F_2$ and A be filters of L. If L is distributive and $F_1 \subseteq A$, then $\overline{Apr}_{F_1}(\overline{Apr}_{F_2}(A)) = \overline{Apr}_{F_2}(A) = \overline{Apr}_{F_2}(\overline{Apr}_{F_1}(A))$.

**Proof:**

Since $F_1 \subseteq A$ and A is a filter , by theorem 3.12, we have $\overline{Apr}_{F_1}(A) = A$. So $\overline{Apr}_{F_2}(\overline{Apr}_{F_1}(A)) = \overline{Apr}_{F_2}(A)$. Since L is distributive, by Lemma 3.8, we get $\overline{Apr}_{F_2}(A)$ is a filter. Therefore we have $F_1 \subseteq A \subseteq \overline{Apr}_{F_2}(A)$. Hence $\overline{Apr}_{F_1}(\overline{Apr}_{F_2}(A)) = \overline{Apr}_{F_2}(A)$.

**Proposition 3.14:**

Let $F_1$ , $F_2$ and A be filters of L and $F_1 \subseteq A$. Then $(\overline{Apr_{F_1}} \cap Apr_{F_2})(A) = \overline{Apr}_{F_1}(A) \cap \overline{Apr}_{F_2}(A)$

**Proof:**

By Lemma 2.7, we have $\overline{Apr}_{F_1}(A \cap B) \subseteq \overline{Apr}_{F_1}(A) \cap \overline{Apr}_{F_1}(B)$. Because A is a filter and $F_1 \subseteq A$, by Theorem 3.12, we have $x \in A \cap \overline{Apr}_{F_1}(B)$. So $x \in A$ and there exist $x' \in B$, $d \in F_1$ such that $x' * d = x * d$. Since $F_1 \subseteq A$, we have $x' * d = x * d \in A$, then $x' \in A$. Therefore , $x' \in A \cap B$ which implies that $x \in \overline{Apr}_{F_1}(A \cap B)$.

**Proposition 3.15:**

Let $F_1$ be a filter of L and A, B be non-empty subsets of L. Then $\overline{Apr}_{F_1}(A) * \overline{Apr}_{F_1}(B) \subseteq \overline{Apr}_{F_1}(A * B)$.

**Proof:**

Let $x \in \overline{Apr}_{F_1}(A) * \overline{Apr}_{F_1}(B)$, there exist $y \in \overline{Apr}_{F_1}(A)$ and $z \in \overline{Apr}_{F_1}(B)$ such that $x = y * z$, there exist $y'$, $z' \in$ A and d, e $\in F_1$ such that $y * d = y' * $ and $z * e = z' * e$. So $(y' * z') * (d * e) = (y * d) * (d * e) = x * (d * e)$. Since $y' * z' \in A * B$, $d * e \in F_1$, we have $x \in \overline{Apr}_{F_1}(A * B)$.

**4.The generalized roughness in a residuated lattice**

**Definition 4.1:**

Let L and K be residuated lattices and $\Sigma : L \to \rho(K)$ a set-valued mapping. Let $F_1$ be a filter of K and A a non-empty subset of K. We define $\Sigma_F(x) = \{b \in [a]_F \ / \ a \in \Sigma(x) \}$ for $x \in L$. It is obvious that $\Sigma_F$ is a set-valued mapping from L to $\rho(K)$ and $\Sigma(x) \subseteq \Sigma_F(x)$. Then, $\underline{\Sigma_F}(A) = \{x \in L \ / \ \Sigma_F(x) \subseteq A\}$ and $\overline{\Sigma_F}(A) = \{x \in L \ / \ \Sigma_F(x) \cap A \neq \Phi\}$ are called generalized lower and upper approximations of A with respect to F, respectively.

**Definition 4.2:**

A mapping $\Sigma : L \to \rho(K)$ is a set-valued homomorphism if $\Sigma(a) * \Sigma(b) \subseteq \Sigma(a * b)$ and $\Sigma(a) \to \Sigma(b) \subseteq \Sigma(a \to b)$ for all a, b $\in$ L.

**Proposition 4.3:**

Let $F_1$, $F_2$ be filters of K and $\Sigma : L \to \rho(K)$ be a set-valued mapping. If A is a subset of K and $F_1 \subseteq F_2$ , then

1. $\underline{\Sigma_{F_1}}(A) \cup \underline{\Sigma_{F_2}}(A) \subseteq \underline{\Sigma_{F_1 \cap F_2}}(A)$
2. $\overline{\Sigma_{F_1 \cap F_2}}(A) \subseteq \overline{\Sigma_{F_1}}(A) \cap \overline{\Sigma_{F_2}}(A)$.

**Proof:** Follows from definitions

**Lemma 4.4:**

Let $F_1$, $F_2$ be filters of K and $\Sigma : L \to \rho(K)$ be a set-valued mapping, then $(\Sigma_{F_1 \cap F_2})(x) \subseteq \qquad (\Sigma_{F_1} \cap \Sigma_{F_2})(x)$.

**Proof:**

Suppose $y \in (\Sigma_{F_1 \cap F_2})(x)$, then there exists $a \in \Sigma(x)$ such that $y \in [a]_{F_1 \cap F_2}$. Thus there exists $d \in F_1 \cap F_2$ such that $y * d = a * d$. So $y \in [a]_{F_1}$ and $y \in [a]_{F_2}$ which implies that $y \in (\Sigma_{F_1} \cap \Sigma_{F_2})(x)$.

**Proposition 4.5:**

Let $F_1$, $F_2$ be filters of K and $\Sigma : L \to \rho(K)$ be a set-valued mapping. If A is a subset of K, then

1. $\underline{\Sigma_{F_1 \cap F_2}}(A) \supseteq \underline{\Sigma_{F_1} \cap \Sigma_{F_2}}(A)$
2. $\overline{\Sigma_{F_1 \cap F_2}}(A) \subseteq \overline{\Sigma_{F_1} \cap \Sigma_{F_2}}(A)$.

**Proof:** Follows from Lemma 4.4.

**Lemma 4.6:**

Let $F_1$ be a filter of K and $\Sigma : L \to \rho*(K)$ be a set-valued mapping. For any $x \in L$, the following statements are equivalent.

1. $\Sigma(x) \subseteq F$
2. $\Sigma_F(x) = F$.

**Proof:**

$(1) \Rightarrow (2)$: Suppose $x' \in \Sigma_F(x)$, there exists $a \in \Sigma(x) \subseteq F$ such that $x' \in [a]_F$. So there exists $d \in F$ such that $x' * d = a * d$, then $x' \in F$. Therefore, $\Sigma_F(x) \subseteq F$. Conversely, let $x' \in F$. Since $\Sigma(x) \neq \Phi$, there exists $a \in \Sigma(x) \subseteq F$. We have $x' * (a * x') = a * (a * x')$, so $x' \in [a]_F$ then $x' \in \Sigma_F(x)$. Therefore, $F \subseteq \Sigma_F(x)$.

$(2) \Rightarrow (1)$: Let $y \in \Sigma(x)$. Since $y \equiv y \pmod{Apr_F}$, we have $y \in \Sigma_F(x) = F$. Therefore, $\Sigma(x) \subseteq F$.

**Theorem 4.7:**

Let $F_1$ be a filter of K and $\Sigma : L \to \rho*(K)$ be a set-valued mapping. If $F \subseteq A \subseteq K$ and $\Sigma(x) \subseteq F$ for all $x \in L$, then $\underline{\Sigma_F}(A) = \overline{\Sigma_F}(A) = L$.

**Proof:** The proof follows from Lemma 4.6.

**Proposition 4.8:**

Let $F_1$ and $F_2$ be filters of K and $F_1 \subseteq F_2$ . Let $\Sigma : L \to \rho*(L)$ be a set-valued mapping. If $x \in \Sigma(x)$ for all $x \in L$, then the following statements are equivalent.

1. $\Sigma(x) \subseteq F_2$ for all $x \in F_2$
2. $\underline{\Sigma_{F_1}}(F_2) = F_2$

**Proof:**

$(1) \Rightarrow (2)$: Let $x \in \underline{\Sigma_{F_1}}(F_2)$, then $\Sigma_F(x) \subseteq F_2$. Since $x \in \Sigma(x) \subseteq \Sigma_F(x)$, we get $x \in F_2$. Let $x \in F_2$. For any $y \in \Sigma_F(x)$, there exist $a \in \Sigma(x)$ and $d \in F$ such that $y * d = a * d$. Since $\Sigma(x) \subseteq F_2$ and $F_1 \subseteq F_2$, we have $y * d = a * d \in F_2$, then $y \in F_2$. Hence $\Sigma_{F_1}(x) \subseteq F_2$ which means $x \in \underline{\Sigma_{F_1}}(F_2)$.

$(2) \Rightarrow (1)$: Let $x \in F_2$ and $y \in \Sigma(x)$, then $y \in \Sigma_F(x)$. Since $F_2 = \underline{\Sigma_{F_1}}(F_2)$. We have $\Sigma_{F_1}(x) \subseteq F_2$. So $y \in F_2$. Therefore, $\Sigma(x) \subseteq F_2$ for all $x \in F_2$.

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