Transmuted Generalized Lindley Distribution

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Abstract:*The Lindley distribution is one of the important for studying stress-strength reliability modeling. Besides, some researchers have proposed new classesof distributions based on modifications of the quasi Lindley distribution. In this paper, a new generalized version of this distribution which is called the transmutedgeneralized Lindley (TGL) distribution is introduced. A comprehensive mathematical treatment of the TGL distribution is provided. We derive the r^{th} moment and moment generating function this distribution. Moreover, we discuss the maximum likelihood estimation of this distribution.*

Keywords: *Generalized Lindley distribution; Survival function; Moments. Maximum likelihood estimation.*

1- *Introduction and Motivation*

Generalized Lindley distribution with parameters a and θ is introduced by Nadarajah et al. (2011) its probability density function (p.d.f) is given by

$$g(x,a,\theta)=\frac{a\theta^2}{\theta+1}(1+x)e^{-\theta x}\left[1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+1}\right)\right]^{a-1};\ x>0,a>0,\theta>-1. \qquad (1.1)$$

The cumulative distribution function (cdf) of GLD is obtained as

$$G(x,a,\theta)=\left[1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+1}\right)\right]^{a},x>0,a>0,\theta>-1. \qquad (1.2)$$

1.1. Transmutation Map

In this subsection we demonstrate transmuted probability distribution. Let F_1 and F_2 be the cumulative distribution functions, of two distributions with a common sample space. The general rank transmutation as given in (2007) is defined as

$$G_{R12}(u)=F_2(F_1^{-1}(u))\ \text{and}\ G_{R21}(u)=F_1(F_2^{-1}(u)).$$

Note that the inverse cumulative distribution function also known as quantile function is defined as

$F^{-1}(y)=\inf_{x\in R}\{F(x)\geq y\}$ for $y\in[0,1]$.

The functions $G_{R12}(u)$ and $G_{R21}(u)$ both map the unit interval $I=[0,1]$ into itself, and under suitable assumptions are mutual inverses and they satisfy $G_{Rij}(0)=0$ and $G_{Rij}(0)=1$. A quadratic Rank Transmutation Map (QRTM) is defined as

$$G_{R12}(u)=u+\lambda u(1-u),|\lambda|\leq 1, \qquad (1.3)$$

from which it follows that the cdf's satisfy the relationship

$$F_2(x)=(1+\lambda)F_1(x)-\lambda F_1(x)^2 \qquad (1.4)$$

which on differentiation yields,

$$f_2(x)=f_1(x)\left[(1+\lambda)-2\lambda F_1(x)\right] \qquad (1.5)$$

where $f_1(x)$ and $f_2(x)$ are the corresponding pdfs associated with cdf $F_1(x)$ and $F_2(x)$ respectively. An extensive information about the quadratic rank transmutation map is given in Shaw and Buckely. (2007). Observe that at $\lambda=0$ we have the distribution of the base random variable. The following Lemma proved that the function $f_2(x)$ in given (1.5) satisfies the property of probability density function.

Lemma: $f_2(x)$ given in (1.5) is a well defined probability density function.

Proof:
Rewriting $f_2(x)$ as $f_2(x) = f_1(x)[(1-\lambda(2F_1(x)-1]$ we observe that $f_2(x)$ is nonnegative. We need to show that the integration over the support of the random variable is equal one. Consider the case when the support of $f_1(x)$ is $(-\infty,\infty)$. In this case we have

$$\int_{-\infty}^{\infty} f_2(x)dx = \int_{-\infty}^{\infty} f_1(x)\left[(1+\lambda)-2\lambda F_1(x)\right]dx$$

$$=(1+\lambda)\int_{-\infty}^{\infty} f_1(x)dx - \lambda\int_{-\infty}^{\infty} 2f_1(x)F_1(x)dx$$

$$=(1+\lambda)-\lambda=1$$

Similarly, other cases where the support of the random variable is a part of real line follows. Hence $f_2(x)$ is a well-defined probability density function. We call $f_2(x)$ the transmuted probability density of a random variable with base density $f_1(x)$. Also note that when $\lambda=0$ then $f_2(x)=f_1(x)$.This proves the required result.

Many authors dealing with the generalization of some well- known distributions. Aryal and Tsokos (2009) defined the transmuted generalized extreme value distribution and they studied some basic mathematical characteristics of transmuted Gumbel probability distribution and it has been observed that the transmuted Gumbel can be used to model climate data. Also Aryal and Tsokos (2011) presented a new generalization of Weibull distribution called the transmuted Weibull distribution. Recently, Aryal (2013) proposed and studied the various structural properties of the transmuted Log- Logistic distribution, and Muhammad khan and king (2013) introduced the transmuted modified Weibull distribution which extends recent development on transmuted Weibull distribution by Aryal et al. (2011). And they studied the mathematical properties and maximum likelihood estimation of the unknown parameters. Elbatal (2013) presented transmuted modified inverse Weibull distribution. Elbatal and Elgarhy (2013) presented transmuted quasi Lindley distribution. The rest of the paper is organized as follows. In Section 2 we demonstrate transmuted probability distribution, the hazard rate and reliability functions of TGL distribution. In Section 3 we studied the statistical properties include quantile functions, expansion of density function, moments, moment generating function. The distribution of order statistics is expressed in Section 4. Finally, In Section5, we demonstrate the maximum likelihood estimatesof the unknown parameters.

2. Transmuted Quasi Lindley Distribution

In this section we studied the transmutedgeneralized Lindley (TGL) distribution. Now using (1.1) and (1.2) we have the cdf of transmutedgeneralized Lindley distribution

$$F_{TGL}(x,a,\theta,\lambda)=\left[1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+1}\right)\right]^{a}\left\{1+\lambda-\lambda\left[1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+1}\right)\right]^{a}\right\} \tag{2.1}$$

where θ scale parameter, a shape parameter and λ is the transmuted parameter. The probability density function (pdf) of the transmutedgeneralized Lindley distribution is given by

$$f_{TGL}(x,a,\theta,\lambda)=\frac{a\theta^2}{\theta+1}(1+x)e^{-\theta x}\left[1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+1}\right)\right]^{a-1}\times\left\{(1+\lambda)-2\lambda\left[1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+1}\right)\right]^{a}\right\}$$

$$=\frac{a\theta^2(1+\lambda)}{\theta+1}(1+x)e^{-\theta x}\left[1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+1}\right)\right]^{a-1}$$
$$-\frac{2a\theta^2\lambda}{\theta+1}(1+x)e^{-\theta x}\left[1-e^{-\theta x}\left(1+\frac{\theta x}{\theta+1}\right)\right]^{2a-1} \tag{2.2}$$

The reliability function of the transmutedgeneralized Lindley distribution is denoted by $R_{TGL}(x)$ also known as the survivor function and is defined as

$$R_{TGL}(x) = 1 - F_{TGL}(x) = 1 - \left[1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta+1}\right)\right]^{a}\left\{1 + \lambda - \lambda\left[1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta+1}\right)\right]^{a}\right\}. \quad (2.3)$$

It is important to note that $R_{TGL}(x) + F_{TGL}(x) = 1$. One of the characteristic in reliability analysis is the hazard rate and the reversed hazard ratefunctions defined by

$$h_{TGL}(x) = \frac{f_{TGL}(x)}{1 - F_{TGL}(x)}$$

$$= \frac{\frac{a\theta^2}{\theta+1}(1+x)e^{-\theta x}\left[1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta+1}\right)\right]^{a-1}\left\{(1+\lambda) - 2\lambda\left[1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta+1}\right)\right]^{a}\right\}}{1 - \left[1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta+1}\right)\right]^{a}\left\{1 + \lambda - \lambda\left[1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta+1}\right)\right]^{a}\right\}}, \quad (2.4)$$

$$\tau_{TGL}(x) = \frac{f_{TGL}(x)}{F_{TGL}(x)} = \frac{\frac{a\theta^2}{\theta+1}(1+x)e^{-\theta x}\left\{(1+\lambda) - 2\lambda\left[1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta+1}\right)\right]^{a}\right\}}{\left[1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta+1}\right)\right]\left\{1 + \lambda - \lambda\left[1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta+1}\right)\right]^{a}\right\}}. \quad (2.5)$$

respectively. It is important to note that the units for $h_{TGL}(x)$ is the probability of failure per unit of time, distance or cycles. These failure rates are defined with different choices of parameters. The cumulative hazard function of the transmuted generalized Lindley distribution is denoted by $H_{TGL}(x)$ and is defined as

$$H_{TGL}(x) = -\ln\left|\left[1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta+1}\right)\right]^{a}\left\{1 + \lambda - \lambda\left[1 - e^{-\theta x}\left(1 + \frac{\theta x}{\theta+1}\right)\right]^{a}\right\}\right|. \quad (2.6)$$

It is important to note that the units for $H_{TGL}(x)$ is the cumulative probability of failure per unit of time, distance or cycles. We can show that. For all choice of parameters the distribution has the decreasing patterns of cumulative instantaneous failure rates.

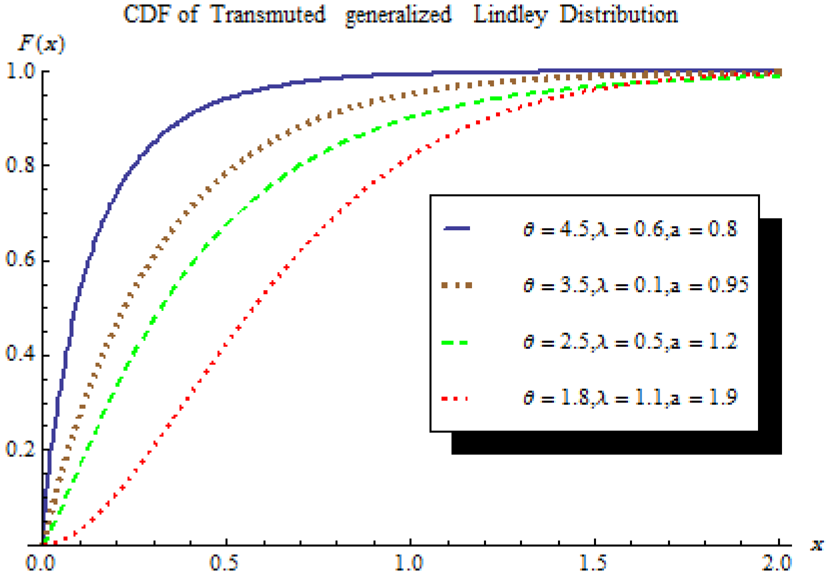

Figure 1: Plot of the cdf of TGL distribution for selected values of the parameters.

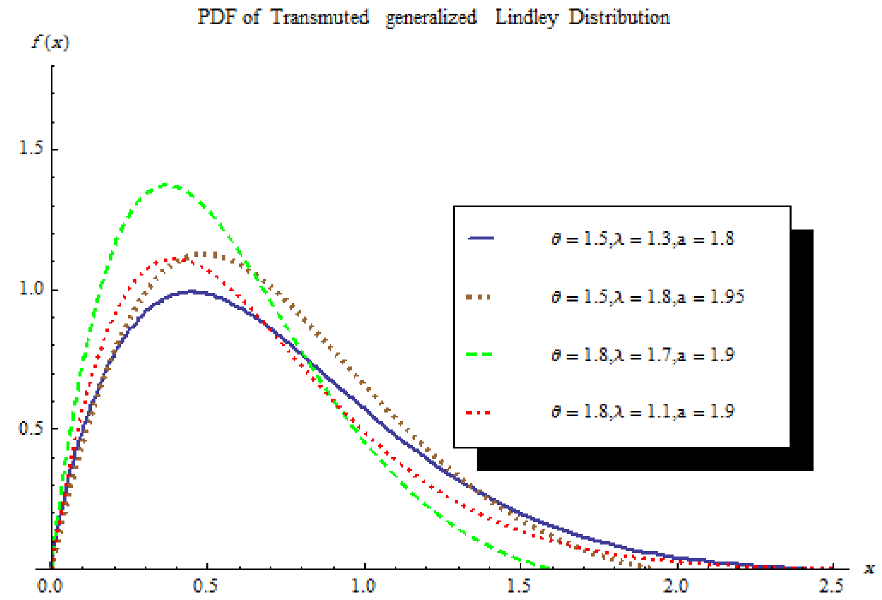

Figure 2: Plot of the pdf of TGL distribution for selected values of the parameters.

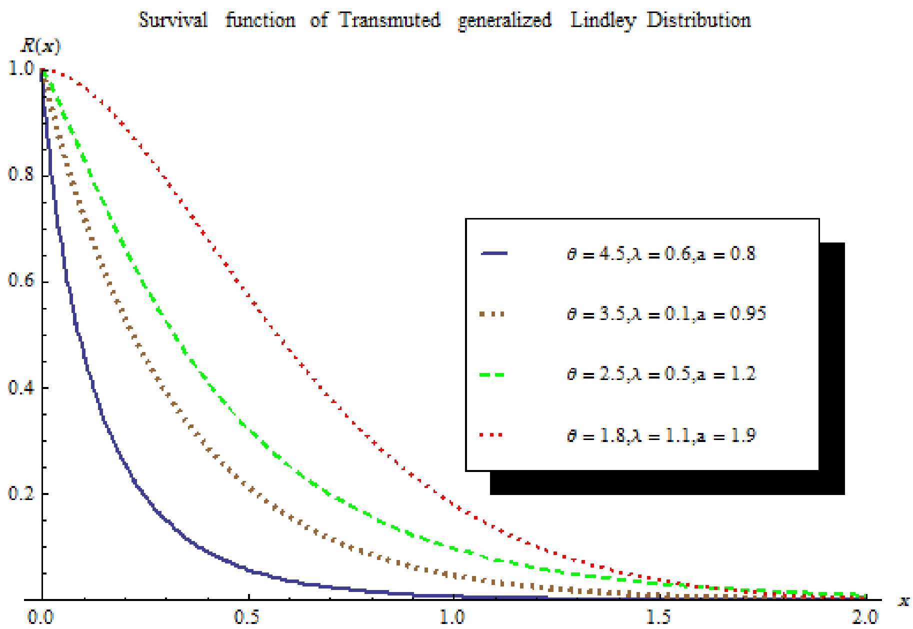

Figure 3: Plot of the survival function of TGL distribution for selected values of the parameters.

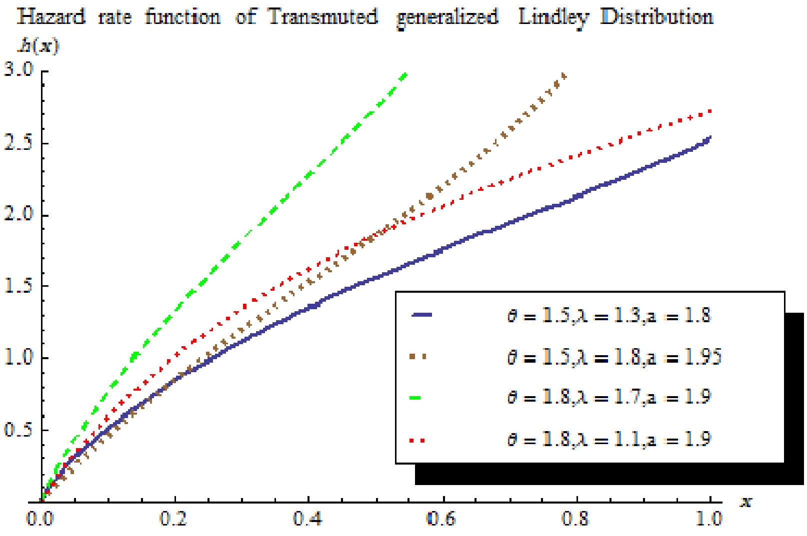

Figure 4: Plot of the hazard rate function of TGL distribution for selected values of the parameters.

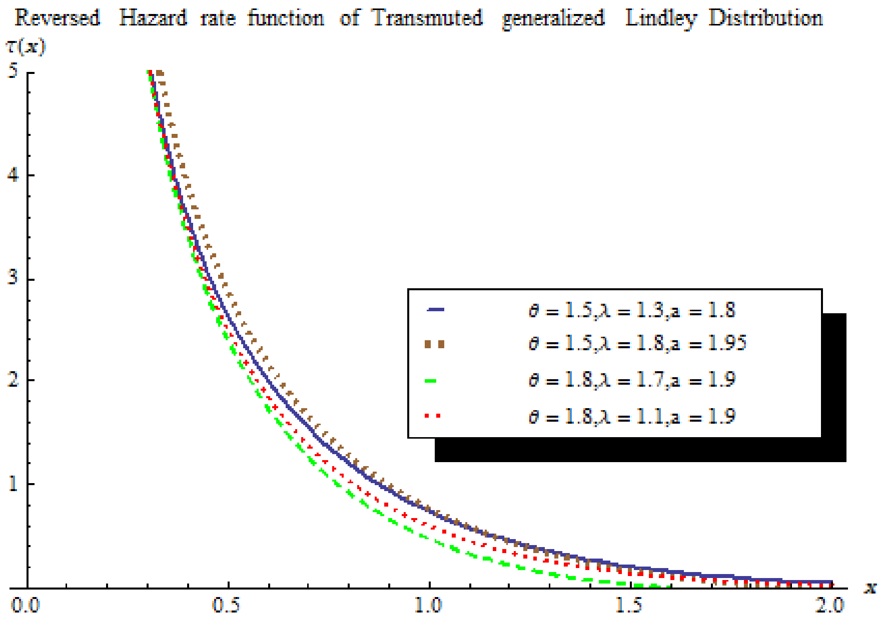

Figure 5: Plot of the reversed hazard rate function of TGL distribution for selected values of the parameters.

Special Cases of the TGL Distribution

The transmuted generalized Lindley is very flexible model that approaches to different distributions when its parameters are changed. The TGL distribution contains as special- models the following well known distributions. If X is a random variable with cdf (2.1), then we have the following cases**:**

1- If $a = 1$ then Equation (2.1) gives transmuted Lindley distribution. That introduced by Faton (2013).

2- If $\lambda = 0$ and $a = 1$ then Equation (2.1) gives Lindley distribution. That introduced by Lindley (1958).

3- If $\lambda = 0$ then Equation (2.1) gives generalized Lindley distribution. That introduced by Nadarajah et al. (2013).

3. Statistical Properties

This section is devoted to studying statistical properties of the (TGL) distribution, specifically quantile function, moments and moment generating function.

3.1. Quantile Function

The qth quantile x_q of the transmuted generalized Lindley distribution can be obtained from (2.1) as

$$e^{-\theta x_q}\left(1+\frac{\theta x_q}{\theta+1}\right)=1-\left\{\frac{(\lambda+1)\pm\sqrt{(\lambda-1)^2-4\lambda q}}{2\lambda}\right\}^{\frac{1}{a}} \tag{3.1}$$

We simulate the TGL distribution by solving the nonlinear equation

$$e^{-\theta x_q}\left(1+\frac{\theta x_q}{\theta+1}\right)=1-\left\{\frac{(\lambda+1)\pm\sqrt{(\lambda-1)^2-4\lambda u}}{2\lambda}\right\}^{\frac{1}{a}}$$

where u has the uniform $U(0,1)$ distribution.

3.2. Expansion of Density Function

In this section representation of pdf for transmuted Kumaraswamy quasi Lindley distribution will be presented. The mathematical relation given below will be useful in this subsection.

It is well-known that, if $\beta>0$ is real non integer and $|z|<1$, the generalized binomial theorem is written as follows

$$(1-z)^{\beta-1}=\sum_{i=0}^{\infty}(-1)^i\binom{\beta-1}{i}z^i. \tag{3.2}$$

Then, by applying the binomial theorem (3.2) in (2.2), the probability density function n of TGL distribution becomes

$$f(x)=\frac{a\theta^2(1+\lambda)}{\alpha+1}(1+x)\sum_{i=0}^{\infty}(-1)^i\binom{a-1}{i}e^{-\theta(i+1)x}\left(1+\frac{\theta x}{\theta+1}\right)^i - \frac{2a\theta^2\lambda}{\theta+1}(1+x)e^{-\theta x}\sum_{i=0}^{\infty}(-1)^i\binom{2a-1}{i}e^{-\theta(i+1)x}\left(1+\frac{\theta x}{\theta+1}\right)^i$$

$$f(x)=\frac{a\theta^2(1+\lambda)M}{\theta+1}\left(x^j+x^{j+1}\right)e^{-\theta(i+1)x}-\frac{2a\theta^2\lambda N}{\theta+1}\left(x^j+x^{j+1}\right)e^{-\theta(i+1)x} \tag{3.3}$$

where

$$\mathrm{M}=\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^i\left(\frac{\theta}{\theta+1}\right)^j\binom{a-1}{i}\binom{i}{j},$$

and

$$N=\sum_{i,j=0}^{\infty}\sum_{k=0}^{j}(-1)^i\left(\frac{\theta}{\theta+1}\right)^j\binom{2a-1}{i}\binom{i}{j}.$$

If β is an integer the index i in the previous sum stops at $\beta-1$.

3.3. Moments

In this subsection we discuss the r^{th} moment for TGL distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

Theorem (3.1).

If X has $TGL(x,\theta,a,\lambda)$, then the r^{th} moment of X is given by the following

$$\mu_r^{'}(x)=\frac{a\theta^2(1+\lambda)M}{\theta+1}\left[\frac{\Gamma(r+j+1)}{[\theta(i+1)]^{r+j+1}}+\frac{\Gamma(r+j+2)}{[\theta(i+1)]^{r+j+2}}\right]-\frac{2a\theta^2\lambda N}{\theta+1}\left[\frac{\Gamma(r+j+1)}{[\theta(i+1)]^{r+j+1}}+\frac{\Gamma(r+j+2)}{[\theta(i+1)]^{r+j+2}}\right] \tag{3.4}$$

Proof:

Let X be a random variable with density function (3.3). The r^{th} ordinary moment of the (TGL) distribution is given by

$$\mu_r^{'}(x)=E(X^{r})=\int_0^\infty x^r f(x)dx$$

$$=\frac{a\theta^2(1+\lambda)M}{\theta+1}\int_0^\infty\left(x^{r+j}+x^{r+j+1}\right)e^{-(i+1)\theta x}dx-\frac{2a\theta^2\lambda N}{\theta+1}\int_0^\infty\left(x^{r+j}+x^{r+j+1}\right)e^{-(i+1)\theta x}dx$$

then

$$\mu_r^{'}(x)=\frac{a\theta^2(1+\lambda)M}{\theta+1}\left[\frac{\Gamma(r+j+1)}{[\theta(i+1)]^{r+j+1}}+\frac{\Gamma(r+j+2)}{[\theta(i+1)]^{r+j+2}}\right]-\frac{2a\theta^2\lambda N}{\theta+1}\left[\frac{\Gamma(r+j+1)}{[\theta(i+1)]^{r+j+1}}+\frac{\Gamma(r+j+2)}{[\theta(i+1)]^{r+j+2}}\right]$$

This completes the proof.

Based on the first four moments of the TGL distribution, the measures of skewness $A(\Phi)$ and kurtosis $k(\Phi)$ of the $TKQL$ distribution can obtained as

$$A(\Phi)=\frac{\mu_3(\theta)-3\mu_1(\theta)\mu_2(\theta)+2\mu_1^3(\theta)}{\left[\mu_2(\theta)-\mu_1^2(\theta)\right]^{\frac{3}{2}}}, \tag{3.5}$$

and

$$k(\Phi)=\frac{\mu_4(\theta)-4\mu_1(\theta)\mu_3(\theta)+6\mu_1^2(\theta)\mu_2(\theta)-3\mu_1^4(\theta)}{\left[\mu_2(\theta)-\mu_1^2(\theta)\right]^2}. \tag{3.6}$$

3.4. Moment generating function

In this subsection we derived the moment generating function of TGL distribution.

Theorem (3.2): If X has TGL distribution, then the moment generating function $M_X(t)$ has the following form

$$M_X(t)=\frac{a\theta^2(1+\lambda)M}{\theta+1}\left[\frac{\Gamma(j+1)}{(\theta+\theta i-t)^{j+1}}+\frac{\Gamma(j+2)}{(\theta+\theta i-t)^{j+2}}\right]-\frac{2a\theta^2\lambda N}{\theta+1}\left[\frac{\Gamma(j+1)}{(\theta+\theta i-t)^{j+1}}+\frac{\Gamma(j+2)}{(\theta+\theta i-t)^{j+2}}\right] \tag{3.7}$$

Proof:

We start with the well known definition of the moment generating function given by

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f_{TGL}(x)dx$$

$$= \frac{a\theta^2(1+\lambda)M}{\theta+1}\int_0^\infty \left(x^j + x^{j+1}\right)e^{-(\theta+\theta i-t)x}dx - \frac{2a\theta^2\lambda N}{\theta+1}\int_0^\infty \left(x^j + x^{j+1}\right)e^{-(\theta+\theta i-t)x}dx$$

Then,

$$M_X(t) = \frac{ab\theta(1-\lambda)\mathrm{M}}{\alpha+1}\left[\frac{\alpha\Gamma(k+1)}{(\theta+\theta j-t)^{k+1}} + \frac{\theta\Gamma(k+2)}{(\theta+\theta j-t)^{k+2}}\right] + \frac{2ab\theta\lambda N}{\alpha+1}\left[\frac{\alpha\Gamma(k+1)}{(\theta+\theta j-t)^{k+1}} + \frac{\theta\Gamma(k+2)}{(\theta+\theta j-t)^{k+2}}\right]$$

This completes the proof.

4. Distribution of the order statistics

In this section, we derive closed form expressions for the pdfs of the r^{th} order statistic of the TGL distribution, also, the measures of skewness and kurtosis of the distribution of the r^{th} order statistic in a sample of size n for different choices of $n; r$ are presented in this section. Let $X_1, X_2, \ldots, X_n$ be a simple random sample from TGL distribution with pdf and cdf given by (2.1) and (2.2), respectively.

Let $X_1, X_2, \ldots, X_n$ denote the order statistics obtained from this sample. We now give the probability density function of $X_{r:n}$, say $f_{r:n}(x,\theta,a,\lambda)$ and the moments of $X_{r:n}, r = 1,2,\ldots,n$. Therefore, the measures of skewness and kurtosis of the distribution of the $X_{r:n}$ are presented. The probability density function of $X_{r:n}$ is given by

$$f_{r:n}(x,\theta,a,\lambda) = \frac{1}{B(r,n-r+1)}\left[F(x,\theta,a,\lambda)\right]^{r-1}\left[1-F(x,\theta,a,\lambda)\right]^{n-r} f(x,\theta,a,\lambda) \qquad (4.1)$$

where $F(x,\theta,a,\lambda)$ and $f(x,\theta,a,\lambda)$ are the cdf and pdf of the TGL distribution given by (2.1), (2.2), respectively, and $B(.,.)$ is the beta function, since $0 < F(x,\theta,a,\lambda) < 1$, for $x > 0$, by using the binomial series expansion of $\left[1-F(x,\theta,a,\lambda)\right]^{n-r}$, given by

$$\left[1-F(x,\theta,a,\lambda)\right]^{n-r} = \sum_{j=0}^{n-r}(-1)^j \binom{n-r}{j}\left[F(x,\theta,a,\lambda)\right]^j, \qquad (4.2)$$

we have

$$f_{r:n}(x,\theta,a,\lambda) = \frac{1}{B(r,n-r+1)}\sum_{j=0}^{n-r}(-1)^j \binom{n-r}{j}\left[F(x,\theta,a,\lambda)\right]^{r+j-1} f(x,\theta,a,\lambda), \qquad (4.3)$$

substituting from (2.1) and (2.2) into (4.3), we can express the k^{th} ordinary moment of the r^{th} order statistics $X_{r:n}$ say $E(X_{r:n}^k)$ as a liner combination of the k^{th} moments of the TGL distribution with different shape parameters. Therefore, the measures of skewness and kurtosis of the distribution of $X_{r:n}$ can be calculated.

5. Estimation and Inference

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the TGL distribution from complete samples only. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from

$TGL(x,\theta,a,\lambda)$ .The likelihood function for the vector of parameters $\Phi=(x,\theta,a,\lambda)$ can be written as

$$L=f(x_i,\Phi)=\prod_{i=1}^{n} f(x_i,\Phi)$$
$$=\left(\frac{a\theta^2}{\theta+1}\right)^n \prod_{i=1}^{n}(1+x_i)e^{-\theta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\left\{(1+\lambda)-2\lambda\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\theta+1}\right)\right]^a\right\}$$
$$\prod_{i=1}^{n}\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\theta+1}\right)\right]^{a-1}. \qquad (5.1)$$

Taking the log-likelihood function for the vector of parameters $\Phi=(x,\theta,a,\lambda)$ we get

$$\ln L=n\ln a+2n\ln\theta-n\ln(\theta+1)+\sum_{i=1}^{n}\ln(1+x_i)-\theta\sum_{i=1}^{n}x_i$$
$$+(a+1)\sum_{i=1}^{n}\ln\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\alpha+1}\right)\right]+\sum_{i=1}^{n}\ln\left[(1+\lambda)-2\lambda\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\theta+1}\right)\right]^a\right]. \qquad (5.2)$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (5.2). The components of the score vector are given by

$$\frac{\partial\ln L}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}\ln\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\theta+1}\right)\right]-\sum_{i=1}^{n}\frac{2\lambda\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\theta+1}\right)\right]^a\ln\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\theta+1}\right)\right]}{(1-\lambda)+2\lambda\left\{1-\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\theta+1}\right)\right]^a\right\}^b}, (5.3)$$

$$\frac{\partial\ln L}{\partial\lambda}=-\sum_{i=1}^{n}\frac{1-2\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\theta+1}\right)\right]^a}{(1+\lambda)-2\lambda\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\theta+1}\right)\right]^a}, \qquad (5.4)$$

and

$$\frac{\partial\ln L}{\partial\theta}=\frac{n}{\theta}-\frac{n}{\theta+1}-\sum_{i=1}^{n}x_i+(a+1)\sum_{i=1}^{n}\frac{x_ie^{-\theta x_i}\left(1+\frac{\theta x_i}{\theta+1}\right)-\frac{x_ie^{-\theta x_i}}{\theta+1}}{1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\theta+1}\right)}$$
$$-2a\sum_{i=1}^{n}\frac{\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\theta+1}\right)\right]^{a-1}\left[x_ie^{-\theta x_i}\left(1+\frac{\theta x_i}{\theta+1}\right)-\frac{x_ie^{-\theta x_i}}{\theta+1}\right]}{(1+\lambda)-2\lambda\left[1-e^{-\theta x_i}\left(1+\frac{\theta x_i}{\theta+1}\right)\right]^a}. \qquad (5.5)$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations to zero and solve them simultaneously. Therefore, we have to use mathematical package to get the MLE of the unknown parameters. Also, all the second order derivatives exist. Thus we have the inverse dispersion matrix is given by

$$\begin{pmatrix}\hat{\theta}\\ \hat{a}\\ \hat{\lambda}\end{pmatrix} \sim N\left[\begin{pmatrix}\hat{\theta}\\ \hat{a}\\ \hat{\lambda}\end{pmatrix},\begin{pmatrix}V_{\theta\theta} & V_{\theta a} & V_{\theta\lambda}\\ V_{a\theta} & V_{aa} & V_{a\lambda}\\ V_{\lambda\theta} & V_{\lambda a} & V_{\lambda\lambda}\end{pmatrix}\right]. \quad (5.6)$$

$$V^{-1} = -E\begin{pmatrix}V_{\theta\theta} & V_{\theta a} & V_{\theta\lambda}\\ V_{a\theta} & V_{aa} & V_{a\lambda}\\ V_{\lambda\theta} & V_{\lambda a} & V_{\lambda\lambda}\end{pmatrix}$$

Where

$$V_{aa} = \frac{\partial^2 L}{\partial a^2}, V_{\lambda\lambda} = \frac{\partial^2 L}{\partial \lambda^2}, V_{\theta\theta} = \frac{\partial^2 L}{\partial \theta^2}, V_{a\lambda} = \frac{\partial^2 L}{\partial a \partial \lambda},, V_{a\theta} = \frac{\partial^2 L}{\partial a \partial \theta}, V_{\lambda\theta} = \frac{\partial^2 L}{\partial \theta \partial \lambda}.$$

By solving this inverse dispersion matrix these solutions will yield asymptotic variance and covariance's of these ML estimators for $\hat{\theta}, \hat{a}$ and $\hat{\lambda}$. Using (5.6), we approximate $100(1-\gamma)\%$ confidence intervals for a, b, λ, α and θ are determined respectively as

$$\hat{\theta} \pm z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{\theta\theta}}, \hat{a} \pm z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{aa}} \text{ and } \hat{\lambda} \pm z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{\lambda\lambda}}.$$

where z_{γ} is the upper 100γ the percentile of the standard normal distribution.

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