Operator Dirac Riemannian Manifolds With Killing Spoinors

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AbstractIn this paper uniform upper and lower continuous Species Dirac Riemannian manifolds $D \subset M^{\pm} \in R$ with curvature bounds on $C^{k} \in M^{\pm} \in R^{k-1}$ as surfaces and applications compact Riemannian boundary $S \in M^{\pm} \subset R$ is complete with charts Riemannianon differential on ∂R

Keywords - *open disk on composters – Locally Euclidean on dim. N-Bundle maps isomorphism-Lower estimates for direct operator – differential Euclidean space.*

I. Introduction

The Riemannian geometry with A set with this property is said to be open that S in Euclidean space E is open if for every P in S there exists a spherical neighborhood $S(P)$ of P , A set in E is compact if and only if is closed and bounded.

A compact set S is also closed , for let Q be an arbitrary point in its complement S^{c} for each P in S and A topological space M is locally Euclidean of dimension n if for every point $x \in M$ there exists on open set $U \in M$ and open set $w \subset R^{n}$ so that U and W are (homeomorphic on Riemannian geometry).

II. A BASIC NOTIONS ON TOPLOGICAL GEOMETRY

An every point in the interior of a circle in the plan can be enclosed in a spherical neighborhood which is also contained in the interior . A set with this property is said to be open that S in Euclidean space E is open if for every P in S there exists a spherical neighborhood $S(P)$ of P which is completely contained in S .for example (i) An open interval $a \le x \le b$ is open set in E^{1}. The interval $a \prec x \le b$ is not open , for every $S(b)$ will contain points $x' \succ b$ and hence not $a \prec x \le b$.(ii) A spherical neighborhood of a point is itself open . In E^{1} it is an open finite interval in E^{2} it is the interior of a sphere called an open sphere.(iii) Euclidean space E itself is open also the null set ϕ is open for otherwise there would be a point P in ϕ such that every $S(P)$ contains points not in ϕ . But there is on P .

2.1 Open disk

Is open set in E^{3}, but it is not open when considered as a subset of a plane in E^{3} contains points off the plan . Thus openness is a relative property of set depending upon the space in which the set is considered to lie .If $\{Q_{\alpha}\}$ is any family of open sets finite or infinite then the union $\cup_{\alpha} Q_{\alpha}$ is open . For let P be in $\cup_{\alpha} Q_{\alpha}$ then P is in same $\{Q_{\alpha 0}\}$ thus for P in $\cup_{\alpha} Q_{\alpha}$ there exists an $S(P)$ in $\cup_{\alpha} Q_{\alpha}$. Hence $\cup_{\alpha} Q_{\alpha}$ is open .If $\{Q_{\alpha}\}, i = 1,..., n$ is a finite family of open sets then also the intersection $\{\cap Q_{\alpha}\}$ is open . For let P be in P then P is in each $\{Q_{\alpha 0}\}$, since the $\{Q_{\alpha}\}, i = 1,..., n$ are open there exist $S_{\varepsilon}(P)$ in each Q_{ε} . Now let $\varepsilon = \min(\varepsilon_{i})$ then $S_{\varepsilon}(P)$ in $\cap Q_{i}$ is open see fig.(1)

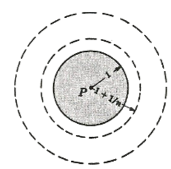


Fig. (1) : open disk

Theorem 2.1.1

Open sets in E have the following properties :

(i) E is open ϕ is open .

(ii) If Q_{α} are open , then $\cup Q_{\alpha}$ is open .

(iii) If Q_{i} , $i = 1,..., n$ are open then $\cap Q_{i}$ is open .

Also let P and Q be distinct points in E . Clearly by taking ε_{1} and ε_{2} sufficiently small say $\varepsilon_{1} = \varepsilon_{2} = 1/s|PQ|$ the neighborhoods $S_{\varepsilon_{1}}(P)$ and $S_{\varepsilon_{2}}(Q)$ are disjoint since neighborhoods are open sets .

Theorem 2.1.2

If Q and P are distinct points in E there exist open sets O_{p} and O_{Q} containing P and Q respectively such that $O_{p} \cap O_{Q} = \phi$.

2.3 Closed sets

A set S in E is closed if the set of points not in S is open , i.e S is closed if its complements S^{c} is open .

Example 2.1.3

(i) The closed interval $a \le x \le b$ in E^1 is closed if its complement is the union of the open sets $x \succ b$ and $x \prec a$ the interval $a \prec x \le b$ is neither open nor closed .
(ii) the set of rational points in the x_1, x_2 plane , i.e the set S of points (p,q) where p and q are rational numbers is neighborhood of a rational number contains an irrational number every $S(p,q)$ contains points not is S . Hence S is not open . Also since every neighborhood of an irrational number contains rational number the complement of S is not open. Hence S is not closed.
(iii) A set in E consisting of a single point is closed . Also a set in E consisting of any finite number of points is closed.
(iv) E is closed since ϕ is open ϕ is closed since E is open .
(v) A torus in E^3 is the surface in E^3 shown in which is obtained by revolving a circle about a line not passing through the circle . A torus in E^3 is closed . A point P is said to be an accumulation point or limit point of a set S in E^3 if every deleted spherical neighborhood $S'(P)$ of P contains at least one point of S .

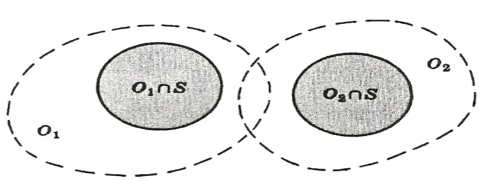
Theorem 2.1.4

A set in E is closed if and only if it contains its limit points the closure of set a set S , denoted by $\hat{S}$ is the set consisting of S and the set of limit points of S . As a solved problem we will show that $\hat{S}$ is the smallest closed set containing S that is (i) $\hat{S}$ is closed (ii) If T is closed and $S \subseteq T$ then $\hat{S} \subseteq T$.

Definition 2.1.5

Let a set S consist of tow disjoint closed disks in E^2 sine there is a nonzero distance between the disks, there exist open sets Q_1 and Q_2 whose union contains S such that their respective intersections with S are nonempty and disjoint . In general a set S in E is said to be disconnected if as above, there exist open sets Q_1 and Q_2 such that .

(1) $S \subseteq O_1 \cup O_2$ and $O_1 \cap S \ne \phi$, $O_2 \cap S \ne \phi$
Have nonempty intersections with S , and .
(2) $(O_1 \cap S) \cap (O_2 \cap S) = O_1 \cap O_2 \cap S = \phi$
The intersections of O_1 and O_2 with S are disjoint a nonempty set S is said to be connected if it is not disconnected see fig. (2).



Example 2.1.6

(i) A set consisting of a single point x_1 is arc wise connected $x_1 = cons\tan = x_1$.(ii) Clearly E itself is arc wise connected for the linear function . $x(t) = x_1 + t(x_2 - x_1)$, $0 \le t \le 1$ is a straight liner connecting any x_1 and x_2 .(iii) It can be shown that in E^n a set arc wise connoted if and if it is an interval . Thus in E^1 the connected and arc wise connected sets are the
If a set P belong to $S \cap Q_1$ let Q belong to $S \cap Q_2$ and let $x = x(t)$, $0 \le t \le 1$ be a continuous arc from P to Q_1 and Q_2 containing S . Now consider the real-valued function $f(t)$ on the interval $0 \le t \le 1$.

(3) $f(t) = \begin{cases} 1\ , & if\ x(t)\, is\ in\ S \cap O_1 \\ -1 & if\ x(t)\, is\ in\ S \cap O_2 \end{cases}$

Since $x(t)$ is in S and $S \subseteq O_1 \cup O_2$ if follows that f is defined for all t in $0 \le t \le 1$. It is also single-valued since $S \cap O_1$ and $S \cap O_2$ are disjoint .

Theorem 2.1.7

If a set S in E is arc wise connected then it connected. Although the converse of the above is true in E^1, it is not true for E in general as that is there are connected sets in E^2 which are not arc wise connected, However if S is connected and open in E then it is arc wise connected in E .

2.2Compact sets

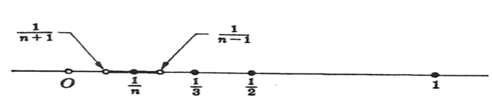
An open covering of set . S in E is a family of open sets whose union contains S . A sub covering is a subset of open covering with the same property and a finite converging is an open covering consisting of a finite number of set . Clearly for every set in E there exists an open covering namely the family consisting of only the set E itself. Now a set S in E is compact if for every open covering Q_α of S in E is compact if for every open covering Q_α of set S there exists a finite sub covering $Q_{\alpha 1},....., Q_{\alpha n}$.

Example 2.2.1

Let S be the infinite set $\{1, 1/2, 1/3, ...\}$ in E^1 shown in this set is not compact for we can exhibit an open covering of S which has no finite sub covering . Namely let $O_1 = \{1/2 \prec x \prec 2\}$ and let O_n denote the open interval .

(4) $\left(\dfrac{1}{n+1} \prec x \prec \dfrac{1}{n-1}\right)$

For $n \ge 2$ clearly O_n contains $1/n$ and so the infinite family O_n , $n = 1,...,$ is open covering of S see fig. (3)



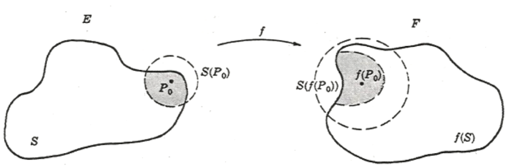
Theorem 2.2.2

A set in E is compact if and only if is closed and bounded.
A compact set S is also closed , for let Q be an arbitrary point in its complement S^c for each P in S there exists an $S(P)$ and an $S^p(Q)$ such that $S(p) \cap S^p(Q) = \phi$ clearly the family $\{S(P)\}$ is open covering of S and since S is compact there exists a finite sub-spending neighborhoods of Q . Note that O and open . But also .
(5)
$O \cap (\cup S\ (P_j) = \cup(O \cap S(P_j)) = \cup(\cap S^P(Q) \cap S(P_j))$

But $S^P(Q) \cap S(P_j) = \phi$ hence $O \cup (\cap S\ (P_j) = \phi$. since the $S(P_j)$ over S it follies that $O \cap S = \phi$ thus $Q \subseteq S^c$.

2.3 Continuous Mappings on Euclidean spaces

Let E and F be Euclidean spaces and S a subset of E . Let f be a mapping of S into F to each P in S there is assigned a point $f(P)$ in F . The mapping f is continuous at a point P_0 in S if as indicated below for every neighborhood $S(f(P_0))$ in F there exists a neighborhood $S(P_0)$ in E such that $f(P)$ is $S(f(P_0))$ for all P in $S(f(P_0)) \cap S$ or equivalently f is continuous at P_0 if for every $S(f(P_0))$ there exists an $S(P_0)$ such that.
(6) $S(f(P_0) \cap S \subseteq S(f(P_0))$
the mapping f is said to be continuous on S or simply continuous if it is continuous at each point in S see fig.(4)



2.3.1Example

(i) The constant mapping $f(p) = Q_0$ which assigns to each P of a set S in E is continuous on S for let be an arbitrary point in S and let $S(f(P_0)) = S(Q_0)$ be an arbitrary neighborhood of $f(P_0)$. But for all in S and hence for all P in any $S(f(P_0))$ of P_0 we have $f(P) = Q_0$ in $S(Q_0)$ thus f is continuous at P_0 is an arbitrary point in S, f is continuous on S .

2.3.2 Theorem

If C is a connected subset of space X and $f : X \to Y$ is continuous then $f(C)$ is connected in the space Y .

Proof:
If we assume $f(C)$ is not connected there exists a continues surjection $g : f(C) \to [a,b]$ where $[a,b]$ has the discrete topology according to $f(C)$ is continuous by the composition $g\ f : C \to [a,b]$ is a continuous surjection , implies C is not connected and thus contradicts the hypothesis of our assumption is false and $f(C)$ is connected .

2.3.3 Corollary

If $f : R^1 \to R^1$ is a continuous injective function then f is a homeomorphism from R^1 to $f(R^1)$.
Proof :
The fact f is injective given f a bijection from R^1 to $f(R^1)$ the fact that f is an open function given $f^{-1} : f(R^1) \to R^1$ continuous there f is a bijection where both f and f^{-1} are continuous implying that f is a homeomorphism by f .

2.3.4 Theorem

The space $X \times Y$ is locally connected if both of the spaces X and Y are locally connected.
Proof:
Suppose first that $X \times Y$ is locally connected . Let $a \in X$ and U be any open set in X containing then $U \times Y$ is open in $X \times Y$ and contain at least one point (a, y) such that $y \in Y$ consequently there is an open connected set V containing (a, y) such that $V \subset U \times Y$ the projection function $p_x : X \times Y \to X$ given $a \in p_x(V) \subset U$ where $p_x(V)$ is open and connected according .

2.3.5 Definition

Let $\{A_\alpha : \alpha \in \Delta\}$ be family of subset of the space X and $B \subset X$ the family $\{A_\alpha : \alpha \in \Delta\}$ cover B if $B \subset \cup A_\alpha$ if $\{\Delta\}$ is finite and $\{A_\alpha : \alpha \in \Delta\}$ covers B then $\{A_\alpha : \alpha \in \Delta\}$ is called a finite cover of B if each $\{A_\alpha : \alpha \in \Delta\}$ is open (called) in X and $\{A_\alpha : \alpha \in \Delta\}$ covers B . Then $\{A_\alpha : \alpha \in \Delta\}$ is called an open (closed) cover of B ,

2.3.6 Definition

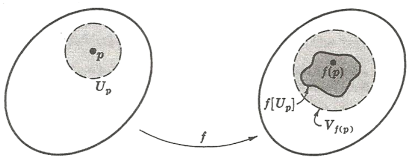
Let $\{A_\alpha : \alpha \in \Delta\}$ be a cover of $B \subset X$ then the family $\{A_\alpha : \beta \in \Omega \subset \Delta\}$ is a cover B .

2.3.7 Theorem

Every compact Hausdorff space is normal .
Proof :
Let A and B be any two disjoint closed subsets of the compact Hausdorff space X both A and B are compact , for $a \in A$ and the set B .

2.3.8 Definition

A function $f : R \to R$ is continuous at a point $p \in R$ if for any open set $V_{f(p)}$ containing $f(p)$ there exists an open set U_p containing p such that $f[U_\alpha] \in V_{f(p)}$ the function f is a continuous function if is continuous at every



2.4. Topological Differential Geometry

In this section is review of basic notions on differential geometry :

Definition 2.4.1 [Hausdrff on topological space]

A topological space M is called (Hausdorff) if for all $x, y \in M$ there exist open sets such that $x \in U$ and $y \in V$ and $U \cap V = \phi$

Definition 2.4.2 [Second countable]

A topological space M is second countable if there exists a countable basis for the topology on M .

Definition 2.4.3[Locally Euclidean of dimension N]

A topological space M is locally Euclidean of dimension n if for every point $x \in M$ there exists on open set $U \in M$ and open set $w \subset R^n$ so that U and W are (homeomorphic).

Definition 2.4.4

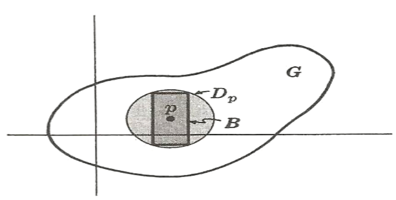
A topological manifold of dimension n is a topological space that is Hausdorff, second countable and locally Euclidean of dimension n .

Example 2.4.5

The open rectangles in plan R^3 bounded by sides parallel to the x-axis and y-axis also form a base β for the usual topology on R^2 for , let $G \in R^2$ be open and $p \in G$. Hence there exists an open disc $p \in D_p \subset G$ then any rectangle $B \in \beta$ whose vertices lie on the boundary of D_p satisfies .

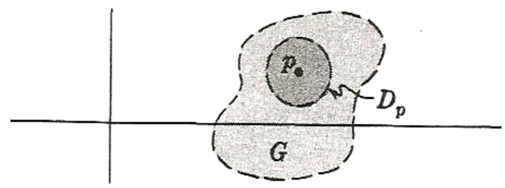
(7) $p \in B \subset D_p \subset G \ or \ p \in B \subset G$

As indicated in the diagram in other words β .fig(2)



Example 2.4.6

Consider the usual topology on the plane R^2 and any pints $p \in R^2$ then the class β_p of all open discs centered at p is a local base at p . For as proven previously any open set G containing p also contains as open disc D_p whose center is p .fig. (3)

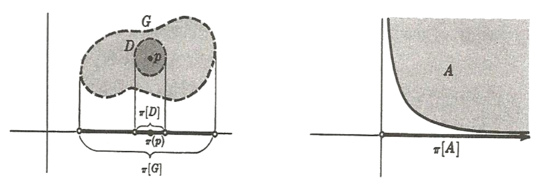


Example2.4.7

The projection mapping $\pi : R^2 \to R$ of the plane R^2 the x-axis , i.e $\pi \langle x, y \rangle = x$ that projection $\pi [D]$ of any open disc $D \subset R^2$ is an open interval . Hence any point $\pi [p]$ in the image $\pi [G]$ of an open set $G \subset R^2$ belongs to an open interval contained in $\pi [G]$ or $\pi [G]$ is open . Accordingly π is an open function . On the other hand π is not a closed function , for set .

(8) $A = \langle x, y \rangle : xy \ge 1, x \ge 0$

Is closed but its projection $\pi [A] = (0, \infty)$ is not close .fig.(4)



Definition 2.4.8

A smooth atlas A of a topological space M is given by : (i) An open covering $\{U\}_{i \in I}$ where $U_i \subset M$ Open and $M = \cup_{i \in I} U_i$

(ii) A family $\{\phi_i : U_i \to W_i\}_{i \in I}$ of homeomorphism ϕ_i onto open subsets $W_i \subset R^n$ so that if $U_i \cap U_j \ne \phi$ then the map

(9) $\phi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$

is (a diffoemorphism)

Definition 2.4.9

If $(U_i \cap U_j) \ne \phi$ then the diffeomorphism $\phi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is known as the transitition map .

Definition 2.4.10

A smooth structure on a Hausdorff topological space is an equivalence class of atlases, with two atlases A

and B being equivalent if for $(U_i,\varphi_i)\in A$ and $(V_j,\Psi_j)\in B$ with $U_i\cap V_j\neq\varphi$ then the transition map.

(10) $\phi_i(U_i\cap V_j)\to \Psi_j(U_i\cap V_j)$

is a diffeomorphism (as a map between open sets of R^n).

Example 2.4.11

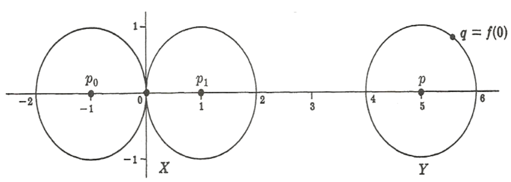
The subset of plan R^2 are homeomorphic , where the topologies are the gelatinized usual topologies fig.(4).

(11)

$X=\{x:d(x,p_0)=1 or\, d(x,p_1)=1,\, p_0=\langle 0,-1\rangle,\, p_1=\langle 0,1\rangle\}$

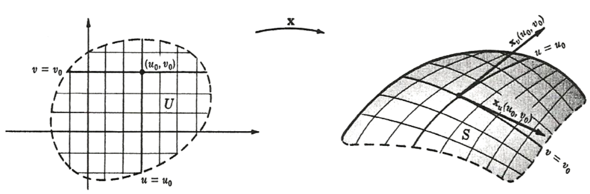
$X=\{x:d(x,p_0)=1 or\, d(x,p_1)=1,\, p_0=\langle 0,-1\rangle,\, p_1=\langle 0,1\rangle\}$

$Y=\{x:d(x,y)=1,\, p=\langle 0,5\rangle\}$



Definition 2.4.12

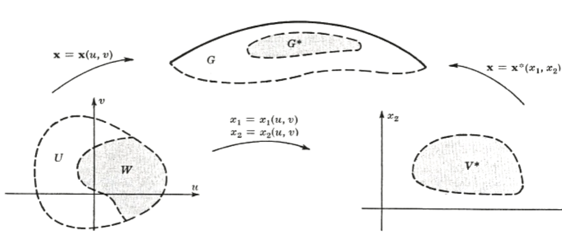
Suppose that $x=x(u,v)$ is a regular parametric representation of S defined on U as indicated in space that the image of the coordinate line $v=v_0$ in U will be a curve $x=x(u,v)$ on S along which u is a parameter . The curve is called the u-parameter on S called the v-parameter curves the image of coordinate lines v (constant) and u (constant)fig.(6).



Example 2.4.13

The there exists an open set W in U and a 1-1 mapping $x_1=x_1(u,v), x_2=x_2(u,v)$ of class C^m of W onto V^* where $\partial(x_1,x_2)/\partial(u,v)\neq 0$ for all (u,v) in U^* such that on W , $x=x(u,v)$ is composite mapping .

(12) $x=x^*(x_1(u,v),x_2(u,v))$ fig (7) .



Definition 2.4.14

A smooth manifold M of dimension n is a topological manifold of dimension n together with a smooth structure

Let M and N be two manifolds of dimension m,n respectively a map $F:M\to N$ is called smooth at $p\in M$ if there exist charts $(U,\phi),(V,\Psi)$ with $p\in U\subset M$ and $F(p)\in V\subset N$ with $F(U)\subset V$ and the composition.

(13) $\Psi\circ F\circ\phi^{-1}:\phi(U)\to\Psi(V)$

is a smooth (as map between open sets of R^n is called smooth if it smooth at every $p\in M$.

Definition 2.4.1.15

A map $F:M\to N$ is called a diffeomorphism if it is smooth objective and inverse $F^{-1}:N\to M$ is also smooth.

Definition 2.4.16

A map F is called an embedding if F is an immersion and homeomorphism onto its image

Definition 2.4.17

If $F:M\to N$ is an embedding then $F(M)$ is an immersed submanifolds of N .

2.5 Tangent space and vector fields

Let $C^\infty(M,N)$ be smooth maps from M and N and let $C^\infty(M)$ smooth functions on M is given a point $p\in M$ denote, $C^\infty(p)$ is functions defined on some open neighborhood of p and smooth at p .

Definition 2.5.1

(i) The tangent vector X to the curve $c:(-\varepsilon,\varepsilon)\to M$ at $t=0$ is the map $c(0):C^\infty(c(0))\to R$ given by the formula .

(14) $X(f)=c(0)(f)=\left(\frac{d(f\circ c)}{dt}\right)_{t=0}$

$\forall f\in C^\infty c(0)$

(ii) A tangent vector X at $p\in M$ is the tangent vector at $t=0$ of some curve $\alpha:(-\varepsilon,\varepsilon)\to M$ with $\alpha(0)=p$ this is

(15) $X=\alpha'(0):C^\infty(p)\to R$.

Remark 2.5.2

A tangent vector at p is known as a liner function defined on $C^\infty(p)$ which satisfies the (Leibniz property).

(16)

$X(f\,g)=X(f)g+f\,X(g)\ ,\forall f,g\in C^\infty(p)$

Definition 2.5.3 [Differential]

Given $F \in C^{\infty}(M,N)$ and $p \in M$ and $X \in T_p M$ choose a curve $\alpha :(-\varepsilon,\varepsilon) \to M$ with $\alpha(0) = p$ and $\alpha'(0) = X$ this is possible due to the theorem about existence of solutions of liner first order ODEs , then consider the map $F_{*p} : T_p M \to T_{F(p)} N$ mapping .

(17) $$X \to F_{*p}(X) = (F \circ \alpha)'(0)$$

this is liner map between two vector spaces and it is independent of the choice of α .

Definition 2.5.4

The liner map F_{*p} defined above is called the derivative or differential of F at p while the image $F_{*p}(X)$ is called the push forward X at $p \in M$.

Definition 2.5.5

Given a smooth manifold M a vector field V is a map $V : M \to TM$ mapping $p \to V(p) \equiv V_p$ and V is called smooth if it is smooth as a map from M to TM .

(Not) $X(M)$ is an R vector space for $Y, Z \in X(M)$, $p \in M$ and $a, b \in R$, $(aY + bZ)_p = aV_p + bZ_p$ and for $f \in C^{\infty}(M)$, $Y \in X(M)$ define $fY : M \to TM$ mapping $p \to (fY)_p = f(p) Y_p$

2.6 Cotangent space and Vector Bundles and Tensor Fields

Let M be a smooth n-manifolds and $p \in M$ .We define cotangent space at p denoted by $T_p^* M$ to be the dual space of the tangent space at $p : T_p^*(M) = \{ f : T_p M \to R \}$, f smooth Element of $T_p^* M$ are called cotangent vectors or tangent convectors at p .

(i) For $f : M \to R$ smooth the composition $T_p^* M \to T_{f(p)} R \cong R$ is called df_p and referred to the differential of f .Not that $df_p \in T_p^* M$ so it is a cotangent vector at p

(ii) For a chart (U, ϕ, x^i) of M and $p \in U$ then $\{dx^i\}_{i=1}^n$ is a basis of $T_p^* M$ in fact $\{dx^i\}$ is the dual basis of $\left\{ \frac{d}{dx^i} \right\}_{i=1}^n$.

Definition 2.6.1

The elements in the tensor product $V_s^r = V \otimes .... \otimes V \otimes V^* \otimes ... \otimes V^*$ are called (r, s) tensors or r-contravariant , s- contravariant tensor

Remark 2.6.2

The Tensor product is bilinear and associative however it is in general not commutative that is $(T_1 \otimes T_2) \neq (T_2 \otimes T_1)$ in general .

Definition 2.6.3

$T \in V_s^r$ is called reducible if it can be written in the form $T = V_1 \otimes ... \otimes V_r \otimes L^1 \otimes ... \otimes L^s$ for.

(18) $$V_i \otimes V_r , L^j \in V^*$$

for $1 \leq i \leq r$, $1 \leq j \leq s$.

Definition 2.6.4

Choose two indices (i, j) where $1 \leq i \leq r$, $1 \leq j \leq s$ for any reducible tensor let $C_i^r(T) \in V_{s-1}^{r-1}$ We extend this linearly to get a linear map $C_i^j : V_s^r \to V_{s-1}^{r-1}$ which is called tensor-contraction.

Remark 2.6.5

An ant symmetric (or alternating $(0,k)$ tensor) $T \in V_k^0$ is called a k-form on V and the space of all k-forms on V is denoted $\wedge^k V^* = \{ T \in V_k^0 : T \}$ alternating

Definition 2.6.6

A smooth real vector boundle of rank k denoted (E, M, π) is a smooth manifold E of dimension $n+1$

The total space a smooth manifold M of dimension n the manifold dimension $n+k$ and a smooth subjective map $\pi : E \to M$ (projection map) with the following properties :

(i) There exists an open cover $\{V_\alpha\}_{\alpha \in I}$ of M and diffeomorphisms $\Psi_\alpha : \pi^{-1}(V_\alpha) \to V_\alpha \times R^k$.

(ii) For any point $p \in M$, $\Psi_\alpha(\pi^{-1}(p)) = \{p\} \times R^k \cong R^k$ and we get a commutative diagram (in this case $\pi_1 : V_\alpha \times R^k \to V_\alpha$ is projection onto the first component .

(iii) whenever $V_\alpha \cap V_\beta \neq \phi$ the diffeomorphism.

(19) $$\psi_\alpha \circ \Psi_\beta^{-1} : (V_\alpha \cap V_\beta) \times R^k \to (V_\alpha \cap V_\beta) \times R^k$$

takes the form

$\Psi_\alpha \circ \Psi_\beta^{-1}(p, a) = (p, A_{\alpha\beta}(p)(a))$, $a \in R^k$ where $A_{\alpha\beta} : V_\alpha \cap V_\beta \to GL(k, R)$ is called transition maps.

2.7 Bundle Maps and isomorphisms

Suppose (E, M, π) and $(\tilde{E}, \tilde{M}, \tilde{\pi})$ are two vector bundles a smooth map $F : E \to \tilde{E}$ is called a smooth bundle map from (E, M, π) to $(\tilde{E}, \tilde{M}, \tilde{\pi})$.

(i) There exists a smooth map $f : M \to \tilde{M}$ such that the following diagram commutes that $\pi(F(q)) = f(\pi(q))$ for all $p \in M$

(ii) F induces a linear map from E_p to $\tilde{E}_{f(p)}$ for any $p \in M$.

Definition 2.7.1 [Dual Bundle]

Take a vector bundle (E, M, π) where $E : \cup_{p \in M} E_p$ replace E_p with its dual $E^*{}_p$ and consider $E^* : \cup_{p \in M} E^*{}_p$. Let $V_\alpha, \psi_\alpha, A_{\alpha\beta}$ by an in the transition maps for the dial bundle E^* are denoted $(A^{dual})_{\alpha\beta} = (A^{-1}_{\alpha\beta})^T$ observe that $(A^{dual})_{\alpha\beta} = (A^{dual})_{\beta\gamma}$.

Definition2.7.2 [Tensor product of vector Bundles]

Suppose (E, M, π) is vector bundle of rank k and $(\tilde{E}, \tilde{M}, \tilde{\pi})$ is vector bundle of rank l over the same base manifold M then define $E \otimes \tilde{E} = \cup_{p \in M} E_p \otimes \tilde{E}_p$, this is well defined because E_p and $\tilde{E}_p$ are vector spaces . Let be an open cover of $M, \Psi_\alpha, \tilde{\Psi}_\alpha, A_{\alpha\beta}, \tilde{A}_{\alpha\beta}$ be the local trivializations and transition maps to E and $\tilde{E}$ respectively then the transudation maps and local trivializations for $E \otimes \tilde{E}$ are given .

(20)

$$a \otimes \tilde{a} \to A_{\alpha\beta} a \otimes \tilde{A}_{\alpha\beta} \tilde{a} \in R^k \otimes R^l \cong R^{k+1}$$

$$, \forall a \in R^k, \tilde{a} \in R^l$$

Definition 2.7.3

Let $F : M \to N$ be a smooth map between two smooth manifolds and $w \in \Gamma(T_k^0 N)$ be a k covariant tensor field we define a k covariant tensor field $F^* w$ over M by

(21)

$$(F^* w)_p (v_1, \ldots, v_k) = w_{F(p)} (F_{*p}(v_1), \ldots, F_{*p}(v_k))$$

$$, \forall v_1, \ldots, v_k \in T_p M$$

In this case $F^* w$ is called the pullback of w by F .

Proposition 2.7.4

Suppose $F : M \to N$ is a smooth map and $G : N \to Q$ a smooth map for M, N, Q smooth manifolds and $w \in T(T_k^0 N), \eta \in T(T_l^0 N)$ and $f \in C^\infty(N)$ then .

(i) $(G \circ F)^* = F^* \circ G^*$.

(ii) $F^*(w \otimes \eta) Fw \otimes F^* w \otimes F^* \eta$ in particular , $F^*(f \circ w) = (f \circ F) F^* w$.

(iii) $F(df) = d(f \circ F)$ (iv) if $p \in M$ and (y^i) are local coordinates in a chart containing the point $F(p) \in N$ then

(22) .

$$F^*(w_{j1,\ldots,jk}\, dy^{i1} \otimes \ldots \otimes dy^{ik})$$

$$= (w_{j1,\ldots jk}, \circ F) d(y^{j1} \circ F) \otimes \ldots \otimes d(y^{ik} \circ F)$$

2.8 Exterior derivative

The exterior derivative is a map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ which is R linear such that $d \circ d = 0$ and if f is a k vector field on k then $(df)(X) = Xf$.

Definition 2.8.1[Integration of differential forms]

$\int_M w$ is well defined only if M is orient able $\dim(M) = n$ and has a partition of unity and w has compact support and is a differential n-form on M .

2.9 Riemannian Manifolds

An inner product (or scalar product) on a vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to R$ that is :

(i)symmetric $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$.

(ii)Bilinear $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$ and $\langle u, av + bw \rangle = a\langle u, v \rangle + b\langle u, w \rangle$ for all $a, b \in R$ and $u, v, w, \in V$.

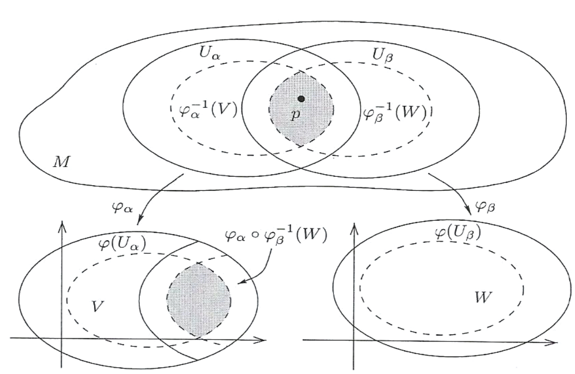
(iii) positive definite $\langle u, v \rangle \succ 0$ for all $u \neq 0$.

Definition2.9.1

A pair (M, g) of a manifold M equipped with a Riemannian metric g is called a Riemannian manifold.

Example 2.9.2

A Topological manifolds M can be covered by a single chart the smooth compatibilility condition is trivially satisfied so any that requirement that each transition map $\psi \circ \varphi^{-1}$ and its inverse of class C^K we obtain the definition of a C^K structure C^W is real manifolds fig(8) .



2.10 Length and Angle between tangent vectors

Suppose (M, g) is a Riemannian manifold and $p \in M$ we define the length (or norm) of a tangent vector $v \in T_p M$ to be $|v| = \sqrt{\langle v, v \rangle_p}$ Recall $g(\cdot,\cdot) = \langle \cdot,\cdot \rangle$ and the angle v, w between $v, w \in T_p M$ $(v \neq 0 \neq w)$ by

$$\cos(v, w) = \frac{\langle v, w \rangle_p}{|v||w|} \quad (3.13)$$

Definition 2.10.1

Let M and N be two manifolds of dimension m, n respectively a map $F : M \to N$ is called smooth at $p \in M$ if there exist charts $(U, \phi), (V, \Psi)$ with $p \in U \subset M$ and $F(p) \in V \subset N$ with $F(U) \subset V$ and the composition $\Psi \circ F \circ \phi^{-1} : \phi(U) \to \Psi(V)$ is a smooth (as map between open sets of R^n is called smooth if it smooth at every $p \in M$.

Definition 2.10.2

A map $F : M \to N$ is called a diffeomorphism if it is smooth objective and inverse $F^{-1} : N \to M$ is also smooth.

Definition 2.10.3

A map F is called an embedding if F is an immersion and homeomorphic onto its image

3.6.4 Definition

If $F : M \to N$ is an embedding then $F(M)$ is an immersed submanifolds of N .

2.11 [Tangent space and vector fields]

Let $C^{\infty}(M, N)$ be smooth maps from M and N and let $C^{\infty}(M)$ smooth functions on M is given a point $p \in M$ denote, $C^{\infty}(p)$ is functions defined on some open neighbourhood of p and smooth at p .

Definition 2.11.1

(i) The tangent vector X to the curve $c : (-\varepsilon, \varepsilon) \to M$ at $t = 0$ is the map $c(0) : C^{\infty}(c(0)) \to R$ given by the formula

(23) $X(f) = c(0)(f) = \left(\frac{d(f \circ c)}{dt} \right)_{t=0} \quad \forall f \in C^{\infty} c(0)$

(ii) A tangent vector X at $p \in M$ is the tangent vector at $t = 0$ of some curve $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$ this is $X = \alpha'(0) : C^{\infty}(p) \to R$.

Remark 2.11.2

A tangent vector at p is known as a liner function defined on $C^{\infty}(p)$ which satisfies the (Leibniz property)

$X(fg) = X(f)g + f X(g) \quad , \forall f, g \in C^{\infty}(p)$
(3.15)

Defintion 2.11.3 [Differential duel spaces]

Given $F \in C^{\infty}(M, N)$ and $p \in M$ and $X \in T_p M$ choose a curve $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$ and $\alpha'(0) = X$ this is possible duel to the theorem about existence of solutions of liner first order ODEs , then consider the map $F_{*p} : T_p M \to T_{F(p)} N$ mapping $X \to F_{*p}(X) = (F \circ \alpha)'(0)$, this is liner map between two vector spaces and it is independent of the choice of α .

Definition 2.11.4

The liner map F_{*p} defined above is called the derivative or differential of F at p while the image $F_{*p}(X)$ is called the push forward X at $p \in M$.

2.12 [Locally Euclidean of dimension N]

A topological space M is locally Euclidean of dimension n if for every point $x \in M$ there exists on open set $U \in M$ and open set $w \subset R^n$ so that U and W are (homeomorphic).

Definition 2.12.1

A topological manifold of dimension n is a topological space that is Hausdorff, second countable and locally Euclidean of dimension N .

Definition 2.12.2

A smooth atlas A of a topological space M is given by : (i) An open covering $\{U\}_{i \in I}$ where $U_i \subset M$ Open and $M = \cup_{i \in I} U_i$

ii) A family $\{\phi_i : U_i \to W_i\}_{i \in I}$ of homeomorphism ϕ_i onto open subsets $W_i \subset R^n$ so that if $U_i \cap U_j \neq \phi$ then the map

(24) $\phi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$

is (a diffoemorphism)

Definition 2.12.3

If $(U_i \cap U_j) \neq \phi$ then the diffeomorphism $\phi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is known as the(transitition map).

III.LOWER ESTIMATES FOR DIRAC OPERATOR

3.1[Lower Estimates For The Eigenvalues Of The Dirac Operator]

In this section we will consider a compact Riemannian manifold (M^n, g) with fixed spin structure and its Dirac operator D which in this case ,is exclusively determined by the *Levi-Givita* connection .by integration from the *schrodinger – Lichnerowicz* formula $D^2 = \Delta + \frac{1}{4}R$ We immediately obtain inequality $\lambda^2 \geq \frac{1}{4}R_0$,for every eigenvalue λ of the Dirac operator ,where $R_0 = \min\{R(m) : m \in M^n\}$ is the minimum of the scalar curvature . However ,this estimateis not optimal

Proposition 3.1.1 :[The Spin of Compact Riemannian Manifold]

Let (M^n, g) be a compact Riemannian manifold with spin structure , and λ an eigenvalue of Dirac operator D ,then

(25) $$\lambda^2 \geq \frac{1}{4}\frac{n}{n-1}R_0$$

Moreover if $\lambda = \pm\frac{1}{2}\sqrt{\frac{n}{n-1}R_0}$ is an eigenvalue ,then Dirac operator and ψ a corresponding eigenvalue , then ψ as a solution of field equation .

(26) $$\nabla_x \psi = \pm\frac{1}{2}\sqrt{\frac{R_0}{n(n-1)}}\, X.\psi$$

And the scalar curvature R is constant

Proof :

The proof is based on not using the *Levi-civita* connection but , instead considering a suitably modified covariant derivative in the spin or bundle , to this end fix a real –valued function $f : M^n \to R^1$ and introduce the covariant derivative ∇f in the spinor bundle S by the formula introduce the covariant derivative ∇f in the spinor bundle S by formula .

(27) $$\nabla f_x \psi = \nabla_x \psi + f(X.\psi)$$

The algebraic properties of multiplication imply that ∇f is metric covariant derivative in the spinor bundle S .

(28) $$X(\psi,\psi) = \nabla f_x(\psi_1,\psi) + \psi_1(f_x \nabla,\psi)$$

Let ,$\nabla f = -\sum_{i=1}^{n} \nabla^j_{e^i} \nabla^j_{e^i} - \sum_{i=1}^{n} div(e^i)\nabla^j_{e^i}$ be the corresponding Laplace operator , and denote by

(29)

$$|\nabla f \psi| = \sum_{i=1}^{n} |\nabla^f_{e^i}\psi| = |\nabla^f_{e^i}\psi|^2 = |\nabla_{e^i}\psi + f_{e^i}\psi|^2$$

The length of the 1-form $\nabla^f \psi$ we will compute the operator $(D-f)^2$ first .

$$(D-f)^2 = (D-f)(D-f) = D - 2fD - \mu$$

And the *schrodinger – Lichnerowicez* formula implies .

$$(D-f)^2 = \Delta + \frac{1}{4}R - 2fD - grad(f) + f^2$$

On the other hand.

(30)

$$\nabla^f = -\sum_{i=1}^{n}(\nabla_{e^i} + f_{e^i})(\nabla_{e^i} + f_{e^i}) - \sum_{i=1}^{n} div(e^i)(\nabla_{e^i} + f_{e^i})$$
$$= \Delta - 2fD - (-grad(f)) + nf^2$$

Summing up , this yields .

$$(D-f)^2 = \Delta^f + \frac{1}{4}R + (1-n)f^2$$

and by integration over M^n ,we obtain the formula Suppose now that $D\psi = \lambda\psi$,then we can insert the function $f = \frac{\lambda}{n}$ into the last formula and obtain

$$\lambda^2\left(\frac{n-1}{n}\right)^2 \|\psi\|^2_{L_2} = \left\|\nabla^{\frac{\lambda}{n}}\psi\right\|^2_{L2} + \lambda^2\left(\frac{n-1}{n^2}\right)$$

An algebraic transformation yields .

(31)

$$\lambda^2\frac{n-1}{n}\|\psi\|^2_{L2} = \left\|\nabla^{\frac{\lambda}{n}}\psi\right\|^2_{L_2} + \frac{1}{4}\int_{M^n} R|\psi|^2 \geq \frac{1}{4}R_0\|\psi\|^2_{L_2}$$

$\lambda^2 \geq \frac{1}{4}\frac{n-1}{n}R_0$, discussing the boundary case in estimate we immediately obtain the remaining assertions of the proposition the method of proof applied here may be refined in various ways . Consider ,for example for fixed smooth real –valued function $f : M^n \to R^1$, the (non-metric)covariant derivative

$$\nabla_x \psi = \nabla_x \psi + \frac{\lambda}{n}X,\psi + \mu X(grad(f))\psi + vd$$

With the " optimal " parameters

$$\mu = -\frac{1}{n-1}, v = -\frac{n}{n-1}$$

and perform a calculation with the length $\|e^{\mu f}\nabla_x \psi\|^2_{L_2}$ similar to the one in the proof a above then one obtains the inequality .

Proposition 3.1.2

$$\lambda^2\frac{n}{n-1}\min\left\{\frac{1}{4}R + \Delta(f) - \frac{n-2}{n-1}|grad|^2\right\}$$

In particular , in dimension the summand $|grad|^2$ drops out .then the formula simplifies to

$\lambda^2 \geq \min\left\{\frac{1}{4}R + 2\Delta(f)\right\}$ The curvature K of the Riemann surface $\left(M^2, g\right)$ is equal to , and we can choose f as a solution to the differential equate .
(32)

$$2\Delta(f) = -k + \frac{1}{vol(M^2, g)}\int_{M^2} k = -k + \frac{2\pi X(M^2)}{vol(M^2)}$$

Thus $\frac{1}{2}R + 2\Delta(f) = \frac{2\pi X(M^2)}{vol(M^2)}$ is constant ,and we obtain $\lambda^2 \geq \frac{2\pi X(M^2)}{vol(M^2)}$. Of course , the last inequality is interesting only for 2-dimensional Riemannian manifolds which ,topologically are sphere . Summarizing, we obtain the following proposition originally due to, Hijazi ,and Bar

Proposition3.1.3 [Dirac Operator]
If $\left(S^2, g\right)$ is a Riemannian metric on S^2 ,then for the first eigenvalue of the Dirac operator ,we have .

(33) $$\frac{\lambda^2 \geq 4\pi}{vol(S^2, g)}$$

The method we have outlined for estimating the eigenvalues of the Dirac operator may be refined even further when the Riemannian manifold carries additional geometric structures. Let us consider e.g .the case of kahler manifold $\left(M^{2k}, J, g\right)$ with complex structure $J = T\left(M^{2k}\right)$ .In this situation consider the covariant derivative .

(34) $$\nabla_x \psi = \nabla_x \psi + f\, X, \psi + h_j(X).\psi$$

Depending on two parameters f and h which can be chosen freely .Elaborating on the weitzenbock formulas for Riemannian manifolds with additional geometric structures one will in general case of a Riemannian manifold .For example the following inequality first proved by $k - D$, *kirchberg* ,holds for killer manifolds

Proposition 3.1.4
Let $\left(M^{2k}, J, g\right)$ be a compact kahler spin manifold and λ an eigenvalue of the Dirac operator ,then
(35)

$$\lambda^2 \geq \left\{\frac{1}{4}\frac{k+1}{k}R_0 \quad if \ k = \dim M_2\right.$$
$$,\left\{\frac{1}{4}\frac{k}{k-1} \quad if \ k = \dim M_2\right.$$

Remark 3.1.5
The kahler case has been investigated by Kramer , semmelmann ,and Weingarten

3.2 [Riemannian Manifolds With Killing Spinors]
By the proposition manifold in (2.5). A spinor field ψ which is an eigenspoinor for the eigenvalue . $\pm\frac{1}{2}\sqrt{\frac{n}{n(n-1)}}R_0$ solves the stronger field equation.

(36) $$\nabla_x \psi = \pm\frac{1}{2}\sqrt{\frac{R_0}{n(n-1)}}X, \psi$$

This leads to general notion of killing spinors .

Definition 3.2.1 [Riemannian Spin manifolds is called Killing]
A spinor field ψ defined on a Riemannian spin manifolds (M^2, g) is called a killing spinor , if there exists a complex number μ such that $\nabla_x \psi = \mu X.\psi$. For all vector $X \in T_\mu$, it self is called killing number of ψ we begin by listing afew elementary properties of killing spinors .

Proposition 3.2.2
Let (M^2, g) be a connect Riemannian manifold
(i)A not identically vanishing killing spinor has no zeroes
(ii) Every killing spinor ψ belongs to the kernel of twistor operator T. Moreover , ψ is an eigenspinor of the Dirac operator $D(\psi) = -n\mu\psi_0$ If ψ is killing spinor cooresponding to a real killing number $\mu \in R^1$, then the vector field .

(37) $$V^\psi = \sum_{i=1}^{n}\left(e_i, \psi, \psi\right)e_i$$

Is a killing vector field of the Riemannian manifold (M^2, g)

Proof : A killing spinor restricted to the curve $r(t), \psi(t) = \psi(r(t))$ satisfies the following first ordinary differential equation a long curve $\frac{d}{dt}\psi(t) = \mu\, r(t).\psi(t)$

Now : $\psi(0) = 0$, immediately implies $\psi(r(t)) \equiv 0$, and this in turn yields starting form $\nabla_x \psi = \mu X, \psi$ .we compute $D\psi = \Sigma_{i=1}^{n} e_i \nabla_{e_i}\psi = \mu\sum_{i=1}^{n} e_i.e_i.\Psi = -n\mu\psi$

And thus obtain .
(38)

$$T(\psi) = \Sigma_{i=1}^{n} e_i \otimes \left(\nabla_{e_i}\psi + \frac{1}{n}e_i D\psi\right)$$

For a fixed point $m_0 \in M^2$,and a local orthonormal frame $e_1, \ldots, e_n$ with $\nabla_{e_i}(m_0) = 0$,we copute the covariant derivative $\nabla_x V^\psi$.

$$\nabla_x V^{\psi} = \Sigma\left(e_i, \nabla_x \psi, \psi\right) e_i + \Sigma\left(e_i, \psi, \nabla_x \psi\right)$$
$$= \left(\mu \Sigma_{i=1}^{n} \left(e_i . X - X_{e_i}\right) \psi, \psi\right)_{e_i}$$

This implies $g(\nabla_x V^{\psi}, Y) = \mu(YX - XY)\psi, \psi$ .hence $g\left(\nabla_x V^{\psi}, \psi\right)$ is antisymmetric in X, Y . But this property characterizes killing vector field on a Riemannian manifold .

Not : every Riemannian manifold allows killing spinners $\psi \neq 0$ and not every number $\nabla_x \mu \in c$,occurs as a killing number we derive a series of necessary conditions .To this end , recall ,weg1 tensor of a Riemannian manifold .

Let .

(40)

$$R_{i,j,k,l} = g\left(\nabla_{e_i} \nabla_{e_j} e_k - \nabla_{e_j} \nabla_{e_i}\right) - \nabla_{[e_e\, e_j]} e_k e_j$$

Be the components of curvature tensor and .

$$R_{i,j} = \sum_{\alpha=1}^{n} R_{\alpha_i} R_{\alpha_j}$$

Those of the Ricci tensor . Then define two new tensor k ,and W by.

(41) $$k_{i,j} = \frac{1}{n-2}\left\{\frac{R}{2(n-1)} g_{i,j} - R_{i,j}\right\}$$

$$W_{\alpha,\beta,\delta} = R_{\alpha,\beta,r,\delta} - g_{\beta,\delta} \cdot k_{\alpha,r} - g_{\alpha,r} \cdot k_{\alpha,r} - g_{\alpha,r} \cdot k_{\beta,\delta} + g_{\beta,r} k_{\alpha,\delta,\beta}$$

W is called the "weg1 tensor " of the Riemannian manifolds because of its symmetry properties the tensor can be considered as a bundle morphism defined on the z-forms of

$$(M^n, g)\ W : \wedge^2 (M^n) \rightarrow \wedge^2 (M^n)$$
$$W(e_i \wedge e_j) = \sum_{k \leq 1} W_{ijkl} e_k \wedge e_i$$

With these notations we have the following

proposition 3.2.3

Let (M^n, g) be a connected Riemannian manifolds spin with a non-trivial killing spinor ψ , for the killing number μ . Then.(i) $\mu^2 = \frac{1}{4}\frac{1}{4n(n-1)} R$ at each point , in particular ,the scalar curvature of (M^n, g) is constant and μ is either real or purely imaginary (ii) (M^n, g) is an Einstein space (iii) $W(w^2)\psi$ or every 2-form , $W^2 \in \wedge(M^n)$

Proof :

$\nabla_x \psi = \mu X \psi$ implies**.**

(42) $\nabla_x \nabla_y \psi = \mu(\nabla_x Y)\psi + \mu^2 Y X \psi$

and hence .

(43)

$$\left(\nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]}\right)\psi = \mu^2 (X Y - Y X)\psi$$

Computing

$$\sum_{\alpha=1}^{n} e_\alpha R(X, e_\alpha)\psi$$

now yields

$$\sum_{\alpha=1}^{n} e_\alpha R(X, e_\alpha)\psi = \mu^2 \sum_{\alpha=1}^{n} e_\alpha \left(e_\alpha X - X e_\alpha\right)\psi$$

On the other hand , we proved the formula

(44) $$\sum_{\alpha=1} e_\alpha R(X, e_\alpha)\psi = \frac{1}{2} Ric(X)\psi$$

Hence , $Ric(X)\psi = 4(n-1)\mu^2 X \psi$, and since ψ does vanish at any point ,this implies $Ric(X)\psi = 4(n-1)\mu^2 X\psi$. Thus (M^n, g) is an Einstein space of scalar curvature tensor $R(X,Y)Z$ of the Riemannian manifold (M^n, g) the formula .

$R(X,Y)\psi = \frac{1}{4}\sum_{\alpha=1}^{n} e_\alpha R(X,Y)_{e_\alpha}\psi$ Hence ,because $4\mu^2 = \frac{R}{n(n-1)}$ equation

(45)

$$\nabla_x \nabla_y \psi - \nabla_y \nabla_x \psi - \nabla_{[x,y]}\psi = \mu^2 (X Y - Y X)$$

Can also written as

$$\left\{\sum_{\alpha=1}^{n} e_\alpha R(X,Y)_{e_\alpha} + \frac{R}{n(n-1)}(XY - YX)\psi = 0\right\}$$

And ,for an Einstein space

$$\sum_{\alpha=1}^{n} e_\alpha \wedge R(X,Y)_{e_\alpha} + \frac{R}{n(n-1)}(X \wedge Y - Y \wedge X)$$

Coincides with $W(X \wedge Y)$ .This ,eventually ,implies $W(w^2)\psi = 0$

From the proof of the preceding we can also deduce the following geometrical property of manifolds with killing spinors .

Proposition 3.2.4 [Riemannian Spin Manifold is Killing Spinor]

A Riemannian spin manifold admitting a killing spinor $\psi \neq 0$, with killing number $\mu \neq 0$ is locally irreducible

Proof :

If M^2 is locally the Riemannian product $M = M_1^k \times M_2^{n-k}$ then we may consider vector X, Y tangent to M_1^k and M_2^{n-k} , respectively this implies $R(X,Y)Z = 0$,and from

(46)

$$\left[\sum_{\alpha=1}^{n} e_\alpha R(X,Y)_{e_\alpha} + \frac{R}{n(n-1)}(X Y - Y X)\right]\psi = 0$$

We obtain $R.X.Y.\psi = 0$ Since $\mu \neq 0$,the scalar curvature is different from zero moreover , X and Y ,are orthogonal vector . But this implies

$\psi = 0$,hence a contradiction.The next-to-last proposition shows that killing are divided into two types , depending on whether the killing number μ real or imaginary $\mu \neq 0$ Real killing spinors $\overset{MER}{\rightarrow} M^n$ is an Einstein space of scalar curvature $R \geq 0$ (ii) Imaginary killing spinors $\overset{MEIR}{\rightarrow} M^n$ is an Einstein space scalar curvature $R \leq 0$ (iii)since $R = 4n(n-1)\mu^2$ real killing spinors precisely correspond to eigenspinors of the Dirac operator for the eigenvalue $\pm \frac{1}{2}\sqrt{\frac{n}{n-1}}R$ .The field equation $\nabla_X \psi = \mu X, Y$ could be generalized by allowing $\mu = M^n \rightarrow c$, to a result by " A.Licherowicz" this does not to an actual generalization

Proposition 3.2.5

Let (M^n, g) ,be a connected spin manifold $\mu : M^n \rightarrow c$ a smooth function and Ψ a non-trivial solution of the equation,

(47) $\nabla_x \psi = \mu X, Y$

If the real part , $Re(\mu) \neq 0$, is not identically zero ,then μ is constant and real Hence Ψ is real killing spinor in low dimension $n = \dim (M^n)$, the geometrical conditions for existence of real or imaginary killing spinors , respectively ,are rather restrictive and for $n \leq 4$ only Riemannian space of constant sectional curvature admit ,this kind of spinor fields . Consider e.g .the case $n = 3$ .then (M^3, g) is necessarily a 3-dimensional . Einstein space ,hence a space form we meet the same situation in dimension $n = 4$.

Proposition 3.2.6

Let (M^4, g) be a connected spin manifold with a non-trivial killing spinor Ψ ,for the killing number $\mu \neq 0$. Then (M^4, g) is space of constant sectional curvature .

Proof :

Decompose the killing spinor $\psi = \psi^+ + \psi^-$, according the splitting of the spinor bundle $S = S^+ \oplus S^-$,the equation for the killing spinor then takes the form

$$\nabla_x \psi = \mu X \psi^-, \mu X \psi^+$$

Defined the

(48)

$$N = \{ m \in M^4 : \psi^+(m) = 0 \; or \; \psi^-(m) = 0 \}$$

set $N \subset M^4$ is a closed subset without inner points . Indeed if N has inner points , then there is an open subset $U \subset N \subset M^n$ e.g ψ^+ vanishes , $\psi|_u^+ = 0$ this implies $\psi|_u^+ \equiv 0$ and sine $\mu \neq 0$,from the killing equation we obtain $\psi|_u^+ \equiv 0$,Hence $\psi = \psi^+ + \psi^-$, vanishes identically on the subset U , a contradiction to the fact prove above that non-trivial killing spinors have no zeroes thus $U = M^4 / N$,is a dense open subset of M^4 the condition on the "wey| tensor" w of M^4 now take the form $W(w^2)\psi^- = 0$ and $W(w^2)\psi^+ = 0$ However , a simple algebraic computation using the realization of the C_4 module $\Delta_4 = \Delta_4^+ \oplus \Delta_4^-$ .explicitly given following fact .

If $\psi^2 \in \wedge^2(R^4)$ is 2-form and $\psi^+ \in \Delta_4^+, \psi^- \in \Delta_4^-$ are two non-trivial spinors , then $\psi^+ = 0$ implies that the 2-form ψ^2 is trivial $\psi^2 = 0$

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