SOME RESULTS ON KRONECKER PRODUCT OF TWO GRAPHS

Dr.P.BHASKARUDU
Head of the Department of Mathematics,
S.V.Arts College, Tirupati,
S.V.University, Andhra Pradesh.
e-mail:bhaskarudu1964@yahoo.co.in

ABSTRACT -- We consider product graphs and recall the results associated to the product graphs. Paul M.Weichsel [3] defined the Kronecker product of graphs. He has proved a characterization for the product graphs to be connected graphs. He also obtained "If G_1 and G_2 are connected graphs with no odd cycles then has exactly two connected components."

E Sampath Kumar [2] has proved that for a connected graph & with no odd cycles $G_1(K)G_2 = 2G$.

We prove that following:

(i) *In* $G_1(K)G_2$, *the* $\deg(u_i, v_j) = \deg(u_i)\deg(v_j)$.

(ii) *If* $\delta_1 \geq 2$ *or* $\delta_2 \geq 2$ $(\delta_1, \delta_2 \neq 0)$ (*where* $\delta_1 = \delta_{G_1}$ *and* $\delta_2 = \delta_{G_2}$) *then* $G_1(K)G_2$ *contains a cycle.*

(iii) (a) $\left|V_{G_1(K)G_2}\right| = \left|V_{G_1}\right| \left|V_{G_2}\right|$ *and*
(b) $\left|E_{G_1(K)G_2}\right| = \left|E_{G_1}\right| \left|E_{G_2}\right|$

(iv) *If* G_1 *and* G_2 *are regular graphs then* $G_1(K)G_2$ *is also a regular graph.*

(v) *If* G_1 *or* G_2 *is a bipartite graph then* $G_1(K)G_2$ *is bipartite graph.*

Key words -Kronecker product of graphs, connected graphs, odd cycles, degree, regular graphs, bipartite graph.

I INTRODUCTION

<u>DEFINITION</u> : (KRONECKER PRODUCT OF TWO GRAPHS)

If G_1 and G_2 are two graphs with vertex sets V_1 and V_2 respectively then their product graph is a graph denoted by $G_1(K)G_2$ with its vertex set as $V_1 \times V_2$ where (u_1, v_1) is adjacent with (u_2, v_2) if and only if $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$ or equivalently $G_1(K)G_2$ is a graph with vertex set $\{(u_i, v_j) / u_i \in V_1, v_j \in V_2\}$ such that the adjacency number of the pair (u_i, v_j) and (u_k, v_l) is the product of adjacency number of u_i, u_k in G_1 and v_j, v_l in G_2.

NOTATION: We use the filling notation $p_1 \to p_k$ to denote a chain in G from vertex p_1 to vertex p_k and $n(p_1 \to p_k)$ is the number of lines (not necessary distinct) in $p_1 \to p_k$.

THEOREM: 1

Let G_1 and G_2 be connected graphs. $G_1(K)G_2$ is connected if and only if either G_1 or G_2 contains an odd cycle.

PROOF:

Let the vertex sets of G_1 and G_2 be $\{p_1\}$ and $\{q_1\}$ respectively. Let q_1 and q_r be adjacent vertices in G_2 and assume that $G_1(K)G_2$ is connected. Hence there exists

$$((p_1, q_1)) \to ((p_1, q_r)) = \begin{pmatrix} (p_1, q_1), (p_2, q_2) - - - - - - - - - - \\ - - - - - - - - (p_u, q_v), (p_1, q_r) \end{pmatrix} \text{ in } G_1(k)G_2$$

Associate with each vertex (p_i, q_j) in this chain an ordered pair of non negative integers $\{\alpha_i, \beta_j\}$ with α_i equals to the length of a shortest chain from p_i to p_l in G_1 and β_j the length of a shortest path from q_j to q_r in G_2. It can be seen there $(p_1, q_1) \sim \{0, 1\}$ *and* $(p_u, q_v) \sim \{1, 1\}$. If (p_w, q_x) and (p_y, q_z) are adjacent vertices in the chain above then $\left|\alpha_y - \alpha_w\right| \leq 1$ *and* $\left|\beta_z - \beta_x\right| \leq 1$;

If at each step in the path both components undergo a non zero change then the sums $\alpha_i + \beta_j$ would have the same parity. Since $(p_1, q_1) \sim \{0, 1\}$ *and* $(p_u, q_v) \sim \{1, 1\}$

it follows that at one step in the chain either $\alpha_y - \alpha_w = 0$ or $\beta_z - \beta_x = 0$.

Let $\beta_z - \beta_x = 0$. Then there are the minimal chains $q_z \to q_r$ and $q_x - q_r$. Let q_m be the first vertex common to both of these chains. Now $n(q_x \to q_m) = n(q_z \to q_m)$.

For, otherwise, the original chain would not be minimal. Hence the cycle $(q_z \to q_m,\ q_m \to q_x,\ q_z)$ is odd.

Similar is the case when $\alpha_y = \alpha_w$. Therefore either G_1 or G_2 contains an odd cycle.

Conversely, let G contain an odd cycle. Let (p_1, q_1) and (p_2, q_2) be arbitrary vertices of $G_1(K)G_2$. Since G_1, G_2 are connected, there are chains $p_1 \to p_2$, and $q_1 \to q_2$ in G_1 and G_2 respectively.

Suppose $n(p_1 \to p_2) + n(q_1 \to q_2)$ is even.

Then from Lemma of Weischsel [3] we have the following result. Suppose G_1 and G_2 be graphs with vertex sets $\{p_i\}$ and $\{q_j\}$ respectively. If (p_1, q_1) and (p_2, q_2) are two vertices of $G_1(K)G_2$ and if there exist $p_1 \to p_2$ in G_1 and $q_1 \to q_2$ in G_2 suchthat $n(p_1 \to p_2) + n(q_1 \to q_2)$ is even then there exist $(p_1, q_1) \to (p_1, q_2)$ in $G_1(K)G_2$. It follows from this result that (p_1, q_1) is connected to (p_2, q_2) in $G_1(K)G_2$. Suppose $n(p_1 \to p_2) + n(q_1 \to q_2)$ is odd. Let $p_1 \to p_1$ be a chain which include exactly one odd cycle of G.

Then $n(p_1 \to p_1)$ is odd and since $n(p_1 \to p_1, p_1 \to p_2)$

$$= n(p_1 \to p1) + n(p_1 \to p_2) \text{ it follows that } n(p_1 \to p_1;\ p_1 \to p_2) + n(q_1 \to q_2) \text{ is even.}$$

Hence (p_1, q_1) is connected to (p_2, q_2) in $G_1(K)G_2$.

If G_2 contains an odd cycle, we can prove similarly.

He has also obtained the following result as a corollary.

COROLLARY: 1

If G_1 and G_2 are connected graphs with no odd cycles then $G_1(K)G_2$ has exactly two connected components.

PROOF:

If q_1 is adjacent to q_2 in G_2 then (p_1, q_1) and (p_2, q_2) are not connected. For if they are connected then it follows that either G_1 or G_2 will have an odd cycle, contrary to hypothesis. Let (p_k, q_k) be an arbitrary vertex of $G_1(K)G_2$. It is enough if we show that (p_k, q_k) is connected either to (p_1, q_1) or (p_1, q_2). Consider the chains $p_k \to p_1$ and $q_k \to p_1$.

If $n(p_k \to p_1) + n(q_k \to q_1)$ is even then (p_k, q_k) is connected to (p_1, q_2).

This completes the proof of corollary.

E.Sampath Kumar [2] has proved the following:

THEOREM: 2

For a connected graph G with no odd cycles $G(K)K_2 = 2G$.

PROOF:

Let $\{a_i\}$ be the vertex set of G and K_2 be the line b_1b_2. By corollary 1, $G(K)K_2$ has exactly two components. Say G_1 and G_2. If for some point a_{i_0} in G, $(a_{i_o}, b_1) \in G_1$ then $(a_{i_o}, b_2) \in G_2$.

For if there is a path $(a_{i_0}, b_1)(a_{i_1}, b_2) - - - - - - - (a_{i_k}, b_1)(a_{i_0}, b_2)$ then k is even and G has odd cycle $a_{i_0}a_{i_1} - - - - - - - a_{i_k}a_{i_0}$, provided the points $a_{i_t}, t = 0, 1, 2, 3, - - - - - - - k$ are all distinct. Suppose $a_{i_t} = a_{i_s}$ for some t and s, where t<s. Let s be the smallest such integer. Clearly, if t is even then s is odd or if t is odd, s is even, G has an odd cycle $a_{i_t}a_{i_{t+1}} - - - - - - - - a_{i_{s-1}}a_{i_s}$. Now the function $f: G \to G_1$ defined by

$$\begin{aligned} f(a_i) &= (a_i, b_1) \text{ if } (a_i, b_1) \in G_1 \\ &= (a_i, b_2) \text{ if } (a_i, b_1) \notin G_1 \end{aligned}$$

Is an isomorphism. Similarly $G \cong G_2$

Now we prove the following.

THEOREM: 3

$In\ G_1(K)G_2\ the\ \deg(u_i,v_j)=\deg(u_i)\deg(v_j).$

PROOF:

$Suppose\ \deg(u_i)=m\ and\ \deg(v_j)=n,$
$i.e.,\ u_i\ is\ adjacent\ with\ vertices$
$u_1,u_2,-----u_m\ \ in\ G_1\ and\ v_j\ is$
$adjacent\ with\ vertices\ v_1,v_2,------v_n$
$in\ G_2.$

Then in the product graph $G_1(K)G_2$ the vertex (u_i, v_j) is adjacent with following vertices:

$$(u_1,v_1)(u_1,v_2)-----------(u_1,v_n)$$
$$(u_2,v_1)(u_2,v_2)------\ \ ----(u_2,v_n)$$
$$.$$
$$.$$
$$(u_m,v_1)(u_m,v_2)----------(u_m,v_n)$$

Also any other vertex (u_k, v_l) is not adjacent with (u_i, v_j) in $G_1(K)G_2$. If $k > m$ or $l > n$, since u_i is not adjacent with u_k if $k > m$ and v_j is not adjacent with v_l if $l > n$.

$Hence\ \ \deg(u_i,v_j)=\deg(u_i)\deg(v_j).$

THEOREM: 4

$If\ \ \delta_1 \ge 2\ \ or\ \ \delta_2 \ge 2\ \ (\delta_1,\delta_2 \ne 0)$
$(where\ \ \delta_1=\delta_{G_1}\ \ and\ \ \delta_2=\delta_{G_2}\ then$
$G_1(K)G_2\ \ contains\ \ a\ \ cycle.$

PROOF:

$By\ Theorem\ 3,\ we\ know\ that$
$\deg(u_i,v_j)=\deg(u_i)\deg(v_j)\ge 2$
$(Since\ \delta_1\ \delta_2 \ne 0\ and\ \delta_1 \ge 2\ \delta_2 \ge 2)$
$Thus\ \delta_{G_1(k)G_2} \ge 2$
$Hence\ G_1(K)G_2\ contains\ a\ cycle.$

THEOREM: 5

$$(a)\ \left|V_{G_1(K)G_2}\right|=\left|V_{G_1}\right|\cdot\left|V_{G_2}\right|\ \ and$$
$$(b)\ \left|E_{G_1(K)G_2}\right|=\left|E_{G_1}\right|\cdot\left|E_{G_2}\right|$$

PROOF:

$$We\ know\ that\ \left|E_{G_1}\right|=e_1=\frac{1}{2}\sum_{i\in v_1} d(u_i)$$
$$and\ \ \left|E_{G_2}\right|=e_2=\frac{1}{2}\sum_{j\in v_2} d(v_j)$$
$$
\begin{aligned}
Now\quad \left|E_{G_1(K)G_2}\right| &= \frac{1}{2}\sum_{i,j} d(u_i,v_j)\\
&= \frac{1}{2}\Big\{\sum_{i,j} d(u_i)\,d(v_j)\Big\} \quad (by\ \ theorem\ 3)\\
&= \frac{1}{2}\Big\{\sum_{i} d(u_i)\Big\}\Big\{\sum_{j} d(v_j)\Big\}\\
&= \frac{1}{2}\{\,2e_1\}\{\,2e_2\}=2\,e_1e_2
\end{aligned}
$$

THEOREM:6

If G_1 and G_2 are regular graphs then $G_1(K)G_2$ is also a regular graph.

PROOF:

Suppose G_1 is a K_1 – regular graph and G_2 is a K_2 regular graph then

$$\deg(u_i)=k_1,\ \forall\ u_i\in V_1\ and\ \deg(v_j)=k_2,\ \forall\ v_j\in V_2$$
$$Let\ (u_i,v_j)\ be\ any\ vertex\ in\ G_1(K)G_2\ then$$
$$\deg\ (u_i,v_j)=\deg\ (u_i)\cdot\deg\ (v_j)$$
$$=k_1k_2$$

Thus every vertex in $G_1(K)G_2$ is of degree k_1k_2, i.e., $G_1(K)G_2$ is k_1k_2 - regular.

REMARK:

However it is to be noted that if G_1, G_2 are simple graphs then $G_1(K)G_2$ can never be a complete graph, for, (u_i, v_j) is not adjacent with (u_i, v_k) for any $j \ne k$ (by definition)

THEOREM: 7

If G_1 or G_2 is a bipartite graph then $G_1(K)G_2$ is bipartite graph.

PROOF:

Suppose G_1 is bipartite graph with bipartition (X, Y)

$$Where\ X = \{x_1, x_2, ------------.x_m\}$$
$$Y = \{y_1, y_2, ------------.y_n\}$$
$$Let\ \ V_2 = \{v_1, v_2, ------------.v_r\}$$
$$Then\ in\ G_1(K)G_2\ the\ vertex\ set\ is$$
$$\{(x_1, v_1)(x_1, v_2) - - - - ------- .(x_1, v_r)$$
$$(x_2, v_1)(x_2, v_2) ----------- .(x_2, v_r)$$
$$.$$
$$. \qquad .$$
$$(x_m, v_1)(x_m, v_2) - - - - ------- (x_m, v_r)$$
$$(y_1, v_1)\ (y_1, v_2) ----------- .(y_1, v_r)$$
$$(y_2, v_1)(y_2, v_2) ----------- .(y_2, v_r)$$
$$. \qquad .$$
$$. \qquad .$$
$$(y_n, v_1)(y_n, v_2) ----- ---- .(y_n, v_r)\}$$

Now, no two vertices of the form (x_i, v_j) and (x_k, v_l) are adjacent since x_i and x_k are not adjacent.

Similarly, no two vertices of the form (y_i, v_j) and (y_k, v_l) are adjacent since y_i and y_k are not adjacent.

Thus $G_1(K)G_2$ is a bipartite graph with bipartition

$$X_{G_1(K)G_2}\ and\ Y_{G_1(K)G_2}\ where\ X_{G_1(K)G_2} = \left\{(x_i, v_j) \middle| \begin{array}{l} i = 1, 2, ----m \\ j = 1, 2, ----r \end{array}\right\}$$
$$and\ \ Y_{G_1(K)G_2} = \left\{(y_i, v_j) \middle| \begin{array}{l} i = 1, 2, ----n \\ j = 1, 2, ----r \end{array}\right\}$$

THEOREM: 8

Suppose G_1 and G_2 are connected graphs. If $G_1(K)G_2$ is a connected bipartite graph then either G_1 or G_2 must be a unipartite graph.

PROOF:

If G_1, G_2 are bipartite graphs then by the previous theorem $G_1(K)G_2$ is also a bipartite graph. However G_1 and G_2 do not contain any odd cycle.

Hence by theorem 1, $G_1(K)G_2$ is not connected a contradiction.

Hence the theorem.

II. CONCLUSION

The study of the product graphs has been providing us sufficient stimulation for obtaining some in-depth knowledge of the various properties of the graphs. It is hoped that the encouragement provided by this study of these product graphs will be a good straight point for further research.

III. REFERENCES

[1] Bondy, J.A., and Murty, US. Graph theory with Applications, Macmillan, London (1976).

[2] Sampathkumar, E. On tensor product graphs, J. Austraial, Math. Soc. 20 (series A) (1975), pp 268-273.

[3] Weichsel, P.M. The kronecker product of Graphs, Proc. Am. Math. Soc. 13(1962), pp 47-52.