# The Upper Connected Monophonic Number and Forcing Connected Monophonic Number of a Graph

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*Abstract* — **A connected monophonic set $M$ in a connected graph $G = (V, E)$ is called a *minimal connected monophonic set* if no proper subset of $M$ is a connected monophonic set of $G$. The *upper connected monophonic number* $m_c^+(G)$ is the maximum cardinality of a *minimal connected monophonic set* of $G$. Connected graphs of order $p$ with upper connected monophonic number 2 and $p$ are characterized. It is shown that for any positive integers $2 \leq a < b \leq c$, there exists a connected graph $G$ with $m(G) = a$, $m_c(G) = b$ and $m_c^+(G) = c$, where $m(G)$ is the monophonic number and $m_c(G)$ is the connected monophonic number of a graph $G$. Let $M$ be a minimum connected monophonic set of $G$. A subset $T \subseteq M$ is called a *forcing subset* for $M$ if $M$ is the unique minimum connected monophonic set containing $T$. A forcing subset for $M$ of minimum cardinality is a *minimum forcing subset* of $M$. The *forcing connected* monophonic *number* of $M$, denoted by $f_{mc}(M)$, is the cardinality of a minimum forcing subset of $M$. The *forcing connected monophonic number* of $G$, denoted by $f_{mc}(G)$, is $f_{mc}(G) = \min\{f_{mc}(M)\}$, where the minimum is taken over all minimum connected monophonic set $M$ in $G$. It is shown that for every integers $a$ and $b$ with $a < b$, and $b - 2a - 2 > 0$, there exists a connected graph $G$ such that, $f_{mc}(G) = a$ and $m_c(G) = b$.**

*Keywords* — **monophonic number, connected monophonic number, upper connected monophonic number, forcing connected monophonic number.**

**AMS Subject Classification : 05C05**

## I. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. For basic graph theoretic terminology we refer to Harary [1]. The *distance* $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u$-$v$ path in $G$. An $u$-$v$ path of length $d(u, v)$ is called an *$u$-$v$ geodesic*. For a vertex $v$ of $G$, the eccentricity $e(v)$ is the distance between $v$ and a vertex farthest from $v$. The minimum eccentricity among the vertices of $G$ is the radius, rad($G$), and the maximum eccentricity is its diameter, diam($G$) of $G$. $N(v) = \{u \in V(G) : uv \in E(G)\}$ is called the neighborhood of the vertex $v$ in $G$. For any set $M$ of vertices of $G$, the *induced subgraph* $\langle M \rangle$ is the maximal subgraph of $G$ with vertex set $M$. A vertex $v$ is an *extreme vertex* of a graph $G$ if $\langle N(v) \rangle$ is complete. A *geodetic set* of $G$ is a set $S \subseteq V(G)$ such that every vertex of $G$ is contained in a geodesic joining some pair of vertices in $S$. The *geodetic number* $g(G)$ of $G$ is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is a *geodetic basis*. The geodetic number of a graph was introduced in [2, 3] and further studied in [4]. A *connected geodetic set* of a graph $G$ is a geodetic set $S$ such that the subgraph $G[S]$ induced by $S$ is connected. The minimum cardinality of a connected geodetic set of $G$ is the *connected geodetic number* of $G$ and is denoted by $g_c(G)$. A connected geodetic set of cardinality $g_c(G)$ is called a *$g_c$-set* of $G$ or a *connected geodetic basis* of $G$. The connected geodetic number of a graph is studied in [11]. A *chord* of a path $u_0, u_1, u_2, \ldots, u_h$ is an edge $u_iu_j$ with $j \geq i + 2$. An $u$-$v$ path is called a *monophonic path* if it is a chordless path. For two vertices $u$ and $v$ in a connected graph $G$, the *monophonic distance* $d_m(u, v)$ is the length of the longest $u - v$ monophonic path in $G$. An $u - v$ monophonic path of length $d_m(u, v)$ is called an *$u - v$ monophonic*. For a vertex $v$ of $G$, the *monophonic eccentricity* $e_m(v)$ is the monophonic distance between $v$ and a vertex farthest from $v$. The minimum monophonic eccentricity among the vertices in the *monophonic radius*, $rad_m(G)$ and the maximum monophonic eccentricity is the *monophonic diameter* $diam_m(G)$ of $G$. A *monophonic set* of $G$ is a set $M \subseteq V(G)$ such that every vertex of $G$ is contained in a monophonic path joining some pair of vertices in $M$. The *monophonic number* $m(G)$ of $G$ is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a *minimum monophonic set* of $G$. The monophonic number of a graph $G$ is introduced in [5] and further studied in [6,7,9]. A *connected monophonic set* of a graph $G$ is a monophonic set $M$ such that the subgraph $\langle M \rangle$ induced by $M$ is connected. The minimum cardinality of a connected monophonic set of $G$ is the *connected monophonic number* of $G$ and is denoted by $m_c(G)$. A connected monophonic set of cardinality $m_c(G)$ is called a *$m_c$-set* of $G$ or a *minimum connected monophonic set* of $G$. The connected monophonic number of a graph is studied in [8]. A subset $T$ of a $g_c$-set $S$ is called a forcing subset for $S$ if $S$ is the unique $g_c$-set containing $T$. A forcing subset for $S$ of minimum cardinality is a minimum forcing subset of $S$. The forcing connected geodetic number of $S$, denoted by $f_c(S)$, is the cardinality of a minimum forcing subset of $S$. The forcing connected geodetic number of $G$, denoted by $f_c(G)$ is $f_c(G) = \min\{f_c(S)\}$, where the minimum is taken over all $g_c$-sets in $S$. The forcing connected geodetic number is studied in [10]. A subset $T \subseteq M$ is called a forcing subset for $M$ if $M$ is the unique minimum monophonic set containing $T$. A forcing subset for $M$ of minimum cardinality is a minimum forcing subset of $M$. The forcing monophonic number of $M$, denoted by $f(M)$, is the cardinality of a minimum forcing subset of $M$. The forcing monophonic number of $G$, denoted by $f_m(G)$ *is* $f_m(G) = \min\{f_m(M)\}$, where the minimum is taken over all minimum monophonic sets $M$ in $G$.

The following theorems are used in the sequel.

**Theorem 1.1.[5]** Each extreme vertex of a connected graph $G$ belongs to every monophonic set of $G$.

**Theorem 1.2. [5]** The monophonic number of a tree $T$ is the number of end vertices in $T$.

**Corollary 1.3.[8]** For any non-trivial tree $T$ of order $p$, $m_c(T) = p$.

**Theorem 1.4.[8]** For the complete graph $k_p (p \geq 2)$, $m_c(k_p) = p$.

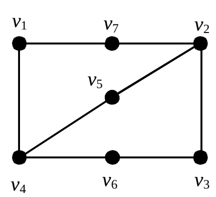
**Theorem 1.5.[8]** Every cut vertex of a connected graph $G$ belongs to every connected monophonic set of $G$.

**Theorem 1.6.[8]** Each extreme vertex of a connected graph $G$ belongs to every connected monophonic set of $G$.

## II. The Upper Connected Monophonic Number of a Graph

**Definition 2.1.** A connected monophonic set $M$ in a connected graph $G$ is called a *minimal connected monophonic set* if no proper subset of $M$ is a connected monophonic set of $G$. The *upper connected monophonic number* $m_c^+(G)$ is the maximum cardinality of a *minimal connected monophonic set* of $G$.

**Example 2.2.** For the graph $G$ given in Figure 2.1, $M_1 = \{v_2, v_4, v_5\}$, $M_2 = \{v_1, v_2, v_7\}$, $M_3 = \{v_3, v_4, v_6\}$, $M_4 = \{v_2, v_3, v_6\}$ and $M_5 = \{v_1, v_4, v_7\}$ are the minimum connected monophonic sets of $G$ so that $m_c(G) = 3$. The sets $M' = \{v_1, v_4, v_5, v_6\}$ and $M'' = \{v_2, v_3, v_5, v_7\}$ are also connected monophonic sets of $G$ and it is clear that no proper subset of $M'$ and $M''$ are connected monophonic sets so that $M'$ and $M''$ are minimal connected monophonic sets of $G$. Hence $m_c^+(G) = 4$.




*G*

Figure 2.1

**Theorem 2.3.** For a connected graph $G$, $2 \leq m_c(G) \leq m_c^+(G) \leq p$.

**Proof.** Any connected monophonic set needs at least two vertices and so $m_c(G) \geq 2$. Since every minimum connected monophonic set is a minimal connected monophonic set, $m_c(G) \leq m_c^+(G)$. Also, since $V(G)$ induces a connected monophonic set of $G$, it is clear that $m_c^+(G) \leq p$. Thus $2 \leq m_c(G) \leq m_c^+(G) \leq p$. ■

**Theorem 2.4.** For a connected graph $G$, $m_c(G) = p$ if and only if $m_c^+(G) = p$.

**Proof.** Let $m_c^+(G) = p$. Then $M = V(G)$ is the unique minimal connected monophonic set of $G$. Since no proper subset of $M$ is a connected monophonic set, it is clear that $M$ is the unique minimum connected monophonic set of $G$ and so $m_c(G) = p$. The converse follows from Theorem 2.3. ■

**Theorem 2.5.** Every extreme vertex of a connected graph $G$ belongs to every minimal connected monophonic set of $G$.

**Proof.** Since every minimal connected monophonic set is a monophonic set, the result follows from Theorem 1.1. ■

**Theorem 2.6.** Let $G$ be a connected graph containing a cut vertex $v$. Let $M$ be a minimal connected monophonic set of $G$, then every component of $G - v$ contains an element $M$.

**Proof.** Let $v$ be a cut vertex of $G$ and $M$ be a minimal connected monophonic set of $G$. Suppose there exists a component $G_1$ of $G - v$ such that $G$ contains no vertex of $M$. By Theorem 2.5, $M$ contains all extreme vertices of $G$ and hence it follows that $G_1$ does not contain any extreme vertex of $G$. Thus $G_1$ contains at least one edge say $xy$. Since $M$ is the minimal connected monophonic set, $xy$ lies on the $u - w$ monophonic path $: u, u_1, u_2, \ldots, v, \ldots, x, y, \ldots, v_1, \ldots, v, \ldots, w$. Since $v$ is a cut vertex of $G$, the $u - x$ and $y - w$ sub paths of $P$ both contains $v$ and so $P$ is not a path, which is a contradiction. ■

**Theorem 2.7.** Every cut-vertex of a connected graph $G$ belongs to every minimal connected monophonic set of $G$.

**Proof.** Let $v$ be any cut-vertex of $G$ and let $G_1, G_2, \ldots, G_r$ $(r > 2)$ be the components of $G - \{v\}$. Let $M$ be any connected monophonic set of $G$. Then $M$ contains atleast one element from each $G_i (1 \leq i \leq r)$. Since $G[M]$ is connected, it follows that $v \in M$. ■

**Corollary 2.8.** For a connected graph $G$ with $k$ extreme vertices and $l$ cut-vertices, $m_c^+(G) \geq \max\{2, k + l\}$.

**Proof.** This follows from Theorems 2.5 and 2.7. ■

**Corollary 2.9.** For the complete graph $G = K_p$, $m_c^+(G) = p$.

**Proof.** This follows from Theorem 2.5. ■

**Corollary 2.10.** For any tree $T$, $m_c^+(T) = p$.

**Proof.** This follows from Corollary 2.9. ■

**Theorem 2.11.** For any positive integers $2 \leq a < b \leq c$, there exists a connected graph $G$ such that $m(G) = a$, $m_c(G) = b$ and $m_c^+(G) = c$.

**Proof.** If $2 \leq a < b = c$, let $G$ be any tree of order $b$ with $a$ end-vertices. Then by Theorem 1.2, $m(G) = a$, by Corollary 1.3, $m_c(G) = b$ and by Corollary 2.9, $m_c^+(G) = b$. Let $2 \leq a < b < c$. Now, we consider four cases.

**Case 1**. $a > 2$ and $b - a \geq 2$. Then $b - a + 2 \geq 4$, let $P_{b-a+2}$: $v_1, v_2, \ldots, v_{b-a+2}$ be a path of length $b - a + 1$. Add $c - b + a - 1$ new vertices $w_1, w_2, \ldots, w_{c-b}, u_1, u_2, \ldots, u_{a-1}$ to $P_{b-a+2}$ and join $w_1, w_2, \ldots, w_{c-b}$ to both $v_1$ and $v_3$ and also join $u_1, u_2, \ldots, u_{a-1}$ to both $v_1$ and $v_2$, there by producing the graph $G$ of Figure 2.2. Let $M = \{u_1, u_2, \ldots, u_{a-1}, v_{b-a+2}\}$ be the set of all extreme vertices of $G$. By Theorem 1.1, every monophonic set of $G$ contains $M$. It is clear that $M$ is a monophonic set of $G$ so that $m(G) = a$. Let $M_1 = M \cup \{v_2, v_3, v_4, \ldots, v_{b-a+1}\}$. By Theorems 1.5 and 1.6, each connected monophonic set contains $M_1$. It is clear that $M_1$ is a connected monophonic set of $G$ so that $m_c(G) = b$.

Let $M_2 = M_1 \cup \{w_1, w_2, \ldots, w_{c-b}\}$. It is clear that $M_2$ is a connected mon show that $M_2$ is a minimal connected monophonic set of $G$.

Assume, to the contrary, that $M_2$ is not a minimal connected monophonic set. Then there is a proper subset $T$ of $M_2$ such that $T$ is a connected monophonic set of $G$. Let $v \in M_2$ and $v \notin T$. By Theorems 1.5 and 1.6, it is clear $v = w_i$, for some $i = 1, 2, ..., c - b$. Clearly, this $w_i$ does not lie on a monophonic path joining any pair of vertices of $T$ and so $T$ is not a connected monophonic set of $G$, which is a contradiction. Thus $M_2$ is a minimal connected monophonic set of $G$ and so $m_c^+(G) \geq c$. Since the order of the graph is $c + 1$, it follows that $m_c^+(G) = c$.

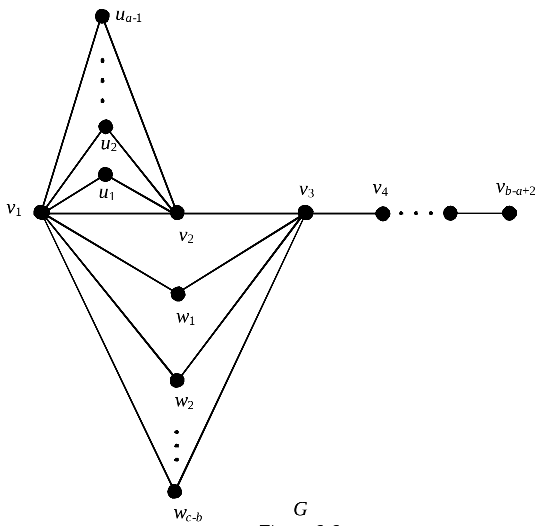


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Figure 2.2

**Case 2.** Let $a > 2$ and $b - a = 1$. Since $c > b$, we have $c - b + 1 \geq 2$. Consider the graph $G$ given in by Figure 2.3. Then as in case 1, $M = \{u_1, u_2, ..., u_{a-1}, v_3\}$ is a minimum monophonic set, $M_1 = M \cup \{v_2\}$ is a minimum connected monophonic set and $M_2 = V(G) - \{v_1\}$ is a minimal connected monophonic set of $G$ so that $m(G) = a$, $m_c(G) = b$ and $m_c^+(G) = c$.

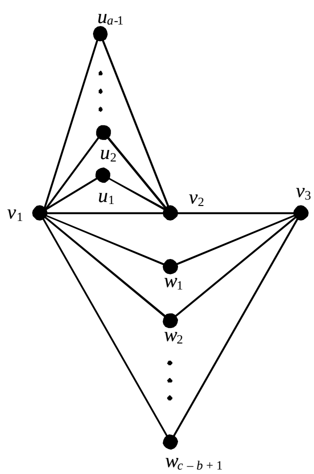

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Figure 2.3

**Case 3.** Let $a = 2$ and $b - a = 1$. Then $b = 3$. Consider the graph $G$ given in Figure 2.4. Then as in case 1, $M = \{v_1, v_3\}$ is a minimum monophonic set, $M_1 = \{v_1, v_2, v_3\}$ is a minimum connected monophonic set and $M_2 = V(G) - \{v_1\}$ is a minimal connected monophonic set of $G$ so that $m(G) = a$, $m_c(G) = b$ and $m_c^+(G) = c$.

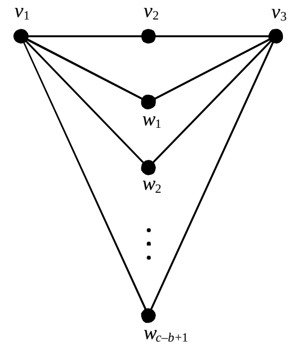

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Figure 2.4

**Case 4.** Let $a = 2$ and $b - a \geq 2$. Then $b \geq 4$. Consider the graph $G$ given in Figure 2.5. Then as in Case 1, $M = \{v_1, v_b\}$ is a minimum monophonic set, $M_1 = \{v_1, v_2, ..., v_b\}$ is a minimum connected monophonic set, $M_2 = V(G) - \{v_1\}$ is a minimal connected monophonic set of $G$ so that $m(G) = a$, $m_c(G) = b$ and $m_c^+(G) = c$. ∎

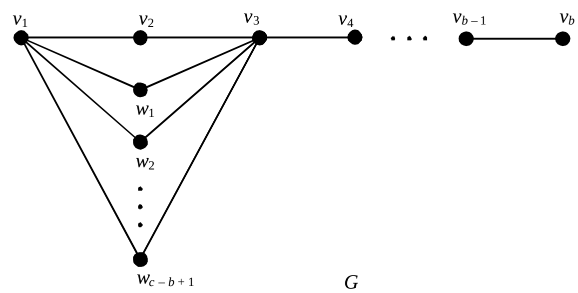

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Figure 2.5

**Theorem 2.12.** For positive integers $r, d$ and $l > d - r + 3$ with $r < d \leq 2r$, there exists a connected graph $G$ with $rad_m(G) = r, diam_m(G) = d$ and $m_c^+(G) = l$.

**Proof.** When $r = 1$, we let $G = k_{1,l-1}$. Then the result follows from Theorem 1.3.

Let $r \geq 2$, let $C_{r+2} : v_1, v_2, \ldots, v_{r+2}, v_1$ be a cycle of length $r + 2$ and let $P_{d-r+1} : u_0, u_1, u_2, \ldots, u_{d-r}$ be a path of length $d_{d-r+1}$. Let $H$ be a graph obtained from $C_{r+2}$ and $P_{d-r+1}$ by identifying $v_1$ in $C_{r+2}$ and $u_0$ in $P_{d-r+1}$. Now add $l - d + r - 3$ new vertices $w_1, w_2, \ldots, w_{l-d+r-3}$ to $H$ and join each $w_i$ $(1 \leq i < l - d + r - 3)$ to the vertex $u_{d-r-1}$ and obtain the graph $G$ as shown in Figure 2.6. Then $rad_m(G) = r$ and $diam_m(G) = d$. Let $M = \{u_0, u_1, u_2, \ldots, u_{d-r}, w_1, w_2, \ldots, w_{l-d+r-3}\}$ be the set of cut-vertices and end-vertices of $G$. By Theorem 1.1 and Theorem 1.5, $M$ is a subset of every connected monophonic set of $G$. It is clear that $M$ is not a connected monophonic set of $G$. Also $M \cup \{x\}$, where $x \notin M$ is not a connected monophonic set of $G$. However $M_1 = M \cup \{v_2, v_3\}$ is a connected monophonic set of $G$. Now, we show that $M_1$ is a minimal connected monophonic set of $G$. Assume, to the contrary, that $M_1$ is not a minimal connected monophonic set. Then there is a proper subset $T$ of $M_1$ such that $T$ is connected monophonic set of $G$. Let $y \in M_1$ and $y \notin T$. By Theorems 1.5 and 1.6, it is clear that $x = u_i$ for some $i = 0, 1, 2, \ldots, d - r$. Clearly this $u_i$ does not lie on a monophonic path joining any pair of vertices of $T$ and so $T$ is not a connected monophonic set of $G$, which is a contradiction. Thus, $M_1$ is a minimal connected monophonic set of $G$ and so $m_c^+(G) \geq l$. Let $M'$ be a minimal connected monophonic set such that $|M'| > l$. By Theorems 1.1 and 1.5, $M'$ contains $M$. Since, $M_1 = M \cup \{v_2, v_3\}$ or $M_2 = M \cup \{v_2, v_{r+2}\}$ and $M_3 = M \cup \{v_{r+1}, v_{r+2}\}$ are also connected monophonic sets of $G$ and $< M' >$ is connected, it follows that $M'$ contains either $M_1$ or $M_2$ or $M_3$, which is a contradiction to $M'$ is a minimal connected monophonic set of $G$. Therefore $m_c^+(G) = l$. ∎

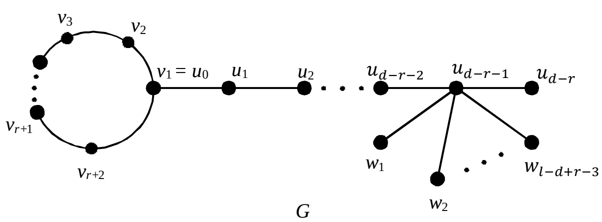


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Figure 2.6

## III. The Forcing Connected Monophonic Number of a Graph

**Definition 3.1.** Let $G$ be a connected graph and $M$ a minimum connected monophonic set of $G$. A subset $T \subseteq M$ is called a *forcing subset* for $M$ if $M$ is the unique minimum connected monophonic set containing $T$. A forcing subset for $M$ of minimum cardinality is a *minimum forcing subset* of $M$. The *forcing connected* monophonic *number* of $M$, denoted by $f_{mc}(M)$, is the cardinality of a minimum forcing subset of $M$. The *forcing connected monophonic number* of $G$, denoted by $f_{mc}(G)$, is $f_{mc}(G) = \min\{f_{mc}(M)\}$, where the minimum is taken over all minimum connected monophonic set $M$ in $G$.

**Example 3.2.** For the graph $G$ given in Figure 2.1, $M_1 = \{v_2, v_4, v_5\}$, $M_2 = \{v_2, v_3, v_6\}$, $M_3 = \{v_1, v_4, v_7\}$, $M_4 = \{v_1, v_2, v_7\}$ and $M_5 = \{v_3, v_4, v_6\}$ are the only four $m_c$-sets so that $m_c(G) = 3$. Also $f_{mc}(M_1) = 1$, $f_{mc}(M_2) = f_{mc}(M_3) = f_{mc}(M_4) = f_{mc}(M_5) = 2$ so that $f_{mc}(G) = 1$.

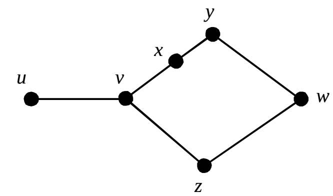
The next theorem follows immediately from the definition of the connected monophonic number and the forcing connected monophonic number of a connected graph $G$.

**Theorem 3.3.** For any connected graph $G$, $0 \leq f_{mc}(G) \leq m_c(G) \leq p$.

**Remark 3.4.** For any non-trivial tree $T$, by Corollary 1.3, the set of all vertices is the unique $m_c$-set of $G$. It follows that $f_{mc}(T) = 0$ and $m_c(T) = p$. For the cycle $C_4$: $u_1, u_2, u_3, u_4, u_1$ of order 4, $M_1 = \{u_1, u_2, u_3\}$, $M_2 = \{u_2, u_3, u_4\}$, $M_3 = \{u_3, u_4, u_1\}$ and $M_4 = \{u_4, u_1, u_2\}$ are the $m_c$-sets of $C_4$ so that $m_c(C_4) = 3$. Also, it is easily seen that $f_{mc}(C_4) = 3$. Thus $f_{mc}(C_4) = m_c(C_4)$. Also, the inequality in the theorem can be strict. For the graph $G$ given in Figure 2.1, $f_{mc}(G) = 1$, $m_c(G) = 3$ and $p = 7$ as in Example 3.2. Thus $0 < f_{mc}(G) < m_c(G) < p$.

**Definition 3.5.** A vertex $v$ of a connected graph $G$ is said to be a *connected monophonic vertex* of $G$ if $v$ belongs to every minimum connected monophonic set of $G$.

**Example 3.6.** For the graph $G$ given in Figure 3.1, $M_1 = \{u, v, y, x\}$, $M_2 = \{u, v, z, w\}$ and $M_3 = \{u, v, x, z\}$ are the only minimum connected monophonic sets of $G$. It is clear that $u$ and $v$ are the connected monophonic vertices of $G$.




*G*
Figure 3.1

**Theorem 3.7.** Let $G$ be a connected graph. Then

a) $f_{mc}(G) = 0$ if and only if $G$ has a unique minimum monophonic set.
b) $f_{mc}(G) = 1$ if and only if $G$ has at least two minimum connected monophonic sets, one of which is a unique minimum connected monophonic set containing one of its elements, and
c) $f_{mc}(G) = m_c(G)$ if and only if no minimum connected monophonic set of $G$ is the unique minimum connected monophonic set containing any of its proper subsets.

**Theorem 3.8.** Let $G$ be a connected graph and let $\Im$ be the set of relative complements of the minimum forcing subsets in their respective minimum connected monophonic sets in $G$. Then $\bigcap_{F \in \Im} F$ is the set of connected monophonic vertices of $G$.

**Corollary 3.9.** Let $G$ be a connected graph and $M$ a minimum connected monophonic set of $G$. Then no connected monophonic vertex of $G$ belongs to any minimum forcing set of $M$.

**Theorem 3.10.** Let $G$ be a connected graph and $W$ be the set of all connected monophonic vertices of $G$. Then $f_{mc}(G) \leq m_c(G) - |W|$.

**Proof.** Let $M$ be any minimum connected monophonic set of $G$. Then $m_c(G) = |M|$, $W \subseteq M$ and $M$ is the unique minimum forcing connected monophonic set containing $M - W$. Then $f_{mc}(G) \leq |M - W| = |M| - |W| = m_c(G) - |W|$. ∎

**Corollary 3.11.** If $G$ is a connected graph with $k$ extreme vertices and $l$ cut-vertices, then $f_{mc}(G) \leq m_c(G) - (k + l)$.

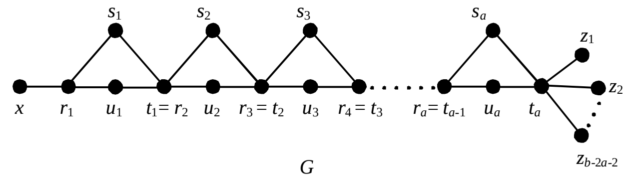
**Proof**. This follows from Theorems 1.5, 1.6 and 3.10.

**Remark 3.12.** The bounds in Theorem 3.10 is sharp. For the graph $G$ given in Figure 3.1, $M_1 = \{u, v, y, x\}$, $M_2 = \{u, v, z, w\}$ and $M_3 = \{u, v, x, z\}$ are the $m_c$-sets so that $m_c(G) = 4$. Also, it is easily seen that $f_{mc}(G) = 2$ and $W = \{u, v\}$ is the set of connected monophonic vertices of $G$. Thus $f_{mc}(G) = m_c(G) - |W|$.

**Theorem 3.13.** For every integers $a$ and $b$ with $a < b$, and $b - 2a - 2 > 0$, there exists a connected graph $G$ such that, $f_{mc}(G) = a$ and $m_c(G) = b$.

**Proof. Case 1.** $a = 0, b \geq 2$. Let $G = k_{1,b-1}$. Then by Theorem 3.7(a), $f_{mc}(G) = 0$ and $m_c(G) = b$.

**Case 2.** $0 < a < b$. Let $F_i: r_i, s_i, u_i, t_i, r_i$ be a copy of $C_4$. Let $H$ be a graph obtained from $F_i$`s by identifying $t_{i-1}$ of $F_{i-1}$ and $r_i$ of $F_i (2 \leq i \leq a)$. Let $G$ be a graph obtained from $H$ by adding $b - 2a - 1$ new vertices $x, z_1, z_2, \dots, z_{b-2a-2}$ and joining the edges $xr_1, t_a z_1, \dots, t_a z_{b-2a-2}$ as shown in Figure 3.2. Let $Z = \{x, z_1, z_2, \dots, z_{b-2a-2}\}$ be the set of end vetices of $G$. It is clear that $Z$ is not a connected monophonic set of $G$. By Theorem 2.7, $Z' = Z \cup \{r_1, r_2, \dots, r_a, t_a\}$ is a subset of every connected monophonic set of $G$. we see that $Z'$ is not a connected monophonic set of $G$. Let $H_i = \{u_i, s_i\} (1 \leq i \leq a)$. We observe that every $m_c$-set of $G$ must contain at least one vertex from each $H_i$ so that $m_c(G) \geq b - 2a - 1 + a + 1 + a = b$. Now, $M = Z' \cup \{s_1, s_2, \dots, s_a\}$ is a connected monophonic set of $G$ so that $m_c(G) \leq b - 2a - 1 + a + 1 + a = b$. Thus $m_c(G) = b$. Next, we show $f_{mc}(G) = a$. Since every $m_c$-set contains $Z'$, it follows from Theorem 3.10 that $f_{mc}(G) \leq m_c(G) - (b - 2a - 1 + a + 1) = a$. It is easily seen that every $m_c$-set of $G$ is of the form $Z' \cup \{s_1, s_2, \dots, s_a\}$ where $s_i \in H_i (1 \leq i \leq a)$. Let $T$ be any proper subset of $M$ with $|T| < a$. Then there exist $s_i (1 \leq i \leq a)$ such that $s_i \notin T$. Let $e_i$ be the vertex of $H_i$ distinct from $s_i$. Then $W = (M - \{s_i\}) \cup \{e_i\}$ is a $m_c$-set properly containing $T$. Thus $M$ is not the unique $m_c$-set containing $T$ so that $T$ is not a forcing subset of $M$. This is true for all $m_c$-sets so that $f_{mc}(G) = a$. ■




*G*

Figure 3.2

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