On Some New Hadamard Type Inequalities for Co-Ordinated (α, β) -Convex Functions

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Abstract

In this paper, we establish some new Hermite-Hadamard type inequalities for m-convex and (α, β)-convex functions of 2-variables on theco-ordinates.

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1. Introduction.

The following definition is well known in literature:

A function $\emptyset$: $I \rightarrow$ R, $\emptyset \neq I \subseteq$ R, is said to be convex on *I* if the inequality

$\emptyset\, ((\lambda x + 1(1-\lambda) y) \leq \lambda \emptyset\, (x) + \; (1 - \; \lambda) \emptyset\, (y),$

hold for all x, y ϵI and $\lambda \epsilon$ [0,1].

Many important inequalities have been established for the class of convex functions but the most famous is the Hermite –Hadamard's inequality. This double inequality is stated as:

Let $\emptyset$: I $\subseteq R \rightarrow R$ be a convex mapping defined on the interval I of real numbers, and a, b $\epsilon\, I$ with a < b. the following double inequality is well known in the literature as the Hermite – Hadamard inequality [5]:

$$\emptyset \left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \emptyset\,(x) dx \;\leq\; \frac{\emptyset\,(a) +\; \emptyset\,(b)}{2}$$

The concept of usually used convexity has been generalized by a number of mathematicians. Some of them can be recited as follows:

Definition 1.1. ([13]). Let $\emptyset$: [0, b] → R be a function and $\beta \;\in$ (0, 1]. If (1.3)

$$\emptyset\,(\lambda x + \beta(1-\lambda) y) \leq \; \lambda \emptyset(x) + \beta(1-\lambda)\emptyset(y)$$

Holds for all x, y ∈ [0, b] and$\lambda \;\in$ [0, 1]., then we say that the function $\emptyset$ (x) is β - convex on [0. b].

Definition 1.2. ([13]). Let $\emptyset$: [0, b] → R be a function and $(\alpha, \beta) \in (0,1]^2$. If (1.4)

$$\emptyset\,(\lambda x + \beta(1-\lambda) y) \leq \; \lambda^{\alpha}\, f\,(x) + \beta(1-\lambda^{\alpha})\emptyset(y)$$

Holds for all x, y ∈ [0, b] and$\lambda \;\in$ [0, 1]., then we say that the function $\emptyset$ (x) is (α, β) - convex on [0. b].

In recent years, some other kinds of Hermite – Hardamard type inequalities were generated in, for example, [1, 2, 3, 5, 7, 9]. For more systematic information, please refer to monographs [4, 6] and related references therein.

In this paper, we will established some new inequalities of Hermite – Hadamard type for functions whose derivatives of *n*-th order are (α, β) – convex and deduce some known results in terms of corollaries.

Main Results:

To establish our main result, we need the following lemma:

Lemma 2.1. Let $0 < \beta \leq 1$ and $b > a > 0$ satisfying $a \neq \beta b$. If $\emptyset^{(n)}(x)$ for $n \in \{0\} \cup N$ exists and is, integrable on the closed interval $[0, b]$, then

$$(2.1)\ \frac{\emptyset(a) + \emptyset(\beta b)}{2} - \frac{1}{\beta b - a}\int_a^{\beta b} \emptyset(x)\,dx - \frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(\beta b - a)^k}{(k+1)!}\emptyset^k(a) = \frac{1}{2}\frac{(\beta b - a)^n}{n!}\int_0^{\beta b} t^{n-1}(n-2t)\,\emptyset^n(ta + \beta(1-t)b)dt,$$

Where the sum above takes 0 when n = 1 and n = 2.
Proof: When n = 1, it is easy to deduce identity (2.1) by performing an integration by parts in the integrals from the wright side and changing the variable.
When n = 2 we have

$$(2.2)\ \frac{\emptyset(a) + \emptyset(\beta b)}{2} - \frac{1}{\beta b - a}\int_a^{\beta b} \emptyset(x)\,dx = \frac{1}{2}\frac{(\beta b - a)^2}{2}\int_0^1 t(1-t)\,\emptyset^n(ta + \beta(1-t)b)dt,$$

This result is same as [8.Lemma 2].
When n = 3, the identity (2.1) is equivalent to

$$(2.3)\ \frac{\emptyset(a) + \emptyset(\beta b)}{2} - \frac{1}{\beta b - a}\int_a^{\beta b} \emptyset(x)\,dx - \frac{(\beta b - a)^2}{12}\emptyset^k(a) = \frac{(\beta b - a)^3}{12}\int_0^1 t^2(3-2t)\,\emptyset^3(ta + \beta(1-t)b)dt,$$

which may be derived from integrating the integral in the second line of (2.3) and utilizing the identity (2.2).
When $n \geq 4$, computing the second line in (2.1) by integration- by parts yields

$$\frac{(\beta b - a)^n}{n!}\int_0^1 t^{n-1}(n-2t)\,\emptyset^n(ta + \beta(1-t)b)dt = -\frac{(n-2)(\beta b - a)^{n-1}}{n!}\emptyset^{n-1}(a) + \frac{(\beta b - a)^{n-1}}{12}\int_0^1 t^{n-2}(n-1-2t)\,\emptyset^{n-1}(ta + \beta(1-t)b)dt,$$

This is recurrent formula

$$S_{a,\beta b}(n) = -T_{a,\beta b}(n-1) + S_{a,\beta b}(n-1)$$

On n, where

$$S_{a,\beta b}(n) = \frac{1}{2}\frac{(\beta b - a)^n}{n!}\int_0^1 t^{n-1}(n-2t)\,\emptyset^n(ta + \beta(1-t)b)dt.$$

And

$$T_{a,\beta b}(n-1) = \frac{1}{2}\frac{(n-2)(\beta b - a)^{n-1}}{n!}\emptyset^{n-1}(a)$$

For $n \geq 4$. Bu mathematical induction, the proof of Lemma 2.1 is complete.

Now we are in a position to establish some integral inequalities of Hermite-Handamard type for function whose derivatives of n-th order are (α, β) convex.

Theorem 3.1.let$(\alpha, \beta) \in (0.1)^2$ and $b > a > 0$ with $a \neq \beta b$. If (x) is $n-$time defferentiable on $[0, b]$, such that $|\emptyset^{(n)}| L[0, b]$ and $|\emptyset^{(n)}(x)|^p$ is $(\alpha, \beta)-$ convex on $[0, b]$ for $n \geq 2$ and $p \geq 1$

(3.1) $$\left|\frac{\emptyset(a)+\emptyset(\beta b)}{2}-\frac{1}{\beta b-a}\int_a^{\beta b}\emptyset(x)\,dx-\frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(\beta b-a)^k}{(k+1)!}\emptyset^k(a)\right|$$
$$\leq \frac{1}{2}\frac{|\beta b-a|^n}{n!}\left(\frac{n-1}{n+1}\right)^{1-1/p}\left\{\frac{n(n-1)+\alpha(n-2)}{(n+\alpha)(n+\alpha+1)}\left|\emptyset^{(n)}(b)\right|^p\right.$$
$$\left.+\beta\left\{\frac{n-1}{n+1}-\frac{n(n-1)+\alpha(n-2)}{(n+\alpha)(n+\alpha+1)}\left|\emptyset^{(n)}(a)\right|^p\right\}^{1/p}\right.$$

Where the sum above takes 0 when n = 2.
Proof: It follows from Lemma 2.1 that

(3.2) $$\left|\frac{\emptyset(a)+\emptyset(\beta b)}{2}-\frac{1}{\beta b-a}\int_a^{\beta b}\emptyset(x)\,dx-\frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(\beta b-a)^k}{(k+1)!}\emptyset^k(a)\right|$$
$$\leq \frac{1}{2}\frac{|\beta b-a|^n}{n!}\int_0^{\beta b}t^{n-1}(n-2t)\,\emptyset^n(ta+\beta(1-t)b)dt.$$

When P =1, since $\left|\emptyset^{(n)}(x)\right|$ $is\ (\alpha.b)-convex,$ we have
$$|\emptyset^n(ta+\beta(1-t)b)|t^\alpha\left|\emptyset^{(n)}(a)+\beta(1-t^\alpha)\right|\emptyset^{(n)}(b)|.$$
Multiplying by the factor $t^{n-1}(n-2t)$on the both sides of the above inequality and integrating with respect to $t\in[0.1]$ lead to
$$|\emptyset^n(ta+\beta(1-t)b)|\int_0^1 t^{n-1}(n-2t)\,dt$$
$$\leq \int_0^1 t^{n-1}(n-2t)\,[t^\alpha|\emptyset^n(a)|+\beta(1-t^\alpha)|\emptyset^n(b)|]dt$$
$$=|\emptyset^n(a)|\int_0^1 t^{n+\alpha-1}(n-2t)\,dt+\beta|\emptyset^n(b)|\int_0^1 t^{n-1}(n-2t)(1-t^\alpha)\,dt$$
$$=\left(\frac{n}{n+\alpha}-\frac{2}{n+\alpha+1}\right)|\emptyset^n(a)|+\beta|\emptyset^n(b)|\left(\frac{n-1}{n+1}-\frac{n}{n+\alpha}-\frac{2}{n+\alpha+1}\right)$$
$$=\frac{n(n-1)+\alpha(n-2)}{(n+\alpha)(n+\alpha+1)}|\emptyset^n(a)|+\beta\frac{n(n-1)+\alpha(n-2)}{(n+\alpha)(n+\alpha+1)}|\emptyset^n(b)|$$
The proof for the case P=1 is complete.
When$P>1$, by the well-known Holders integer inequality, we obtain

(3.3) $$\int_0^1 t^{n-1}(n-2t)\,|\emptyset^n(ta+\beta(1-t)b)|dt$$
$$\leq \int_0^1 t^{n-1}(n-2t)\,|\,\emptyset^n(ta+\beta(1-t)b)|dt\left[\int_0^1 t^{n-1}(n-2t)\,dt\right]^{1-1/p}$$
$$\times\left[\int_0^1 t^{n-1}(n-2t)\,|\,\emptyset^n(ta+\beta(1-t)b)|^p dt\right]^{1/p}$$
Using the$(\alpha.b)$-convexity of $|\,\emptyset^n(x)|^p$, we have
(3.4)
$$\int_0^1 t^{n-1}(n-2t)\,|\,\emptyset^n(ta+\beta(1-t)b)|^p dt\leq\int_0^1 t^{n-1}(n-2t)\,[t^\alpha|\emptyset^n(a)|+\beta(1-t^\alpha)|\emptyset^n(b)|^p]dt$$
$$=\frac{n(n-1)+\alpha(n-2)}{(n+\alpha)(n+\alpha+1)}|\emptyset^n(b)|^p+\beta\left|\frac{n-1}{n+1}-\frac{n}{n+\alpha}-\frac{2}{n+\alpha+1}\right||\emptyset^n(b)|^p$$
Combining (3.2), (3.3), and (3.4) yields
$$\left|\frac{\emptyset(a)+\emptyset(\beta b)}{2}-\frac{1}{\beta b-a}\int_a^{\beta b}\emptyset(x)\,dx-\frac{1}{2}\sum_{k=2}^{n-1}\frac{(k-1)(\beta b-a)^k}{(k+1)!}\emptyset^k(a)\right|$$
$$\leq\frac{1}{2}\frac{|\beta b-a|^n}{n!}\left(\frac{n-1}{n+1}\right)^{1-1/p}\left\{\frac{n(n-1)+\alpha(n-2)}{(n+\alpha)(n+\alpha+1)}|\emptyset^n(b)|^p+\beta\left[\frac{n-1}{n+1}-\frac{n}{n+\alpha}-\frac{2}{n+\alpha+1}\right]|\emptyset^n(b)|^p\right\}^{1/p}$$

This completes the proof.

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