Applications Geometry on Hypes faces in Riemannian Manifolds

Mohamed M.Osman

Department of mathematics faculty of science
University of Al-Baha – Kingdom of Saudi Arabia

Abstract *In this paper uniform contractible Geometry on Euclidean space is mod k ,we also construct a pair uniformly contractible Riemannian metrics on R^n so that the resulting manifolds M and M' are bounded is close to a homeomorphism and a proof the Laplace operator on compact Riemannian manifolds.*

Keywords *Basic differential geometry – The injective manifolds – geometric maximum principles – Lorentzian manifolds .*

I. Introduction

This paper is a contribution to the collection of problems that Riemannian geometry with boundary, in the Euclidean domain the interior eigenvalue problem for the Laplacian boundary condition and Neumann eigenvalue geometry is given ,flat and trivial, and the interesting phenomena come from the shape of the boundary ,Riemannian manifolds have no boundary, and the geometric phenomena are those of the interior is called differential geometry .

II. A BASIC NOTIONS ON DIFFERENTIAL GEOMETRY

2.1 First principles a basic notions

Definition 2.1.1 Topological Spaces

A topological space M is called (Hausdorff) if for all $x, y \in M$ there exist open sets such that $x \in U$ and $y \in V$ and $U \cap V = \phi$

Definition 2.1.2

A topological space M is second countable if there exists a countable basis for the topology on M .

Definition 2.1.3

A topological space M is locally Euclidean of dimension n if for every point $x \in M$ there exists on open set $U \in M$ and open set $w \subset R^n$ so that U and W are (homeomorphic).

Definition 2.1.4

A topological manifold of dimension n is a topological space that is Hausdorff, second countable and locally Euclidean of dimension n .

Definition 2.1.5

A smooth atlas A of a topological space M is given by :

(i) An open covering $\{U\}_{i\in I}$ where $U_i \subset M$ Open and $M = \cup_{i\in I} U_i$

(ii) A family $\{\phi_i : U_i \to W_i\}_{i\in I}$ of homeomorphism ϕ_i onto open subsets $W_i \subset R^n$ so that if $U_i \cap U_j \neq \phi$ then the map $\phi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is (a diffoeomorphism)

Definition 2.1.6 Diffeomorphism

If $(U_i \cap U_j) \neq \phi$ then the diffeomorphism $\phi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is known as the (transitition map).

Definition 2.1.7

A smooth structure on a Hausdorff topological space is an equivalence class of atlases, with two atlases A and B being equivalent if for $(U_i, \varphi_i) \in A$ and $(V_j, \Psi_j) \in B$ with $U_i \cap V_j \neq \varphi$ then the transition map.

(1) $$\phi_i(U_i \cap V_j) \to \Psi_j(U_i \cap V_j)$$

is a diffeomorphism (as a map between open sets of R^n).

Definition 2.1.8 Smooth manifolds

A smooth manifold M of dimension n is a topological manifold of dimension n together with a smooth structure .

Definition 2.1.9

Let M and N be two manifolds of dimension m, n respectively a map $F : M \to N$ is called smooth at $p \in M$ if there exist charts $(U, \phi), (V, \Psi)$ with $p \in U \subset M$ and $F(p) \in V \subset N$ with $F(U) \subset V$ and the composition $\Psi \circ F \circ \phi^{-1} : \phi(U) \to \Psi(V)$ is a smooth (as map between open sets of R^n is called smooth if it smooth at every $p \in M$.

Definition 2.1.10

A map $F : M \to N$ is called (a diffeomorphism)if it is smooth bijective and inverse $F^{-1} : N \to M$ is also smooth.

Definition 2.1.11

A map F is called an embedding if F is an immersion and homeomorphic onto its image

Definition 2.1.12

If $F : M \to N$ is an embedding then $F(M)$ is an immersed submanifolds of N .

Definition 2.1.13

Let $C^{\infty}(M, N)$ be smooth maps from M and N and let $C^{\infty}(M)$ smooth functions on M is given a point $p \in M$ denote, $C^{\infty}(p)$ is functions defined on some open neighbourhood of p and smooth at p .

Definition 2.1.14

(i) The tangent vector X to the curve $c:(-\varepsilon, \varepsilon) \to M$ at $t = 0$ is the map $c(0): C^{\infty}(c(0)) \to R$ given by the formula .

(2) $$X(f) = c(0)(f) = \left(\frac{d(f \circ c)}{dt}\right)_{t=0} \quad \forall f \in C^{\infty} c(0)$$

(ii) A tangent vector X at $p \in M$ is the tangent vector at $t = 0$ of some curve $\alpha :(-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$ this is $X = \alpha'(0): C^{\infty}(p) \to R$.

Remark 2.1.15

A tangent vector at p is known as a liner function defined on $C^{\infty}(p)$ which satisfies the (Leibniz property)

(3)

$$X(fg) = X(f)g + fX(g) \quad , \forall f, g \in C^{\infty}(p)$$
.

Definition 2.1.16

Given $F \in C^{\infty}(M, N)$ and $p \in M$ and $X \in T_p M$ choose a curve $\alpha :(-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$ and $\alpha'(0) = X$ this is possible due to the theorem about existence of solutions of liner first order ODEs , then consider the map $F_{*p} : T_p M \to T_{F(p)} N$ mapping $X \to F_{*p}(X) = (F \circ \alpha)'(0)$, this is liner map between two vector spaces and it is independent of the choice of α .

Definition 2.1.16

The liner map F_{*p} defined above is called the derivative or differential of F at p while the image $F_{*p}(X)$ is called the push forward X at $p \in M$.

Definition 2.1.17

Given a smooth manifold M a vector field V is a map $V : M \to TM$ mapping $p \to V(p) \equiv V_p$ and V is called smooth if it is smooth as a map from M to TM .

$X(M)$ is an R vector space for $Y, Z \in X(M)$, $p \in M$ and $a, b \in R$, $(aY + bZ)_p = aV_p + bZ_p$ and for $f \in C^{\infty}(M)$, $Y \in X(M)$ define $fY : M \to TM$ mapping $p \to (fY)_p = f(p)Y_p$

Definition 2.1.18 Cotangent space and Bundles

Let M be a smooth n-manifolds and $p \in M$ .We define cotangent space at p denoted by $T_p^* M$ to be the dual space of the tangent space at $p : T_p^*(M) = \{ f : T_p M \to R \}$, f smooth Element of $T_p^* M$ are called cotangent vectors or tangent convectors at p .

(i) For $f : M \to R$ smooth the composition $T_p^* M \to T_{f(p)} R \cong R$ is called df_p and referred to the differential of f .Not that $df_p \in T_p^* M$ so it is a cotangent vector at p

(ii) For a chart (U, ϕ, x^i) of M and $p \in U$ then $\{dx^i\}_{i=1}^n$ is a basis of $T_p^* M$ in fact $\{dx^i\}$ is the dual basis (4) $\left\{ \frac{d}{dx^i} \right\}_{i=1}^n$

Definition 2.1.19

The elements in the tensor product

$$V_s^r = V \otimes .... \otimes V \otimes V^* \otimes ... \otimes V^*$$

are called (r, s) tensors or r-contravariant , s-contravariant tensor .

Remark 2.1.20

The Tensor product is bilinear and associative however it is in general not commutative that is

$$(T_1 \otimes T_2) \neq (T_2 \otimes T_1)$$

in general .

Definition 2.1.21

$T \in V_s^r$ is called reducible if it can be written in the form

$$T = V_1 \otimes ... \otimes V_r \otimes L^1 \otimes ... \otimes L^s$$

for.

(5) $V_i \otimes V_r , L^j \in V^* \quad 1 \le i \le r , 1 \le j \le s$

Definition 2.1.22

Choose two indices (i, j) where $1 \le i \le r , 1 \le j \le s$ for any reducible tensor $T = V_1 \otimes ... \otimes V_r \otimes L^1 \otimes .... \otimes L^2$ let

$C_i^r(T) \in V_{s-1}^{r-1}$ We extend this linearly to get a linear map $C_i^j : V_s^r \to V_{s-1}^{r-1}$ which is called tensor-contraction.

Remark 2.1.23

An ant symmetric (or alternating $(0,k)$ tensor) $T \in V_k^0$ is called a k-form on V and the space of all k-forms on V is denoted . $\wedge^k V^* = \{T \in V_k^0 : T\}$ alternating .

Definition 2.1.14

A smooth real vector boundle of rank k denoted (E, M, π) is a smooth manifold E of dimension $n+1$

The total space a smooth manifold M of dimension n the manifold dimension $n+k$ and a smooth subjective map $\pi : E \to M$ (projection map) with the following properties :

(i) There exists an open cover $\{V_\alpha\}_{\alpha \in I}$ of M and diffeomorphisms $\Psi_\alpha : \pi^{-1}(V_\alpha) \to V_\alpha \times R^k$.

(ii) For any point

(6) $p \in M$, $\Psi_\alpha(\pi^{-1}(p)) = \{p\} \times R^k \cong R^k$

and we get a commutative diagram (in this case $\pi_1 : V_\alpha \times R^k \to V_\alpha$ is projection onto the first component .

(iii) whenever $V_\alpha \cap V_\beta \neq \phi$ the diffeomorphism.

$$\psi_\alpha \circ \Psi_\beta^{-1} : (V_\alpha \cap V_\beta) \times R^k \to (V_\alpha \cap V_\beta) \times R^k$$

takes the form .

(7)

$\Psi_\alpha \circ \Psi_\beta^{-1}(p, a) = (p, A_{\alpha\beta}(p)(a)), a \in R^k$

where $A_{\alpha\beta} : V_\alpha \cap V_\beta \to GL(k, R)$ is called transition maps.

2.2 Bundle Maps and isomorphisms

Suppose (E, M, π) and $(\tilde{E}, \tilde{M}, \tilde{\pi})$ are two vector bundles a smooth map $F : E \to \tilde{E}$ is called a smooth bundle map from (E, M, π) to $(\tilde{E}, \tilde{M}, \tilde{\pi})$.

(i) There exists a smooth map $f : M \to \tilde{M}$ such that the following diagram commutes that $\pi(F(q)) = f(\pi(q))$ for all $p \in M$

(ii) F induces a linear map from E_p to $\tilde{E}_{f(p)}$ for any $p \in M$.

Definition 2.2.1

Let X and Y be topological spaces we say that X and Y are homeomorphic if there exist continuous function such that $f \circ g = id_y$ and $g \circ f = id_X$ we write $X \cong Y$ and say that f and g are homeomorphisms between X and Y , by the definition a function $f : X \to Y$ is a homeomorphisms if and only if

(i) f is a bijective .(ii) f is continuous (iii) f^{-1} is also continuous.

Definition 2.2.2 Differentiable manifolds

A differentiable manifolds is necessary for extending the methods of differential calculus to spaces more general R^n a subset $S \subset R^3$ is regular surface if for every point $p \in S$ the a neighborhood V of P is R^3 and mapping $x : u \subset R^2 \to V \cap S$ open set $U \subset R^2$ such that.

(i) x is differentiable homomorphism.

(ii) the differentiable $(dx)_q : R^2 \to R^3$, the mapping x is called a parametnzation of S at P the important consequence of differentiable of regular surface is the fact that the transition also example below if $x_\alpha : U_\alpha \to S^1$ and $x_\beta : U_\beta \to S^1$ are $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = w \neq \phi$, the mappings $x_\beta^{-1} \circ x_\alpha : x^{-1}(w) \to R^2$ and .

(10) $x_\alpha^{-1} \circ x_\beta = x_\beta^{-1}(w) \to R$

Are differentiable . A differentiable structure on a set M induces a natural topology on M it suffices to $A \subset M$ to be an open set in M if and only if $x_\alpha^{-1}(A \cap x_\alpha(U_\alpha))$ is an open set in R^n for all α it is easy to verify that M and the empty set are open sets that a union of open sets is again set and that the finite intersection of open sets remains an open set. Manifold is necessary for the methods of differential calculus to spaces more general than de R^n , a differential structure on a manifolds M induces a differential structure on every open subset of M , in particular writing the entries of an $n \times k$ matrix in succession identifies the set of all matrices with $R^{n,k}$, an $n \times k$ matrix of rank k can be viewed as a k-frame that is set of k linearly independent vectors in R^n , $V_{n,k} K \leq n$ is called the steels manifold ,the general linear group $GL(n)$ by the foregoing $V_{n,k}$ is differential structure on the group n of orthogonal matrices, we define the smooth maps function $f : M \to N$ where M , N are differential manifolds we will say that f is smooth if there are atlases (U_α, h_α) on M , (V_B, g_B) on N , such that the maps $g_B f h_\alpha^{-1}$ are smooth wherever they are defined f is a homeomorphism if is smooth and a smooth inverse. A differentiable structures is topological is a manifold it an open covering U_α where each set U_α is homeoomorphic, via some homeomorphism

h_α to an open subset of Euclidean space R^n, let M be a topological space , a chart in M consists of an open subset $U \subset M$ and a homeomorphism h of U onto an open subset of R^m , a C^r atlas on M is a collection (U_α, h_α) of charts such that the U_α cover M and h_B, h_α^{-1} the differentiable

Definition 2.2.3 The injective manifold

A differentiable manifold of dimension N is a set M and a family of injective mapping $x_\alpha \subset R^n \to M$ of open sets $u_\alpha \in R^n$ into M such that.

(i) $u_\alpha x_\alpha(u_\alpha) = M$

(ii) for any α, β with $x_\alpha(u_\alpha) \cap x_\beta(u_\beta)$

(iii) the family (u_α, x_α) is maximal relative to conditions (i),(ii) the pair (u_α, x_α) or the mapping x_α with $p \in x_\alpha(u_\alpha)$ is called a parameterization , or system of coordinates of M , $u_\alpha x_\alpha(u_\alpha) = M$ the coordinate charts (U, φ) where U are coordinate neighborhoods or charts , and φ are coordinate homeomorphisms transitions are between different choices of coordinates are called transitions maps.

(11) $$\varphi_{i,j} : (\varphi_j \circ \varphi_i^{-1})$$

Which are anise homeomorphisms by definition , we usually write $x = \varphi(p), \varphi : U \to V \subset R^n$ collection U and $p = \varphi^{-1}(x), \varphi^{-1} : V \to U \subset M$ for coordinate charts with is $M = \cup U_i$ called an atlas for M of topological manifolds. A topological manifold M for which the transition maps $\varphi_{i,j} = (\varphi_j \circ \varphi_i)$ for all pairs φ_i, φ_j in the atlas are homeomorphisms is called a differentiable , or smooth manifold , the transition maps are mapping between open subset of R^m , homeomorphisms between open subsets of R^m are C^∞ maps whose inverses are also C^∞ maps , for two charts U_i and U_j the transitions mapping.

(12) $$\varphi_{i,j} = (\varphi_j \circ \varphi_i^{-1}) : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

And as such are homeomorphisms between these open of R^m , for example the differentiability $(\varphi'' \circ \varphi^{-1})$ is achieved the mapping $(\varphi'' \circ (\tilde{\varphi})^{-1})$ and $(\tilde{\varphi} \circ \varphi^{-1})$ which are homeomorphisms since $(A \approx A'')$ by assumption this establishes the equivalence $(A \approx A'')$, for example let N and M be smooth manifolds n and m respecpectively , and let $f : N \to M$ be smooth mapping in local coordinates $f' = (\psi \circ f \circ \varphi^{-1}) : \varphi(U) \to \psi(V)$,with respects charts (U, φ) and (V, ψ) , the rank of f at $p \in N$ is defined as the rank of f' at $\varphi(p)$ (i.e) $rk(f)_p = rk(J f')_{\varphi(p)}$ is the Jacobean of f at p this definition is independent of the chosen chart , the commutative diagram in that.

(13) $$f'' = (\psi' \circ \psi^{-1}) \circ \tilde{f} \circ (\varphi' \circ \varphi^{-1})^{-1}$$

Since $(\psi' \circ \psi^{-1})$ and $(\varphi' \circ \varphi^{-1})$ are homeomorphisms it easily follows that which show that our notion of rank is well defined $(J f'')_{x'} = J(\psi' \circ \psi^{-1})_{y'} J f' (\varphi' \circ \varphi^{-1})^{-1}$, if a map has constant rank for all $p \in N$ we simply write $rk(f)$, these are called constant rank mapping. The product two manifolds M_1 and M_2 be two C^k - manifolds of dimension n_1 and n_2 respectively the topological space are arbitral unions of sets of the form $U \times V$ where U is open in M_1 and V is open in M_2 , can be given the structure C^k manifolds of dimension n_1, n_2 by defining charts as follows for any charts M_1 on (V_j, ψ_j) on M_2 we declare that $(U_i \times V_j , \varphi_i \times \psi_j)$ is chart on $M_1 \times M_2$ where $\varphi_i \times \psi_j : U_i \times V_j \to R^{(n_1+n_2)}$ is defined so that.

(14) $$\varphi_i \times \psi_j(p, q) = (\varphi_i(p), \psi_j(q))$$

for all $(p, q) \in U_i \times V_j$. A given a C^k n-atlas, A on M for any other chart (U, φ) we say that (U, φ) is compatible with the atlas A if every map $(\varphi_i \circ \varphi^{-1})$ and $(\varphi \circ \varphi_i^{-1})$ is C^k whenever $U \cap U_i \neq 0$ the two atlases A and $\tilde{A}$ is compatible if every chart of one is compatible with other atlas . A sub manifolds of others of R^n for instance S^2 is sub manifolds of R^3 it can be obtained as the image of map into R^3 or as the level set of function with domain R^3 we shall examine both methods below first to develop the basic concepts of the theory of Riemannian sub manifolds and then to use these concepts to derive a equantitive interpretation of curvature tensor , some basic definitions and terminology concerning sub manifolds, we define a tensor field called the second fundamental form which measures the way a sub manifold curves with the ambient manifold , for example X be a sub manifold of Y of $\pi : E \to X$ and $g : E_1 \to Y$ be two vector brindled and assume that E is compressible , let $f : E \to Y$ and $g : E_1 \to Y$ be two tubular neighborhoods of X in Y then there exists a C^{p-1} . The smooth manifold , an n-dimensional manifolds is a set that looks like R^n . It is a union of subsets each of which may be equipped with a coordinate system with coordinates running over an open subset of R^n . Here is a precise definition.

Definition 2.2.4

Let M be a metric space we now define what is meant by the statement that M is an n-dimensional C^∞ manifold.

(i). A chart on M is a pair (U,φ) with U an open subset of M and φ a homeomorphism a (1-1) onto, continuous function with continuous inverse from U to an open subset of R^n , think of as assigning coordinates to each point of U .

(ii) Two charts (U,φ) and (V,ψ) are said to be compatible if the transition functions .

(15)

$$(\psi\circ\varphi^{-1}):\varphi(U\cap V)\subset R^n\to\psi(U\cap V)\subset R^n$$
$$(\varphi\circ\psi^{-1}):\psi(U\cap V)\subset R^n\to\varphi(U\cap V)\subset R^n$$

Are C^∞ that is all partial derivatives of all orders of $\psi\circ\varphi^{-1}$ and $\varphi\circ\psi^{-1}$ exist and are continuous.

(iii) An atlas for M is a family $A=\{(U_i,\varphi_i):i\in I\}$ of charts on M such that $\{U_i\}_{i\in I}$ is an open cover of M and such that every pair of charts in A are compatible . The index set I is completely arbitrary . It could consist of just a single index. It could consist of uncountable many indices . An atlas A is called maximal if every chart (U,φ) on M that is compatible with every chat of A .

(iv) An n-dimensional manifold consists of a metric space M together with a maximal atlas A

Example 2.2.5

Let I_n be the identity map on R^n , then $\{R^n,I_n\}$ is an atlas for R^n indeed , if U is any nonempty open subset of R^n , then $\{U,I_n\}$ is an atlas for U so every open subset of R^n is naturally a C^∞ manifold.

Definition 2.2.6

Let M a Riemannian manifold and $\gamma:[0,1]\to M$ a smooth map i,e a smooth curve in M . The length of curve is $L(\gamma)=\int_0^1\sqrt{g(\gamma,\gamma)}\,dt$. Where $\gamma'(t)=D_{\gamma_t}\left(\frac{d}{dt}\right)$, with this definition , any Riemannian manifold is metric space define .

(16) $$d(x,y)=\inf\{L(\gamma)\in R:\gamma(t)=y\}$$

are Riemannian an manifold space.

Proposition 2.2.7

Any manifold a demits a Riemannian metric

Proof :

Take a converging by coordinate neighborhoods and a partition of unit subordinate to covering on each open set U_α we have a metric $g_\alpha=\sum_i dx_i^2$. In the local coordinates , define $g=\sum_i\varphi_i g_{\alpha(i)}$ this sum is well-defined because the support of φ_i . Are locally finite . Since $\varphi_i\geq 0$ at each point every term in the sum is positive definite or zero, but at least one is positive definite so that sum is positive definite.

Proposition 2.2.8

Consider any manifold M and its cotangent bundle $T^*(M)$, with projection to the base $p:T^*(M)\to M$, let X be tangent vector to $T^*(M)$ at the point $\zeta\in T_a^*M$ then $D_p(X)\in T^*(M)$ so that $\varphi(X)=\zeta_a(D_p(x))$ defines a conical a conical 1-form φ on $T^*(M)$ in coordinates $(x,y)\to\sum_i y_i\,dy$ the projection p is $p(x,y)=x$ so if

(17) $$X=\sum a_i\frac{\partial}{\partial x_i}+\sum b_i\frac{\partial}{\partial y_i}$$

so if given take the exterior derivative $w=-d\varphi=\sum dx_i\wedge dy_i$ which is the canonical 2-from on the cotangent bundle it is non-degenerate, so that the map $X\to(i\times w)$ from the tangent bundle of $T^*(M)$ to its contingent bundle is isomorphism. Now suppose f is smooth function an $T^*(M)$ its derivative is a 1-form df .Because of the isomorphism a above there is a unique vector field X on $T^*(M)$ such that $df=(i\times w)$ from the g another function with vector field Y , then .

$$Y(t)=df\ (Y)=i_Y\cdot i^iX^w=-i\,X^iY^w=-(X)_g$$

On a Riemannian manifold we shall see next there is natural function on $T^*(M)$. In fact a metric defines an inner on T^* as well as on T for the map $X\to g(X,-)$ defines an isomorphism form T to T^* then .

(18) $$g\left(\sum_j g_{ij}dx_j\times\sum_k g_{kl}\,dx_l\right)=g_{ik}$$

which means that $g^*(dx_j,dx_k)=g^{ik}$ where g^{ik} denotes the matrix to g_{ik} we consider the function $T^*(M)$ defined by $H(\zeta_a)=g^*(\zeta_a,\zeta_a)$.

Definition 2.2.9

The vector field X on $T^*(M)$ given by $I_i w=dH$ is called the geodesist flow of the metric g .

Definition 2.2.10

If $\gamma:(a,b)\to T^*(M)$ Is an integral curve of the geodesic flow. Then the curve $P(\gamma)$ in (M) is called a geodesic .

In locally coordinates, if the geodesic flow .

(19) $$X = \left(a_i \frac{\partial}{\partial x_i} \right) + \left(b_j \frac{\partial}{\partial y_j} \right)$$

Proposition 2.2.11
The function f a above is If $f(\zeta_x) = \zeta_x(X_x)$
Proof :

Write in coordinates If $X = \sum a_i \left(\frac{\partial}{\partial x_i} \right) b_j \left(\frac{\partial}{\partial y_j} \right)$ where If $\phi = \sum_i y_i dx_i$ since $\tilde{X}$ projects on M then $X = \sum a_i \frac{\partial}{\partial x_i}$ by the definition of ϕ .Now let M be a Riemannian manifold and H , the function on $T^*(M)$ defined by the metric as a above , if φ_t is an one parameter group of isometrics , then the induced diffeomorphisms of $T^*(M)$ will preserve the function H so the vector field $\tilde{Y}$ will satisfy $\tilde{Y}(H) = 0$. that $X(f) = 0$ where X is the geodesic flow a long the geodesic flow, and is therefore a constant of integration of the geodesic equations

Definition 2.2.12
We mentioned a above that a metric g , defines an inner product not just on T_a but also an inner product g^* on T_a^* , with this we can define an inner product on pth exterior power $T_a^*(\wedge^p)$:
$(\alpha_1 \Lambda \alpha_2 \Lambda ....\Lambda \alpha_p, \beta_1 \Lambda \beta_2 \Lambda ....\Lambda \beta_p) = Det\ g^*(\alpha_i, \beta_i)$
Thus if $dx_1 \Lambda dx_2 \Lambda ....\Lambda dx_n$ defines the orientation $w = \sqrt{\det g_{ij}}\, dx_1 \Lambda, ..., \Lambda dx_n$ on a compact manifold we can integrate this to obtain total volume – so a metric defines not only length but also volumes, Now take $\alpha \in \Lambda^P T_a^*, \beta \in \Lambda^{n-P} T_a^*$ and
define $f_\beta : \wedge^p T_a^* \to R, by\ \ f_\beta(\alpha) w = \beta \wedge \alpha$ .But we have an inner product , so any liner map on $\Lambda^P T_a^*$ is of the form $\alpha \to (\alpha, \gamma)$ for some $\gamma \in \Lambda^P(T_a^*)$ so we have a well –defined liner map $\beta \to \gamma\beta$ form $\Lambda^{n-P}(T_a^*)$ to $\Lambda^P(T_a^*)$ satisfying $(\gamma_\beta, \alpha)\, w = \beta \Lambda \alpha$.

Definition 2.2.13
The Hodge star operator is the linear map $* : \Omega^P(M) \to \Omega^{n-P}(M)$ with the property that at each point.

(20) $$(\alpha, \beta) w = \alpha \Lambda * \beta$$

Proposition 2.2.14
Let M be an oriented Riemannian manifold with volume for w , and let $\alpha \in \Omega^P(M)$, $\beta \in \Omega^{P-1}(M)$ be forms of compact support then .

(21) $$\int_M (d^*\alpha, \beta)\, w = \int_M (\alpha, d\beta)\ w$$

Definition 2.2.15 Deferential Laplacian on p-forms
Let M be an oriented Riemannian manifold , then the Laplacian on p-forms is the deferential operator .

(22) $\Delta : \Omega^P(M) \to \Omega^P(M)$ defined by $\Delta : dd^* + d^*d$

Definition 2.2.16 Starting Point
A differential form $\alpha \in \Omega^P(M)$ is harmonic if $\Delta\alpha = 0$,on a compact manifold harmonic ply a important role, which there is no time to explore in this course
here is the starting point.

Definition 2.2.17 Harmonic and de Rham Manifold
Let M be a compact oriented Riemannian manifold then :
(i) a p-form is harmonic if and only if $d\alpha = 0$ and $d^*\alpha = 0$
(ii) In each de Rham cohomology class there is at most one harmonic from.

Theorem 2.2.18 The Fundamental Theorem of Riemannian Geometry
Suppose M is An m-dimensional smooth manifold , and G is a symmetric covariant tensor field of rank 2 on M if (U_i, u^i) is a local coordinate system on M then the tensor field G can be expressed as.

(23) $$G = g_{ij} du^i \otimes du^j$$

On U, where $g_{ij} = g_{ji}$ is a smooth function on U . U provides a bilinear function on $T_p(M)$ at every point $p \in M$ .Suppose $X = X^i \frac{\partial}{\partial u^i}, Y = Y^i \frac{\partial}{\partial u^i}$ then $G(X,Y) = g_{ij} X^i Y^J$ .We say that the tensor G is no negated at the point if , whenever $X \in T_p(M)$ and $G(X,Y) = 0$. For all $Y \in T_p(M)$ it must be true that $X = 0$ this implies that G is no degenerate at p if and only if the system of linear equations $g_{ij}(p) X^i = 0 \quad 1 \le j \le m$ has zero as its only solution (i,e) det $(g_{ij}(p)) \ne 0$ if for all $X \in T_p(M)$ we have $G(X,Y) \ge 0$ And the equality holds only if $X = 0$ then we say G is positive definite at p . From liner algebra a necessary and sufficient condition for G to be positive definite that matrix (g_{ij}) is positive definite . Thus a positive definite tensor G is necessarily non degenerate .

2.3 Generalized Tensor is Riemannian
If an m-dimensional smooth manifold M is given a smooth every no degenerate symmetric covariant tensor field of 2-rank , G then M is called a generalized tensor or metric tensor or metric of M . If G is positive definite then M is called Riemannian

manifold.for a generalized Riemannian manifold M $,G = g_{ij}du^i \otimes du^j$ specifies an inner product on the tangent space $T_p(M)$ at every point $p \in M$ for any $X,Y \in T_p(M)$ let .

(24) $$X.Y = G(X.Y) = g_{ij}(p)X^iY^j$$

When G is positive definite, it is meaningful to define the length of a tangent vector and the angle between two tangent vectors at the some point $|X| = \sqrt{g_{ij}X^iX^J}$. Thus a Riemannian manifold is a differentiable manifold which has a positive definite inner product on the tangent space at every point . The inner product is required to smooth X,Y are smooth tangent vector fields then X,Y is a smooth on M

Definition 2.31 Smooth Parametrzel Curve

$dS^2 = g_{ij}du^i du^j$ is independent of the choice of the local coordinate system u^i and usually called the metric form or Riemannian metric (dS) is precisely the length of an infinitesimal tangent vector and is called the element of are length . Suppose a $C = u^i = u^i(t)$ and $t_0 \le t \le t_1$ is a continuous and piecewise smooth parameterized curve on M ,then the are length of C is defined to be .

(25) $$S = \int_{t_0}^{t_1} \sqrt{g_{ij}\frac{du^i du^j}{dt \quad dt}}dt$$

II,GEOMETRIY MAXIMUM PRINCIPLES FOR HYPESURFACES IN LORTZIAN AND RIEMANNIAN MANIFOLDS

2.1 Geometrid Maximum and principle Riemannian manifolds

The version of the analytic principle given by:

(i) U_0 is lower semi – continuous and $M[U_0] \le H_0$ in the sense of support function.

(ii) U_1 is upper – semi – continuous and $M[U_0] \ge H_0$ in the sense function with a one – sided Hessian bound .

(iii) $U_1 \le U_0$ in Ω and $U_1 = U_0$ is locally a $C^{1,1}$ -function in Ω finally if a^{ij} and b are locally $C^{k+2,\alpha}$ function in Ω . In particular if a^{ij} and b are smooth is $U_1 = U_0$, $\Omega \subset R^n$ is specially natural in Lorentzian setting as C^0 space like hyper surfaces in definition.

(26) $$S_{\eta,r} = \{ p : d\,(p,\exp\ (r,\eta) = r) \}$$

them $S_{\eta,r}$ contains $\pi(\eta)$ and neighborhood of $\pi(\eta)$ is smooth , at $\pi(\eta)$ pointing unit normal $r \ge 0$ and $k \subset T(M)$ can a lows be locally represented as a graphs also applies to hyper surfaces in Riemannian manifolds that can be represented locally as graphs. We first state our conventions on the sign of the second fundamental form and the mean curvature to fix choice of signs a Lorentzian manifold $(M.g)$.

Definition 2.1.2 Hypersurface

A subset $N \subset M$ of that space-time $(M.g)$ is C^0 space like hyper surface , if for each $p \in N$, there is a neighborhood U of p in M so that $N \cap V$ is causal and edge less in U .

Remark 2.1.3

In This definition not that if $D(N \cap U,U)$ is the domain of dependence of in U , then $D(N \cap U,U)$ is open in M and $U \cap U$ is a Cauchy hyper surface is globally hyperbolic thus by replacing U by $D(N \cap U,U)$ we can assume the neighborhood U is the last definition is globally hyperbolic and that $U \cap U$ in a Cauchy surface in U In particular a C^0 space like hyper surface is a topological. Let $(M.g)$ be a spacetime and let N_0 and N_1 be two C^0 space like hyper surfaces in $(M.g)$ which meet at a point q. Say that no is locally to the future of N_1 near q iff for some neighborhood U of P in which N_1 is a causal and edgeless .

(27) $$N_0 \cap U \subset J^+(N_1,\cup)$$

where $J^+(N_1,\cup)$ is causal future of N_1 in U .

Definition 2.1.4 Saclike hyper surface is Space-time

(i) Let N be a C^0 space like hyper surface in the space-time $(M.g)$ and H_0 a constant then N has mean curvature $\le H_0$, in the sense of support hyper surfaces for all $q \in N$, $\varepsilon \ge 0$ there is C^2 future support hyper surface $S_{q,\varepsilon}$ to N at q and the mean curvature of $S_{q,\varepsilon}$ at q satisfies $H_q^{S_{q,\varepsilon}} \le H_0 + \varepsilon$.

(ii) N has mean curvature $\ge H_0$ in the sense of support hyper surfaces with one- sided Hessian bounds for all compact sets $K \subseteq N$ there is compact set $K^\wedge \subseteq T(M)$ and constant $C_k \ge 0$, such that for all $q \in N$ so that .

the future pointing unit normal $n^{P_{(q,\varepsilon)}}$ and second fundamental form $h^{P_{(P\varepsilon)}}$ of $P_{q,\varepsilon}$ satisfy .

(28) $$H_q^{P_{(q,\varepsilon)}} \ge H_0 - \varepsilon \quad , \quad h_q^{P_{(q,\varepsilon)}} \ge -C_{kg} \,\Big|_{P_{q,\varepsilon}}$$

Proposition 2.1.5

Let $(M.g)$ be a space-time $r_n \ge 0$ and $K \subset T(M)$ a compact set of future pointing time like unit vectors.

Assume that there is a δ so that for all $\eta \in k$, the geodesic $\gamma_\eta(t) = \exp(t_\eta)$ maximizes the Lorentzian distatance on the interval $[r_0 + \delta]$ for each $\eta \in k$, and $r_0 \geq 0$, let $\pi(\eta)$ be the base point of r_n and set .

(29) $$S_{\eta,r} = \{\, P:\ d(P,\ \exp(r,\eta))\ \} = r$$

Theorem 2.1.6

Let N_0 and N_1 be C^0 spacesurlike hyper surfaces in a specimen $(M\ .g)$ which meet at a point q_0, such that N_0 is locally to future of N_1, near q_0. Assume for some constant.

(i) No has mean curvature $\leq H_0$ in the sense of support hyperspaces .

(ii) N_1 has mean curvature $\geq H_0$ in the sense of support hyperspaces with one- sided Hessian bounds, then $U_0 = U_1$ near q_0 ,(i,e)there is a neighborhood of q_0 such that .

(30) $$N_0 \cap 0 = N_1 \cap 0$$

Moreover is smooth space like hyper surface with mean curvature H_0 .

Remark 2.1.7

If $(M\ .g)$ the metric only has finite differentiability , say g is $C^{k,\alpha}$ with $k \geq 2$, and $0 \leq \alpha \leq 1$ then since the function a^{ij} and b in the definition of the mean curvature operator H , depend on the first derivatives of the metric , they are of class . Thus the regularity part, implies hyper surface.

(31) $$N_0 \cap 0 = N_1 \cap 0$$

in the statement of the last is $C^{k+1,\alpha}$.

2.2. Reduction to Analytic Maximum Principle

Let $(M\ .g)$ be an an-dimesinal pastime and let ∇ be metric connection of metric g then near any point q of M there is a coordinate system $(x^1, x^2, ..., x^n)$ so that the metric takes the form.

$$g = \sum_{A.B=1}^{n} (g_{A,B})\, dx^A\, dx^B = \sum_{i,j}^{n-1} (g_{i,j})\, dx^i dx^j - (dx^n)^2$$

And so that $\partial / \partial x^n$ is future pointing time like unit vector . (To construct such coordinates choose smooth spikelike hypersurface) S in M passing through g and let $(x^1, x^2, ..., x^{n-1})$ be is as required . Let f be a function defined near the origin in R^{n-1} with $f(0) = 0$ the define a map F_f form a neighbourhood of the origin in R^{n-1} to M so that the coordinate system $(x^1, x^2, ..., x^n) F_f$ is given by.

$$F_f\left(x^1, x^2, ..., x^{n-1}\right) = \left(x^1, x^2, ..., x^{n-1}\right)$$

this parameterizes a smooth hyper surface N_f through x_0 and moreover every smooth space like hyper surface x_0 is uniquely parameterized in this manner for unique f satisfying .

(32) $$1 - \sum_{i,j=1}^{n-1} g^{ij}\ (D_i f\, D_j)\quad , \qquad f \geq 0$$

This is exactly the condition that the image of F_f is space like when the image is space like set .

(33)

$$X_i = (\partial/\partial x^i) + D_i f\ (\partial/\partial x^n)\ ,\ W = \left(1 - \sum_{i,j}^{n-1} g^{i,j} D_i f D_j\right)^{1/2}$$

$$n = \frac{1}{w}\left(\partial/\partial x^n + \sum_{i,j}^{n-1} g^{i,j} D_i f\ (\partial/\partial x^j)\right)$$

Then $(x_1, x_2, ..., x_n)$ is a basis for the tangent space to image of N_f and n is the future pointing time like unit normal to N_f . Now tedious calculation shows that the second fundamental form $h$of is given by .

(34) $$h\ (X_i, X_j) = \frac{1}{w}\left(D_{ij} f + \Gamma_{ij}^{m} - V_{ij}\right)$$

Where Γ_{ij}^{k} are the christoffel symbols and V_{ij} are by Solving for the Hessian of f in terms of the second fundamental form of N_f given .

(35) $$D_{ij} f = Wh\ (X_i, X_j) - \Gamma_{ij}^{n} + V_{ij}$$

The induced metric on N_f has its components in the coordinated system $(x^1, x^2, ..., x^{n-1})$ given by .

(36) $$G_{ij} = g(X_i, X_j) = g_{ij} - D_i f\, D_j f$$

Let $[G^{ij}] = [G_{ij}]^{-1}$ then the mean curvature of N_f

$$H = \frac{1}{n-1}\ ,\ h = \frac{1}{n-1} \sum_{i,j=1}^{n-1} G^{ij} h(X_i, X_j)$$

$$= \frac{1}{(n-1)W} \sum_{i,j=1}^{n-1} G^{ij}\left(D_{ij} f + \Gamma_{ij}^{n} - V_{ij}\right)$$

Where $x = (x^1, x^2, ...x^{n-1})$,

$\left[G^{ij}(x, f, Df)\right] = \left[g_{ij}(x, f) - D_i f(x) D_j f(x)\right]^{-1}$ and $V_{ij}(x, f, D f)$ is

$a^{ij}(x, f, Df) = \dfrac{1}{(n-1)w} G^{ij}(x, f, Df)$ and.

(37) $$b(x, f, D f) = \frac{1}{n-1} \sum_{i,j=1}^{n} G^{i,j}\left(\Gamma_{ij}^{n} - V_{ij}\right)$$

Therefore if $H[f]$ is the mean curvature of N_f then the operator $f :\to H[f]$ is quasi-linear .

Lemma 2.2.1 Curvature Tangent Bundle

Let $U_{\alpha}(\Omega_{\alpha}) \subset K$ where K is compact , then there is a compact subset $\hat{K}$ of the tangent bundle $T(M)$ that contains the set $U_{\alpha}\{n_{\alpha}(x):\ x \in \Omega_{\alpha}\}$ if and only if there is a $\rho \geq 0$ so that for all α the lower bound $W_{\alpha}(x) \geq \rho_0$ hold for $x \in \Omega_{\alpha}$, Moreover if this lower bound holds and $0 \leq \rho \leq \rho_0$ there is bound $|f_{\alpha}| \leq \beta$ and if $U = U_{P,B,k} \subset R^{n-1} \times R \times R^{n-1}$ is defined by .

(38)

$$U = U_{\rho,B,K} = \left\{ \begin{array}{c} (x^1, x^2, \ldots\ldots\ldots, x^{n-1}, r) \\ (p_1, p_2, \ldots\ldots\ldots, p_{n-1}) = \ (x, r, p) \end{array} \right\}$$

$x \in \beta$, $|r| \leq \beta$, $\sum_{i\,j=1}^{n-1} g^{i,j}(x,r)P_i\,P_j \leq 1-\rho^2$ then for any $x \in k$ the fiber functions f_{α} are U admissible over and , finally the mean curvature operator H is uniformly elliptic on U .

Definition 2.2.2 Geometric Maximum Principle for Riemannian Manifolds

We now fix our sign conventions on the imbedding invariants of smooth hypersurfaces in Riemannian manifold (M, g). It will be convenient to assume that our hyper surfaces are the boundaries of open sets.An this a lways true Locally it is not a restriction by ∇ let $D \subset M$ be connected open set and let $N \subset \partial D$, be part of all ∂D is smooth , let n be the outward pointing unit normal along N then the second fundamental form of N symmetric bilinear form defined on the tangent space to N by $h^N(X,Y) = \langle \nabla_x, \nabla_n \rangle$ The mean curvature of N is then.

(39)

$$H^N = \frac{1}{n-1}\Big|_{g,N} h^N = \frac{1}{n-1}\sum_{i=1}^{n-1} h^N(e_1, e_j)$$

And where $(e_1, e_2, \ldots, e_{n-1})$ is local orthogonal from for $T(N)$ this is the sign convention so that for the boundary S^{n-1} of the unit ball β^n in R^n the second fundamental form $h^N = -g\,\big|_{S^N}$ is negative definite the mean curvature is $H^{s^{(n-1)}} = -1$.

Definition 2.2.3 Hypersuface on Curvature $\geq H_0$

Let U be an open set in the Riemannian manifold (M, g) then :

(i) ∂U has mean curvature $\geq H_0$ in the sense of contact hypersurfaces iff for all $q \in \partial U$ and $\varepsilon \geq 0$ there is an open set D of M with $\bar{D} \subseteq \bar{U}$ and $q \in \partial D$ near q is a C^2 hypersuface of M and at point $q, H_q^{\partial D} \geq H_0 - \varepsilon$

(ii) ∂U has mean curvature $\geq H_0$ in the sense contact hypersurface is constant $C_k \geq 0$ so that for all $q \in k$ and $\varepsilon \geq 0$ there is open set D of M with $\bar{D} \subseteq \bar{U}$ and $q \in \partial D$ the of ∂D near q , $H_q^{\partial D} \geq H_0 - \varepsilon$ and also .

(40) $$H_q^{\partial D} \geq -C_{kg}\ \big|_{\partial D}.$$

The Hyper-surfaces of manifolds as Let $M \subset K$ be any hyper-surface of quaternion manifold (K, Q),we define $H \subset TM$ to be the maximal Q - invariant distribution on M , if f is any defining function for M .

(iii) If f is any defining function for M , i,e $M = f^{-1}(0)$ and $df\ \big|_M \neq 0$ then .

(41)

$$H = \{X \in TM : df(J_1X) = df(J_2X) = df(J_3X) = 0\}$$

This H is always a smooth co-dimension 3-disribution on M .

(iv) we say that a hyper-surface M of quaternion manifold $(K, Q = \{J_1, J_2, J_3\})$ is a QC - hyper-surface if :

(42)

$$\nabla df(X, X) \neq 0,\ X \in H,\ X = 0$$

$$\nabla df(JX, JY) = \hat{\nabla} df(X, Y),\ X, Y \in H,\ s = 1,2,3$$

Where $H \subset TM$ is the maximal Q -invariant distribution on M,$\hat{\nabla}$ is any torsion – free quaternion of (K, Q) and f is any defining function for M , for example the field of quaternions $H = Sup_R\{1, i, j, k\}$ where $i^2 = j^2 = -k^2 = -1$ and $i.j = -j, i = k$. Consider the flat quaternionic manifold $K = H^{n+1}$ whith its standard quaternionic structure $Q = Span\{J_1, J_2, J_3\}$, $J_1(x) = -x.i$, $J_2(x) = -x.j$, $J_3(x) = -x.k$ is a torsion free quaternionic connection $\hat{\nabla}$ we take the flat connection here . It clearly holds $\hat{\nabla} \times Q \subset Q$, $Let\ x = (q_1, q_2, \ldots, p) \in H^n \times H$ we have the following three basic of QC hyper-surfaces $H^n \times H$.

(43)

$$M_1 : \Sigma_{a=1}^n |q_1|^2 + \mathrm{Re}(p) = 0\ ,\ M_2 : \Sigma_{a=1}^n |q_a|^2 - |p|^2 = -1$$

$$M_3 : \Sigma_{a=1}^n |q_a|^2 + |p|^2 = 1$$

the sphere.

Theorem 2.2.4 Geometric Maximum Principle for Riemannian Manifolds

Let (M, g) be a Riemannian manifold $U_0, U_1 \subset M$ open sets and let H_0 be a constant , assume that .

(i) $U_0 \cap U_1 = 0$

(ii) ∂U_0 has mean curvature $\geq H_0$ in the sense of contact hypersurfaces.

(iii) ∂U_1 has mean curvature $\geq H_0$ in the sense of contact hypersurfaces with a one sided Hessian bound .

(iv) there is a point $p \in \dot{U}_0 \cap \overline{U}$ and a neighborhood N of p that has coordinates $(x^1, x^2, ..., x^n)$ cantered at p so that for some $r \geq 0$ the image of these coordinates is the box $(x^1, x^2, ..., x^n) = |\, x^i \,| \leq r$ and there are Lipschitz continous and there are Lipschitz continuous

function $U_0, U_1 : (x^1, x^2, ..., x^{n-1}) : |\, x^i \,| \leq r \,, (-r, r)$, so that $U_0 \cap N$ are given by .

(44)

$U_0, N = (x^1, x^2, ..., x^n) : x^n \geq U_0 (x^1, x^2, ..., x^{n-1})$

$U_1, N = (x^1, x^2, ..., x^n) : x^n \geq U_1 (x^1, x^2, ..., x^{n-1})$

This implies $U_0 \equiv U_1$ and U_0 is smooth function , therefore $\partial U_0 \cap U_1 = \partial U_1 \cap N$ is a smooth embedded hypersurface with constant mean curvature H_0 (with respect to the outward normal to U_1).

Definition 2.2.5 Lorentzian Mainfolds

Let (M, g) be a Lorentzian manifold and let $q \geq 0$, then (M, g) is globally hyperbolic of order q if and only if M is strongly causal and $x \leq y, d\,(x, y) \leq \frac{\pi}{q}$ implies that $C(x, y)$ is compact where $c(x, y)$ is set of causal curves connecting x and y .

Corollary 2.2.6 Lorentzian Maximal Diameter Theorem

Let (M, g) be connected Lorentzian manifold which is globally hyperbolic of order 1. and assume that $Ric(T,T) \geq (n-1)$ for any time like unit vector T if M a timelike geodesic segment $\gamma : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to M$ of length π connecting x and y , then $D = \{ z : x \leq z \leq y \}$ is isometric to $(S_1^n(-1), g_s)$ .Moreover if M contains a time like geodesic $\gamma = (-\infty, \infty) \to M$ such that each segment $\gamma\,|_{[t, t+\pi]}$ is maximizing then (M, g) is isometric to $(S_1^n(-1), g_g)$. Moreover if M contains a time like geodesic $\gamma = (-\infty, \infty) \to M$ such that each segment $\gamma\,|_{[t, t+\pi]}$ is maximizing then (M, g) is isometric to universal anti-de sitter space $R^n(-1)$.

Definition 2.2.7 Asymptote Curvature

Let $\gamma = (-\pi/2, \pi/2) \to M$ be line in M , and let $s \in (-\pi/2, \pi/2)$ for $P \in \gamma(S)$, let α_S be a maximal geodesic connecting P and $\gamma(S)$, if there is sequence and time like unit vector V such that $S_K \to \pi/2$, $P \leq \gamma(S_K)$, and $\alpha_{S_K}(0) \to V \in T(M)_P$ then the maximal geodesic starting at P in the direction V is called an asymptote to γ and V .

Definition 2.2.8 Timelike Lines I

A strip is a totally geodesic immersion f of $(-\pi/2, \pi/2 \times \{S\})$ is a timelike line for each $S \in I$. We will denote by S the space $(-\pi/2, \pi/2) \times I$, $-dt^2 + \cos^2(t) dt^2$ into M for some interval I so that $f\,|_{[-\pi/2, \pi/2 \times \{S\}]}$ is time like line for each $S \in I$.

Lemma 2.2.9 Parallel Lines

If γ_1 and γ_2 are parallel lines , then $I(\gamma_1) = I(\gamma_2)$, and the Busman function b_1^t and b_2^t of γ_1 and γ_2 through x and parallel to γ_1 .

lemma 2.2.10 Lorentzian Productmtric

Let (N, g_N) be a Riemannian manifold of dimension at least three , set $M = R \times N$ and give M the lorentzian productmtric $g = -dt^2 + g_N$ let $R_{A,B,C,D}$ be the curvature tensor of (M, g) as tensor.

2.3 The Spectrum of the Palladian in Riemannian Manifolds

To any compact Riemannian manifold (M, g) is boundary we associate second- order (P.D.E) , the Laplace operator Δ is defined by : $\Delta(f) = -\,div\,(grad\ f)$ For $f \in L^2(M, g)$. We also sometimes write Δ_g for Δ if we want to emphasize which metric the Laplace operator is associated with the set of eigenvalues of Δ is called the spectrum of Δ or of M which we will write as space Δ (or space (M, g) , they form a discrete sequence $0 = \lambda_1 \leq \lambda_2 \leq ,...., \leq \lambda_n$ for simplicity , we will assume that M is connected . This will for example imply that the smallest eigenvalue λ_0 . Occurs with multiplicity .

Definition 2.3.1

Let $F : M \to N$ be a smooth map between two smooth manifolds and $w \in \Gamma(T_k^0 N)$ be a k covariant tensor field we define a k covariant tensor field $F^* w$ over M by .

(6)

$(F^* w)_p (v_1, ..., v_k) = w_{F(p)} (F_{*p}(v_1), ..., F_{*p}(v_k))$

$, \forall\, v_1, ..., v_k \in T_p M$

In this case $F^{*}w$ is called the pullback of w by F .

Proposition 2.3.2

Suppose $F : M \to N$ is a smooth map and $G : N \to Q$ a smooth map for M, N, Q smooth manifolds and $w \in T\left(T_k^{\,0} N\right), \eta \in T\left(T_l^{\,0} N\right)$ and $f \in C^{\infty}(N)$ then .

(i) $(G \circ F)^{*} = F^{*} \circ G^{*}$.

(ii) $F^{*}(w \otimes \eta) F w \otimes F^{*} w \otimes F^{*} \eta$ in particular , $F^{*}(f \circ w) = (f \circ F) F^{*} w$.

(iii) $F(df) = d(f \circ F)$ (iv) if $p \in M$ and (y^{i}) are local coordinates in a chart containing the point $F(p) \in N$ then.

(45)

$$F^{*}\left(w_{j1,\ldots,jk}\, dy^{i1} \otimes \ldots \otimes dy^{ik}\right) = \left(w_{j1,\ldots jk}, \circ F\right) d\left(y^{j1} \circ F\right) \otimes \ldots \otimes d\left(y^{ik} \circ F\right) \cdot$$

Definition 2.3.3 Exterior derivative

The exterior derivative is a map $d : \Omega^{k}(M) \to \Omega^{k+1}(M)$ which is R linear such that $d \circ d = 0$ and if f is a k vector field on k then $(df)(X) = Xf$.

Definition 2.3.4 Integration of differential forms

$\int_M w$ is well defined only if M is orient able dim(M) $= n$ and has a partition of unity and w has compact support and is a differential n-form on M .

Definition 2.3.5 Riemannian Manifolds

An inner product (or scalar product) on a vector space V is a function $\langle \cdot , \cdot \rangle : V \times V \to R$ that is :

(i)symmetric $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$.

(ii)Bilinear

$\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$ and $\langle u, av + bw \rangle = a\langle u, v \rangle + b\langle u, w \rangle$ for all $a, b \in R$ and $u, v, w, \in V$.

(iii) positive definite $\langle u, v \rangle \succ 0$ for all $u \neq 0$.

Definition 2.3.6

A pair (M, g) of a manifold M equipped with a Riemannian metric g is called a Riemannian manifold.

Definition 2.3.7 Length and Angle between tangent vectors

Suppose (M, g) is a Riemannian manifold and $p \in M$ we define the length (or norm) of a tangent vector $v \in T_p M$ to be $|v| = \sqrt{\langle v, v \rangle_p}$ Recall $g(\cdot,\cdot) = \langle \cdot,\cdot \rangle$ and the angle v, w between $v, w \in T_p M$ $(v \neq 0 \neq w)$ by.

(46) $$\cos\,(v, w) = \frac{\langle v, w \rangle_p}{|v||w|}$$

Definition 2.3.8

If L is liner operator defined on $T_p M$, then the spectrum of L is the set of eigenvalues of L .It is denoted by space (L) . We take the Laplace operator Δ defined as $\Delta = -(d\delta + \delta d)$, where δ is adjoin of d in spectral geometry we consider the following two equations (i) Does the spectrum of M determine the geometry of M (ii) Does the geometry of M determine the spectrum of M .

2.4. Sequences be Spectra

Sequences occur can as the spectra of manifolds a version of this question. Has been answered what finite sequences can occur as the initial part of spectra of manifolds . If M is a closed connected manifold of dimension greater that or Equal p preassigned finite sequence $0 = \lambda_1 \leq \lambda_2, \ldots\ldots, \leq \lambda_k$ is Sequence of first $K+1$ eigenvalues of Δ_g for some choice of the metric g on M . In particular , this means that for closed connected manifolds of 3-dimension or Greater , there are no restrictions on the multiplicities of the eigenvalues λ_i for $i \geq 0$. In 2-dimension , there are some restrictions on the multiplicities of the eigenvalues. Let M be a closed connected 2-manifold with Euler characteristic $\chi(M)$, and let m_j be the multiplicity of the $j-th$ eigenvalue $j \geq 0$ of the laplacian operator associated to a metric on M then :

(i) If M is the unit sphere, then $m_j \leq 2_j + 1$.

(ii) If M is the real projective plane , then $m_j \leq 2_j + 3$

(iii) If M is the torus , then $m_j \leq 2_j + 4$.

(iv) If M is the klen bottle , then $m_j \leq 2_j + 3$

(v) If , $\chi(M) \leq 0$ then $m_j \leq 2_j + 2\chi(M) + 3$. For finite sequences $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2, \ldots\ldots, \leq \lambda_n$ however the result by-Colin de derriere holds – even in 2-dimension .

Definition 2.4.1 Estimates on the first Eigenvalue

The geometry of a manifold affects more that the multiplicities of the eigenvaluees . Here we will focus on bounds on the first non-zero eigenvalue λ_1 imposed by the geometry . the first lower bound is due to lichnzeowicz .

Theorem 2.4.2 Ricci Tensor

Let (M,g) be a closed Riemannian manifold of dimension $n \geq 2$ and let Ric be its Ricci tensor field if Ricci $(X,X) \geq (n-1)k \geq 0$. For some constant $k \geq 0$, and for all $X \in T(M)$, then $\lambda_1 \geq nk$.

Theorem 2.4.3

Let (M,g) be a closed Riemannian manifold , if Ricci $(X,X) \geq (n-1)k \geq 0$ .For some nonnegative constant k and for all $X \in T(M)$ then.

(47) $$\lambda_1 \geq \frac{(n-1)\,k}{4} + \frac{\pi^2}{D^2(M)}$$

It is in general much easier to given upper bounds on λ_1 that it is give lower bounds . The basic result in this area is a comparison theorem due to a complete Riemannian n- manifold whose Ricci curvature is $\geq (n-1)k, k$, is some const.

Theorem 3.4.6 Ricci Curvature

If M is a compact n-manifold with Ricci curvature $\geq (n-1)(-k)k \geq 0$, then

(48) $$\lambda_1 \leq \frac{(n-1)^2}{4}k + \frac{c_2}{D^2(M)}$$

Where c_2 is positive constant depending only on n.

Definition 3.4.7 Geometric Implications Of The spectrum

The spectrum does not in general determine the geometry of a manifold Neverthless earthiness , some geometric information can be extracted from the spectrum . In what follows , we define a spectral invariant to be any thing that is completely determined by the spectrum .

Definition 3.4.8 Invariants From The Heat Equation

Let M be a Riemannian manifold . A heat kernel or alternatively fundamental solution to the heat equation , is a function $K:(0,\infty)\times(M\times)M \to M$

That satisfies $K(t,x,y)\ is\ C^1\ in\ (t)$ and C^2 in x and y . $\frac{\partial K}{\partial t} + \Delta_2(K) = 0$ where Δ_2 is the Laplacian with respect to the second variable . $\lim_{t\to 0^+} \int_M K(t,x,y)\, f(y)\,dy = f(x)$ For any compactly supported function f on M .The heat kernel exists and unique for Riemannian manifold , its importance stems from the fact that the solution to the heat equation .

(49) $$\frac{\partial u}{\partial t} + \Delta(u) = 0, u:[0,\infty]\times M \to R$$

$$\frac{\partial u}{\partial t} + \Delta(u) = 0, u:[0,\infty]\times M \to R$$

Where Δ is Laplacian with respect to second variable , with initial condition $u(0,x) = f(x)$ is given by:

(50) $$u(t,x) = \int_M K(t,x,y)\, f(y)\,dy$$

If $\{\lambda_i\}$ in spectrum of M and $\{\zeta_i\}$ are the associated eigenfunctions (normalized so thst they form an orthonormal basis of $L^2(M)$ then we can write .

(51) $$K(t,x,y) = \sum_i e^{-\lambda_i t}\zeta_i(x)\zeta_i(y)$$

From this it clear that the heat trace $Z(t) = \int_{t\to 0} K(t,x,x) = \sum_M e^{-\lambda_i t_i}$ a spectral invariant . The heat trace has an asymptotic expansion as $t \to 0^+$. $Z(t) = (4\pi t)^{\dim M/2} \sum_{j=1}^{\infty} a_j t^j$. Where the a_j are integrals over M of universal homogenous polynomials in the curvature and covariant derivatives. The first few of these are

(52)
$$a_0 = vol\ (M), a_1 = \frac{1}{6}\int_M S, a_2 = \int_M \left(5S^2 - 2|Ric|^2 - |Rm|^2\right)$$

Where S is the scalar curvature , Ric. is the Ricci tensor , R.m. is the curvature tensor . the dimension the volume and total scalar curvature are thus completely determined by spectrum . If M is a surface then the Gauss Bonnet theorem implies that the Euler characteristic of M is also a spectral invariant . Amore in depth study of the heat trace can yield more information of dimension $n \leq 6$ and if M has same spectrum as the n-sphere S^n with the standard metric (resp . RP^m) then M is in fact isometric to S^n (resp . RP^m) more on this can be found .

Definition 2.4.2 Isospectral Manifolds

As was alluded to earlier, geometry is not in general a spectral in variant two manifolds are said to be isospectral if they have the same spectrum . Of non isometric isospectral manifolds was found too distinct but isospectral manifolds .

Definition 2.4.3 Direct Computation of The Spectrum

The first of those is straightforward: direct computation . it rarely possible to explicitly compute the spectrum of a manifold were actually discovered via this method . Milnor's example mentioned above consists of two isospectral factory-quotients of Euclidean space by lattices of full rank being one of full rank being one of the few examples of Riemannian manifolds whose spectra can be computed explicicitly spherical space forms – quotients of spheres by finite groups of orthogonal transformations acting without fixed points form another class of examples of manifolds isospectral for the Laplaction acting on p-forms for $p \le k$ but not for the Laplaction acting on p-forms for $p \le k+1$ (recall that a lens space is spherical space form where the group is cyclic .

Theorem 2.4.4

Let $m\Gamma_1$ and $m\Gamma_2$ be compact discrete subgroup of a lie group G , and let g be a left invariant metric on G if $m\Gamma_1$ and $m\Gamma_2$ are representation equivalent then .

(53) $$Spec(m_1/G,g) = Spec(m_2/G,g)$$

out .then the formula simplifies to

$\lambda^2 \ge \min\left\{\frac{1}{4}R + 2\Delta(f)\right\}$ The curvature K of the Riemann surface (M^2, g) is equal to , and we can choose f as a solution to the differential equate .

$$2\Delta(f) = -k + \frac{1}{vol(M^2,g)}\int_{M^2} k = -k + \frac{2\pi X(M^2)}{vol(M^2)}$$

Thus $\frac{1}{2}R + 2\Delta(f) = \frac{2\pi X(M^2)}{vol(M^2)}$ is constant ,and we obtain $\lambda^2 \ge \frac{2\pi X(M^2)}{vol(M^2)}$. Of course , the last inequality is interesting only for 2-dimensional Riemannian manifolds which ,topologically are sphere . Summarizing ,we obtain the following proposition originally due to, Hijazi ,and Bar.

Proposition 2.4.5 Dirac Operator

If (S^2, g) is a Riemannian metric on $D\psi = \lambda\psi$,then for the first eigenvalue of the Dirac operator ,we have .

(54) $$\frac{\lambda^2 \ge 4\pi}{vol(S^2,g)}$$

The method we have outlined for estimating the eigenvalues of the Dirac operator may be refined even further when the Riemannian manifold carries additional geometric structures. Let us consider e.g .the case of kahler manifold (M^{2k}, J, g) with complex structure $J = T(M^{2k})$ .In this situation consider the covariant derivative .

(55) $$\nabla_x\psi = \nabla_x\psi + f\,X,\psi + h_j(X).\psi$$

Depending on two parameters f and h which can be chosen freely .Elaborating on the weitzenbock formulas for Riemannian manifolds with additional geometric structures one will in general case of a Riemannian manifold .For example the following inequality first proved by $k - D$, *kirchberg* ,holds for killer manifolds

Proposition 2.4.6

Let (M^{2k}, J, g) be a compact kahler spin manifold and λ an eigenvalue of the Dirac operator ,then

(56)

$$\lambda^2 \ge \left\{\frac{1}{4}\frac{k+1}{k}R_0 \quad if\ k = \dim M_2\right., \left\{\frac{1}{4}\frac{k}{k-1} \quad if\ k = \dim M_2\right.$$

Remark 2.4.7

The kahler case has been investigated by Kramer , semmelmann ,and Weingarten

Defintion 2.4.8 Riemannian Manifolds With Killing Spinors

By the proposition manifold in a spinor field ψ which is an eigenspoinor for the eigenvalue $\frac{1}{2}\sqrt{\frac{n}{n(n-1)}}R_0$ solves the stronger field equation

(57) $$\nabla_x\psi = \pm\frac{1}{2}\sqrt{\frac{R_0}{n(n-1)}}X,\psi$$

This leads to general notion of killing spinors .

Definition 2.4.9 Riemannian Spin manifolds is called Killing

A spinor field ψ defined on a Riemannian spin manifolds (M^2, g) is called a killing spinor , if there exists a complex number μ such that $\nabla_x\psi = \mu X.\psi$. For all vector $X \in T_\mu$, it self is called killing number of ψ we begin by listing afew elementary properties of killing spinors .

Proposition 2.4.10

Let (M^2, g) be a connect Riemannian manifold

(i)A not identically vanishing killing spinor has no zoroes
(ii) Every killing spinor ψ belongs to the kernel of twistor operator T. Moreover , ψ is an eigenspinor of the Dirac operator $D(\psi) = -n\mu\psi_0$ If ψ is killing spinor cooresponding to a real killing number $\mu \in R^1$, then the vector field .

(58) $$V^{\psi} = \sum_{i=1}^{n} (e_i, \psi, \psi) e_i$$

Is a killing vector field of the Riemannian manifold (M^2, g)

Proof : A killing spinor restricted to the curve $r(t), \psi(t) = \psi(r(t))$ satisfies the following first ordinary differential equation a long curve $\frac{d}{dt}\psi(t) = \mu\, r(t).\psi(t)$

Now : $\psi(0) = 0$, immediately implies $\psi(r(t)) \equiv 0$, and this in turn yields starting form $\nabla_x \psi = \mu X, \psi$ .we compute $D\psi = \sum_{i=1}^{n} e_i \nabla_{e_i} \psi = \mu \sum_{i=1}^{n} e_i . e_i . \Psi = -n\mu\psi$

And thus obtain .

(60)

$$T(\psi) = \sum_{i=1}^{n} e_i \otimes \left(\nabla_{e_i} \psi + \frac{1}{n} e_i\, D\psi \right)$$

For a fixed point $m_0 \in M^2$,and a local orthonormal frame $e_1, ..., e_n$ with $\nabla_{e_i}(m_0) = 0$,we copute the covariant derivative $\nabla_x V^{\psi}$.

$$\nabla_x V^{\psi} = \sum (e_i, \nabla_x \psi, \psi) e_i + \sum (e_i, \psi, \nabla_x \psi)$$
$$= \left(\mu \sum_{i=1}^{n} (e_i . X - X_{e_i}) \psi, \psi\right)_{e_i}$$

This implies $g(\nabla_x V^{\psi}, Y) = \mu(YX - XY)\psi, \psi$ .hence $g(\nabla_x V^{\psi}, \psi)$ is antisymmetric in X, Y . But this property characterizes killing vector field on a Riemannian manifold .

Not : every riemannian manifold allows killing spinors $\psi \neq 0$ and not every number $\nabla_x \mu \in c$,occurs as a killing number we derive a series of necessary conditions .To this end , recall ,weg1 tensor of a Riemannian manifold .

Let .
$$R_{i,j,k,l} = g\left(\nabla_{e_i} \nabla_{e_j} e_k - \nabla_{e_j} \nabla_{e_i}\right) - \nabla_{[e_i, e_j]}\, e_k\, e_j$$

Be the components of curvature tensor and .

(61) $$R_{i,j} = \sum_{\alpha=1}^{n} R_{\alpha_i} R_{\alpha_j}$$

Those of the Ricci tensor . Then define two new tensor k ,and W by.

(62) $$(k_{i,j} = \frac{1}{n-2}\left\{ \frac{R}{2(n-1)} g_{i,j} - R_{i,j} \right\}$$

$$W_{\alpha,\beta,\delta} = R_{\alpha,\beta,r,\delta} - g_{\beta,\delta} \cdot k_{\alpha,r} - g_{\alpha,r} \cdot k_{\alpha,r} - g_{\alpha,r} \cdot k_{\beta,\delta} + g_{\beta,r} \cdot k_{\delta,\beta}$$

W is called the “weg1 tensor ” of the Riemannian manifolds because of its symmetry properties the tensor can be considered as a bundle morphism defined on the z-forms of with these notations we have the following

GET PEER REVIEWED

The basic notions on differential geometry calculus , Encluding the geometric formulation φ of the notion of the differential and the inverse function φ^{-1} theorem ∂M . A certain familiarity with the elements of the differential Geometry of surfaces with the basic definition of differentiable manifolds , starting with properties of covering spaces and of the fundamental group and its relation to covering spaces

REFERENCES

[1] Osman.Mohamed M,Basic integration on smooth manifolds and application maps with stokes theorem ,http//www.ijsrp.org-6-januarly2016.

[2] Osman.Mohamed M, fundamental metric tensor fields on Riemannian geometry with application to tangent and cotangent ,http//www.ijsrp.org- 6januarly2016.

[3] Osman.Mohamed M, operate theory Riemannian differentiable manifolds ,http//www.ijsrp.org-6januarly2016.

[4] J.Glover,Z,pop-stojanovic, M.Rao, H.sikic, R.song and Z.vondracek, Harmonic functions of subordinate killed Brownian motion,Journal of functional analysis 215(2004)399-426.

[5] M.Dimitri, P.Patrizia , Maximum principles for inhomogeneous Ellipti inequalities on complete Riemannian Manifolds , Univ. degli studi di Perugia,Via vanvitelli 1,06129 perugia,Italy e-mails : Mugnai@unipg,it , 24July2008.

[6] Noel.J.Hicks . Differential Geometry , Van Nostrand Reinhold Company450 west by Van N.Y10001.

[7] L.Jin,L.Zhiqin , Bounds of Eigenvalues on Riemannian Manifolds, Higher eduationpress and international press Beijing-Boston ,ALM10,pp. 241-264.

[8] S.Robert. Strichaiz, Analysis of the laplacian on the complete Riemannian manifolds. Journal of functional analysis52,48,79(1983).

[9] P.Hariujulehto.P.Hasto, V.Latvala,O.Toivanen , The strong minmum principle for quasisuperminimizers of non-standard growth, preprint submitted to Elsevier , june 16,2011.-Gomez,F,Rniz del potal 2004 .

[10] Cristian , D.Paual, The Classial Maximum principles some of ITS Extensions and Applicationd , Series on Math.and it Applications , Number 2-2011.

[11] H.Amann ,Maximum principles and principlal Eigenvalues , J.Ferrera .J.Lopez

[12] J.Ansgar staudingerweg 9.55099 mainz,Germany e-mail:juengel@mathematik.uni-mainz.de ,U.Andreas institute fiir mathematic,MA6-3,TU Berlin strabe des 17.juni. 136,10623 Berlin ,Germany, e-mail: unterreiter@math.tu-berlin..

[13] R.J.Duffin,The maximum principle and Biharmoni functions, journal of math. Analysis and applications 3.399-405(1961).

First Author
Dr. : Mohamed Mahmoud Osman- (phd)
Studentate the University of Al-Baha –Kingdom of Saudi Arabia
Al-Baha P.O.Box (1988) – Tel.Fax : 00966-7-7274111
Department of mathematics faculty of science
Tel. 00966535126844