On Strongly Multiplicative Graphs

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Abstract — **A graph G with p vertices and q edges is said to be strongly multiplicative if the vertices are assigned distinct numbers 1, 2, 3, …, p such that the labels induced on the edges by the product of the end vertices are distinct. We prove some of the special graphs obtained through graph operations such as C_n^+ (a graph obtained by adding pendent edge for each vertex of the cycle C_n), $(P_n \cup mK_1) + N_2$, $P_n + mK_1$ and C_n^d (cycle C_n with non-intersecting chords) are strongly multiplicative.**

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I. INTRODUCTION

In this paper we deal with only finite, simple, connected and undirected graphs obtained through graph operations. A labeling of a graph G is an assignment of labels to vertices or edges or both following certain rules. Labeling of graphs plays an important role in application of graph theory in Neural Networks, Coding theory, Circuit Analysis etc. A useful survey on graph labeling by J.A. Gallian (2010) can be found in [5]. All graphs considered here are finite, simple and undirected. In most applications labels are positive (or nonnegative) integers, though in general real numbers could be used.

Beineke and Hegde [4] call a graph with p vertices strongly multiplicative if the vertices of G can be labeled with distinct integers 1, 2, . . . , p such that the labels induced on the edges by the product of the end vertices are distinct. They prove the following graphs are strongly multiplicative: trees; cycles; wheels; K_n if and only if $n \le 5$; $K_{r,r}$ if and only if $r \le 4$; and $P_m \times P_n$. Beineke and Hegde [4] obtain an upper bound for the maximum number of edges $\lambda(n)$ for a given strongly multiplicative graph of order n. It was further improved by C. Adiga, H. N. Ramaswamy, and D. D. Somashekara [2] for greater values of n. It remains an open problem to find a nontrivial lower bound for $\lambda(n)$. Seoud and Zid [7] prove the following graphs are strongly multiplicative: wheels; rK_n for all r and n at most 5; rK_n for $r \ge 2$ and n = 6 or 7; rK_n for $r \ge 3$ and n = 8 or 9; $K_{4,r}$ for all r; and the corona of P_n and K_m for all n and $2 \le m \le 8$. Germina and Ajitha [6] prove that $K_2 + K_t$, quadrilateral snakes, Petersen graphs, ladders, and unicyclic graphs are strongly multiplicative. Acharya, Germina, and Ajitha [1] have shown that every graph can be embedded as an induced subgraph of a strongly multiplicative graph. In this paper we study strongly multiplicative labeling for some special classes of graphs.

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Definition 1:

A graph G = (V(G), E(G)) with p vertices is said to be multiplicative if the vertices of G can be labeled with distinct positive integers such that label induced on the edges by the product of labels of end vertices are all distinct.

Definition 2:

A graph G = (V(G), E(G)) with p vertices is said to be strongly multiplicative if the vertices of G can be labeled with p distinct integers 1, 2 ,..., p such that label induced on the edges by the product of labels of the end vertices are all distinct.

II. MAIN RESULTS

Theorem 1:

The graph C_n^+ has strongly multiplicative labeling.

Proof : Let $V = \{v_1, v_2, v_3, ...v_n, v_{n+1}, ...v_{2n}\}$ be the vertex set and $E = E_1 \cup E_2 \cup E_3$ be the edge set where $E_1 = \{v_i v_{i+1},\ 1 \le i < n\}$, $E_2 = \{v_i v_{n+i},\ 1 \le i \le n\}$ and $E_3 = \{v_n v_1\}$ of the graph C_n^+. Define a bijection $f : V \to \{1,2,3,.....2n\}$ such that

Case (i): When n is even

$$f(v_i) = \begin{cases} 4i - 3 & 1 \le i \le \dfrac{n}{2} \\ 4(n-i) + 3 & \dfrac{n}{2} + 1 \le i \le n \end{cases}$$

and

$$f(v_{n+i}) = \begin{cases} 4i - 2 & 1 \le i \le \dfrac{n}{2} \\ 4(n-i) + 4 & \dfrac{n}{2} + 1 \le i \le n \end{cases}$$

Define an induced function $g : E \to N$, such that

$g(v_i v_j) = f(v_i) f(v_j)\ \forall\ v_i v_j \in E$ and $v_i, v_j \in V$

We show that the labeling of edges within the edge sets and among the edge sets are distinct. If it is assumed in each case that the induced label of the edges are same then we arrive at contradiction.

For the edges in E_1:

Sub Case (a) For $i \neq j$ and $1 \leq i,\ j < \frac{n}{2}$

If we assume that $g(v_i v_{i+1}) = g(v_j v_{j+1})$

$\Rightarrow f(v_i) f(v_{i+1}) = f(v_j) f(v_{j+1})$

$(4i-3)\,(4(i+1)-3) = (4j-3)\,(4(j+1)-3)$

gives i = j a contradiction as $i + j = \frac{1}{2}$ is not possible.

Sub Case (b) For $i \neq j$ and $\frac{n}{2}+1 \leq i,\ \ j < n$

If we assume that $g(v_i v_{i+1}) = g(v_j v_{j+1})$

$\Rightarrow f(v_i) f(v_{i+1}) = f(v_j) f(v_{j+1})$

$(4(n-i)+3)(4(n-i-1)+3) = (4(n-j)+3)(4(n-j-1)+3)$

gives i = j a contradiction as $i + j = \frac{4n-1}{2}$ is not possible.

Sub Case (c) For $1 \leq i < \frac{n}{2}$ and $\frac{n}{2}+1 \leq i < n$ also $i \neq j$ in E_1

We have $g(v_i v_{i+1}) = 16i^2 - 8i$ and

$g(v_j v_{j+1}) = (16j^2 - 8j) + (16n^2 + 8n - 32nj)$

Clearly $16i^2 - 8i \neq (16j^2 - 8j) + (16n^2 + 8n - 32nj)$

$\therefore\ g(v_i v_{i+1}) \neq g(v_j v_{j+1})$

For the edges in E_2:

Sub Case (a) For $i \neq j$ and $1 \leq i,\ j \leq \frac{n}{2}$

If we assume that $g(v_i v_{n+i}) = g(v_j v_{n+j})$

$\Rightarrow f(v_i) f(v_{n+i}) = f(v_j) f(v_{n+i})$

$(4i-3)(4i-2) = (4j-3)(4j-2)$

gives i = j a contradiction as $i + j = \frac{5}{4}$ is not possible.

Sub Case (b) For $i \neq j$ and $\frac{n}{2}+1 \leq i,\ \ j \leq n$

If we assume that $g(v_i v_{n+i}) = g(v_j v_{n+j})$

$\Rightarrow f(v_i) f(v_{n+i}) = f(v_j) f(v_{n+j})$

$(4(n-i)+3)(4(n-i)+4) = (4(n-j)+3)(4(n-j)+4)$

gives i = j a contradiction as $i + j = \frac{8n+7}{4}$ is not possible.

Sub Case (c) For $1 \leq i < \frac{n}{2}$ and $\frac{n}{2}+1 \leq j < n$ also $i \neq j$ in E_2:

We see that $g(v_i v_{n+i}) = 16i^2 - 20i + 6$

$g(v_j v_{n+j}) = 16j^2 - 20j + 6 + (16n^2 - 32nj + 28n - 8j + 12)$

$\therefore\ g(v_i v_{n+i}) \neq g(v_j v_{n+j})$

Clearly the labeling of edges of E_1 and that of E_2 are all distinct as the labeling of edges of E_1 are all odd and those of E_2 are even. Also edges of E_1 & E_3 and E_2 & E_3 are also distinct as edge in E_3 is with the minimum odd label 3. C_n^+ has strongly multiplicative labeling for n even.

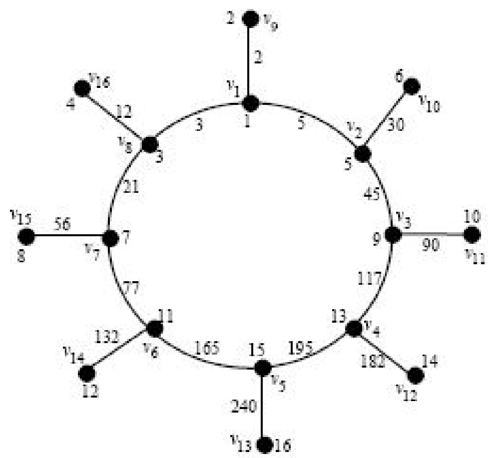

Fig. 1. Strongly multiplicative labeling for C_8^+

Case (ii): When n is odd

The bijection $f : V \rightarrow \{1,2,3,.....2n\}$ is defined as the following

$$f(v_i) = \begin{cases} 4i-3 & 1 \leq i \leq \frac{n+1}{2} \\ 4(n-i)+3 & \frac{n+1}{2}+1 \leq i \leq n \end{cases}$$

and

$$f(v_{n+i}) = \begin{cases} 4i-2 & 1 \leq i \leq \frac{n+1}{2} \\ 4(n-i)+4 & \frac{n+1}{2}+1 \leq i \leq n \end{cases}$$

The proof follows as above, replacing $1 \le i \le \frac{n}{2}$ by $1 \le i \le \frac{n+1}{2}$ and $\frac{n}{2}+1 \le i \le n$ by $\frac{n+1}{2}+1 \le i \le n$.

Thus C_n^+ has strongly multiplicative labeling for n odd.

Hence C_n^+ has strongly multiplicative labeling for all n.

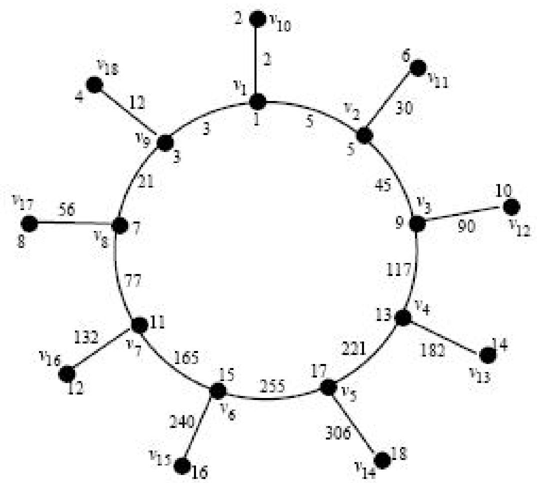

Fig.2. strongly multiplicative labeling for C_9^+

Theorem 2: The graph $(P_2 \cup mk_1)+N_2$ has strongly multiplicative labeling.

Proof: Let $V=\{v_1,v_2,v_3,......v_m,v_{m+1},.....v_{m+4}\}$, where $n=m+4$ be the vertex set and $E=E_1 \cup E_2 \cup E_3$ be the edge set where $E_1=\{v_i v_{n-2},\ 1 \le i \le n-3\}$,

$E_2=\{v_i v_{n-1},\ 1 \le i \le n-3\}$ and

$E_3=\{v_n v_{n-1}, v_n v_{n-2}, v_n v_{n-3}\}$ of the graph $(P_2 \cup mk_1)+N_2$.

Define a bijection $f: V \to \{1,2,3,.....n\}$ such that

$f(v_i)=i,\ 1 \le i \le n$

Define an induced function $g: E \to N$, such that

$g(v_i v_j)=f(v_i)f(v_j)\ \forall\ v_i v_j \in E$ and $v_i, v_j \in V$

We show that the labeling of edges within the edge sets and among the edge sets are distinct. If it is assumed in each case that the induced label of the edges are same then we arrive at contradiction.

For the edges in E_1:

For $i \ne j,\quad 1 \le i,\ j \le n-3$

If we assume $g(v_i v_{n-2})=g(v_j v_{n-2})$

$\Rightarrow f(v_i)f(v_{n-2})=f(v_j)f(v_{n-2})$

$\Rightarrow i(n-2)=j(n-2)$

$\Rightarrow i=j$ a contradiction.

For the edges in E_2:

For $i \ne j,\quad 1 \le i,\ j \le n-3$

If we assume $g(v_i v_{n-1})=g(v_j v_{n-1})$

$\Rightarrow f(v_i)f(v_{n-1})=f(v_j)f(v_{n-1})$

$\Rightarrow i(n-1)=j(n-1)$

$\Rightarrow i=j$ a contradiction

For the edges in E_3:

For $i \ne j,\quad 1 \le i,\ j \le 3$

If we assume $g(v_n v_{n-i})=g(v_n v_{n-j})$

$\Rightarrow f(v_n)f(v_{n-i})=f(v_n)f(v_{n-j})$

$\Rightarrow n(n-i)=n(n-i)$

$\Rightarrow i=j$ a contradiction

Now we have to show that the edges between different edge sets are distinct.

For the edges in E_1 and E_2 :

For $i \ne j,\quad 1 \le i,\ j \le n-3$

If we assume $g(v_i v_{n-2})=g(v_j v_{n-1})$

$\Rightarrow f(v_i)f(v_{n-2})=f(v_j)f(v_{n-1})$

$\Rightarrow i(n-2)=j(n-1)$

$\Rightarrow i-j=\frac{2i-j}{n}$ a contradiction

For the edges in E_1 and E_3:

For $i \ne j,\quad 1 \le i \le n-3,\quad j=1,2,3$

If we assume $g(v_i v_{n-2})=g(v_n v_{n-j})$

$\Rightarrow f(v_i)f(v_{n-2})=f(v_n)f(v_{n-j})$

$\Rightarrow \frac{n-2}{n}=\frac{n-j}{i}$ a contradiction

For the edges in E_2 and E_3:

For $i \ne j,\quad 1 \le i \le n-3,\quad j=1,2,3$

If we assume $g(v_i v_{n-1})=g(v_n v_{n-j})$

$\Rightarrow f(v_i)f(v_{n-1})=f(v_n)f(v_{n-j})$

$\Rightarrow \frac{n-1}{n} = \frac{n-j}{i}$ a contradiction.

This implies all the edge labeling are distinct. Hence the graph $(P_2 \cup mk_1) + N_2$ has strongly multiplicative labeling.

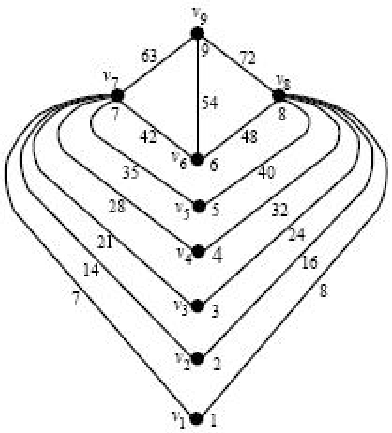

Fig.3. Strongly multiplicative labeling for $(P_2 \cup 5k_1) + N_2$

Theorem 3: The graph $P_2 + mk_1$ has strongly multiplicative labeling.

Proof : Let $V = \{v_1, v_2, v_3, ...v_n\}$, be the vertex set and $E = E_1 \cup E_2 \cup E_3$ be the edge set where $E_1 = \{v_i v_{n-1},\ 1 \le i \le n-2\}$, $E_2 = \{v_i v_n,\ 1 \le i \le n-2\}$ and $E_3 = \{v_n v_{n-1}\}$ of the graph $P_2 + mk_1$.

Define a bijection $f : V \to \{1,2,3,.....n\}$ such that $f(v_i) = i,\ 1 \le i \le n$

Define an induced function $g : E \to N$, such that $g(v_i v_j) = f(v_i) f(v_j)$ forall $v_i v_j \in E$ and $v_i, v_j \in V$

We show that the labeling of edges within the edge sets and among the edge sets are distinct. If it is assumed in each case that the induced label of the edges are same then we arrive at contradiction.

For the edges in E_1 :

For $i \ne j,\ \ 1 \le i,\ j \le n-2$

If we assume $g(v_i v_{n-1}) = g(v_j v_{n-1})$

$\Rightarrow f(v_i) f(v_{n-1}) = f(v_j) f(v_{n-1})$

$\Rightarrow i(n-1) = j(n-1)$

$\Rightarrow i = j$ a contradiction

For the edges in E_2 :

For $i \ne j,\ 1 \le i,\ j \le n-2$

If we assume $g(v_i v_n) = g(v_j v_n)$

$\Rightarrow f(v_i) f(v_n) = f(v_j) f(v_n)$

$\Rightarrow in = jn$

$\Rightarrow i = j$ a contradiction

Now we have to show that the edges between different edge sets are distinct.

For the edges in E_1 and E_2 :

For $i \ne j,\ \ 1 \le i,\ j \le n-2$

If we assume $g(v_i v_{n-1}) = g(v_j v_n)$

$\Rightarrow f(v_i) f(v_{n-1}) = f(v_j) f(v_n)$

$\Rightarrow i(n-1) = jn$

$\Rightarrow n = \frac{i}{i-j}$ a contradiction

For the edges in E_1 and E_3 :

For $1 \le i \le n-2$

If we assume $g(v_i v_{n-1}) = g(v_n v_{n-1})$

$\Rightarrow f(v_i) f(v_{n-1}) = f(v_n) f(v_{n-1})$

$\Rightarrow i(n-1) = n(n-1)$

$\Rightarrow i = n$ a contradiction

For the edges in E_2 and E_3 :

For $1 \le i \le n-2$

If we assume $g(v_i v_n) = g(v_n v_{n-1})$

$\Rightarrow f(v_i) f(v_n) = f(v_n) f(v_{n-1})$

$\Rightarrow in = n(n-1)$

$\Rightarrow i = n-1$ a contradiction

This implies all the edge labeling are distinct. Hence the graph $P_2 + mk_1$ has strongly multiplicative labeling.

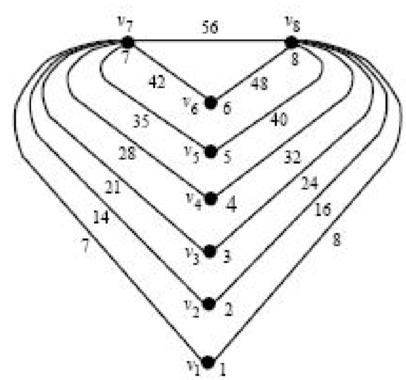

Fig.4. Strongly multiplicative labeling for $P_2 + 6k_1$

Definition 4[3]: Let $G = (V, E : R_1, R_2)$. The vertex set $V \to \{1,2,.....,n\}$ and the edge set is defined by the relations R_1 and R_2 such that $\begin{matrix} R_1 : b = a+1 \quad \forall \quad a,b \in V \\ R_2 : a+b = n+1 \end{matrix}$.

If $n \equiv 0 \pmod 2$, we get cycle C_n with $d = (n-2)/2$ non-intersecting chords and when $n \equiv 1 \pmod 2$ we get cycle C_n with $d = (n-3)/2$ non-intersecting chords. The graph obtained by this relation is $C_n^{\ d}$, $n \ge 5$.

Theorem 5: The graph $C_n^{\ d}$, $n \ge 5$ with non intersecting chords has strongly multiplicative labeling.

Proof: Let $V = \{v_1, v_2, v_3, ...v_n\}$, be the vertex set and $E = E_1 \cup E_2 \cup E_3$ be the edge set of the graph $C_n^{\ d}$ with non intersecting chords, where $E_1 = \{v_i v_{i+1}, 1 \le i < n\}$, $E_2 = \left\{v_{i+1} v_{n-i+1}, 1 \le i < \frac{n}{2} - 1\right\}$ and $E_3 = \{v_n v_1\}$. Define a bijection $f : V \to \{1,2,3,.....n\}$ such that

Case (i):When n is even

$$f(v_i) = \begin{cases} 2i-1 & 1 \le i \le \frac{n}{2} \\ 2(n-i)+2 & \frac{n}{2}+1 \le i \le n \end{cases}$$

Define an induced function $g : E \to N$, such that $g(v_i v_j) = f(v_i) f(v_j) \ \forall \ v_i v_j \in E$ and $v_i, v_j \in V$. We show that the labeling of edges within the edge sets and among the edge sets are distinct.

If it is assumed in each case that the induced label of the edges are same then we arrive to a contradiction.

For the edges in E_1 :

Sub Case (a) For $i \ne j$ and $1 \le i, \ j \le \frac{n}{2} - 1$

If we assume that $g(v_i v_{i+1}) = g(v_j v_{j+1})$

$$\Rightarrow f(v_i) f(v_{i+1}) = f(v_j) f(v_{j+1})$$

$$(2i-1)(2(i+1)-1) = (2j-1)(2(j+1)-1)$$

gives i = j a contradiction.

Sub Case (b) For $i \ne j$ and $\frac{n}{2}+1 \le i, \ j \le n$

If we assume that $g(v_i v_{i+1}) = g(v_j v_{j+1})$

$$\Rightarrow f(v_i) f(v_{i+1}) = f(v_j) f(v_{j+1})$$

$$(2(n-i)+2)(2(n-i-1)+2) = (2(n-j)+2)(2(n-j-1)+2)$$

gives i = j a contradiction as $i + j = 2n+1$ is not possible.

Sub Case (c) Edges of E_1 for $1 \le i < \frac{n}{2}$ are with odd labels and those of $\frac{n}{2}+1 \le j < n$ are with even labels. Thus the edge labels of these categories are distinct.

For the edges in E_2 :

For $i \ne j$ and $1 \le i, \ j \le \frac{n}{2} - 1$

If we assume that $g(v_{i+1} v_{n-i+1}) = g(v_{j+1} v_{n-j+1})$

$$\Rightarrow f(v_{i+1}) f(v_{n-i+1}) = f(v_{j+1}) f(v_{n-j+1})$$

$$(2(i+1)-1)(2(n-n+i-1)+2) = (2(j+1)-1)(2(n-n+j-1)+2)$$

gives i = j a contradiction as $i + j = \frac{-1}{2}$ is not possible

Now we have to show that the edges between different edge sets are distinct.

For the edges in E_1 and E_2 :

The labels of edges of E_2 and those of E_1 for $1 \le i < \frac{n}{2}$ are distinct as the labels of E_2 are with even numbers and those of E_1 for $1 \le i < \frac{n}{2}$ are with odd numbers.

Also $i \ne j$ and for $\frac{n}{2}+1 \le i < n$ and $1 \le j \le \frac{n}{2} - 1$

$$g(v_i v_{i+1}) = f(v_i) f(v_{i+1}) = 2i^2 + i + (2n^2 + 2n + 4ni - 3i)$$

$$g(v_{j+1} v_{n-j+1}) = f(v_{j+1}) f(v_{n-j+1}) = 2j^2 + j$$

$$g(v_i v_{i+1}) \ne g(v_{j+1} v_{n-j+1})$$

Moreover $g(v_n v_1) = f(v_n) f(v_1)$
$= 2$

Thus all the edges of $C_n^{\ d}$, $n \ge 5$ with non intersecting chords are distinct. Hence $C_n^{\ d}$, $n \ge 5$ with non intersecting chords has strongly multiplicative labeling for n even.

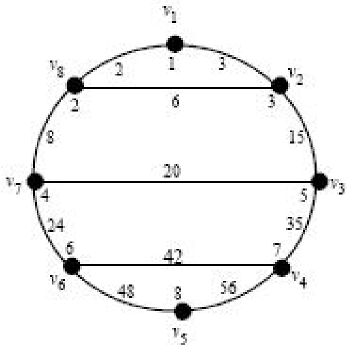

Fig. 5. $C_8^{\ 3}$ with non intersecting chords

Case (ii): When n is odd

We define the bijection $f : V \to \{1,2,3,\ldots..n\}$ such that

$$f(v_i) = \begin{cases} 2i-1 & 1 \le i \le \frac{n+1}{2} \\ 2(n-i)+2 & \frac{n+1}{2}+1 \le i \le n \end{cases}$$

With the edge set being $E = E_1 \cup E_2 \cup E_3$ where $E_1 = \{v_i v_{i+1}, 1 \le i < n\}$, $E_2 = \left\{v_{i+1} v_{n-i+1}, 1 \le i < \frac{n-1}{2} - 1\right\}$ and $E_1 = \{v_n v_1\}$.

The proof follows as above.

Thus $C_n^{\ d}$ with non intersecting chords has strongly multiplicative labeling for n odd.

Hence $C_n^{\ d}$ with non intersecting chords has strongly multiplicative labeling for all n.

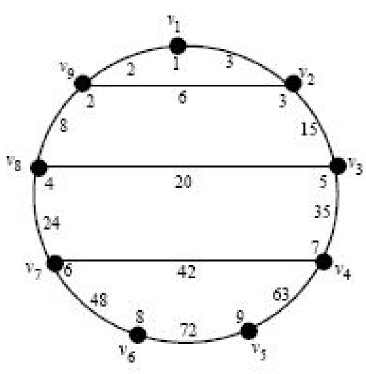

Fig. 6. $C_9^{\ 3}$ with non intersecting chords

Theorem 6: The Union of two strongly multiplicative graphs is also a strongly multiplicative graph.

Proof: Let G_1 and G_2 be two strongly multiplicative graphs with number of vertices n_1 and n_2 respectively. The graph $G_1 \cup G_2$ will have $n_1 + n_2$ vertices. Since G_1 is strongly multiplicative the induced labeling of the edges are distinct. Relabel the vertices of G_2 as $n_1 + 1$ for the vertex with label 1, $n_1 + 2$ for the vertex of label 2 and so on. As vertices are relabeled, the induced edge labeling will be added by the quantity $n_1^{\ 2} + (i+j)\, n_1$ for all $1 \le i,\ j \le n_2$. The addition of the quantity $n_1^{\ 2} + (i+j)\, n_1$ with the labels of the edges of G_2 will still result in the label of edges distinct. As G_2 is also strongly multiplicative graph, $G_1 \cup G_2$ is strongly multiplicative graph.

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