The Forcing Monophonic Hull Number of a Graph

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***Abstract* — For a connected graph $G = (V, E)$, let a set M be a minimum monophonic hull set of G. A subset $T \subseteq M$ is called a forcing subset for M if M is the unique minimum monophonic hull set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing monophonic hull number of M, denoted by $f_{mh}(M)$, is the cardinality of a minimum forcing subset of M. The forcing monophonic hull number of G, denoted by $f_{mh}(G)$, is $f_{mh}(G)=\min\{f_{mh}(M)\}$, where the minimum is taken over all minimum monophonic hull sets in G. Some general properties satisfied by this concept are studied. The forcing monophonic hull numbers of certain classes of graphs are determined. It is shown that, for every pair a, b of integers with $0 \le a \le b$ and $b \ge 2$, there exists a connected graph G such that $f_{mh}(G) = a$ and $mh(G) = b$.**

***Keywords:* hull number, monophonic hull number, forcing hull number, forcing monophonic hull number.**

AMS Subject Classification: 05C12, 05C05.

I. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer to Harary [1, 9]. A convexity on a finite set V is a family C of subsets of V, convex sets which is closed under intersection and which contains both V and the empty set. The pair (V, E) is called a convexity space. A finite graph convexity space is a pair (V, E), formed by a finite connected graph $G = (V, E)$ and a convexity C on V such that (V, E) is a convexity space satisfying that every member of C induces a connected sub graph of G. Thus, classical convexity can be extended to graphs in a natural way. We know that a set X of R^n is convex if every segment joining two points of X is entirely contained in it. Similarly a vertex set W of a finite connected graph is said to be convex set of G if it contains all the vertices lying in a certain kind of path connecting vertices of W[2,8]. The distance $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest $u–v$ path in G. An $u–v$ path of length $d(u,v)$ is called an $u–v$ geodesic. A vertex x is said to lie on a u-v geodesic P if x is a vertex of P including the vertices u and v. For two vertices u and v, let $I[u,v]$ denotes the set of all vertices which lie on $u - v$ geodesic. For a set S of vertices, let $I[S] = \cup_{u,v\in S} I[u, v]$. The set S is *convex* if $I[S] = S$. Clearly if $S = \{v\}$ or $S = V$, then S is convex. The *convexity* number, denoted by $C(G)$, is the cardinality of a maximum proper convex subset of V. The smallest convex set containing S is denoted by $I_h(S)$ and called the *convex hull* of S. Since the intersection of two convex sets is convex, the convex hull is well defined. Note that $S \subseteq I[S] \subseteq I_h(S) \subseteq V$. A subset $S \subseteq V$ is called a *geodetic set* if $I[S] = V$ and a *hull set* if $I_h(S) = V$. The *geodetic number* $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is a *minimum geodetic set* or simply a *g- set* of G. Similarly, the *hull number* $h(G)$ of G is the minimum order of its hull sets and any hull set of order $h(G)$ is a *minimum hull set* or simply a *h- set* of G. The geodetic number of a graph is studied in [1,4,10] and the hull number of a graph is studied in [1,6]. A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique minimum hull set containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset of M*. The *forcing hull number* of S, denoted by $f_h(S)$, is the cardinality of a minimum forcing subset of S. The *forcing hull number* of G, denoted by $f_h(G)$, is $f_h(G) = \min\{f_h(S)\}$, where the minimum is taken over all minimum hull sets S in G. The forcing hull number of a graph is studied in[3,14]. A *chord* of a path $u_o, u_1, u_2, \ldots, u_n$ is an edge u_iu_j with $j \ge i + 2$. $(0 \le i, j \le n)$. A $u - v$ path P is called *monophonic path* if it is a chordless path. A vertex x is said to lie on a $u - v$ monophonic path P if x is a vertex of P including the vertices u and v. For two vertices u and v, let $J[u,v]$ denotes the set of all vertices which lie on $u - v$ monophonic path. For a set M of vertices, let $J[M] = \cup_{u,v\in M} J[u, v]$. The set M is *monophonic convex* or *m-convex* if $J[M] = M$. Clearly if $M = \{v\}$ or $M = V$, then M is m-convex. The *m-convexity* number, denoted by $C_m(G)$, is the cardinality of a maximum proper m-convex subset of V. The smallest m-convex set containing M is denoted by $J_h(M)$ and called the *monophonic convex hull or m-convex hull* of M. Since the intersection of two m-convex set is m-convex, the m-convex hull is well defined. Note that $M \subseteq J[M] \subseteq J_h(M) \subseteq V$. A subset $M \subseteq V$ is called a *monophonic set* if $J[M] = V$ and a *m-hull set* if $J_h(M) = V$. The *monophonic number* $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a *minimum monophonic set* or simply a *m- set* of G. Similarly, the *monophonic hull number* $mh(G)$ of G is the minimum order of its m-hull sets and any m-hull set of order $mh(G)$ is a *minimum monophonic set* or simply a *mh- set* of G. The monophonic number of a graph is studied in [5,7,11,13] and the monophonic hull number of a graph is studied in [12,13]. A vertex v of G is said to be a *monophonic vertex* of a graph G if v belongs to every minimum monophonic set of G. A vertex v is an *extreme vertex* of a graph G if the sub graph induced by its neighbors is complete. Throughout the following G denotes a connected graph with at least two vertices.

The following theorem is used in sequel.

Theorem 1.1.[12] Let G be a connected graph. Then each extreme vertex of G belongs to every monophonic hull set of G. $mh(G) = p$ if and only if $G = K_p$.

II. The Forcing Monophonic Hull Number of a Graph

Definition 2.1. Let G be a connected graph and M a minimum monophonic hull set of G. A subset $T \subseteq M$ is called a *forcing subset* for M if M is the unique minimum monophonic hull set containing T. A forcing subset for M of minimum cardinality is a *minimum forcing subset of M*. The *forcing monophonic hull number* of M, denoted by $f_{mh}(M)$, is the cardinality of a minimum forcing subset of M. The *forcing monophonic hull number* of G, denoted by $f_{mh}(G)$, is $f_{mh}(G) = \min\{f_{mh}(M)\}$, where the minimum is taken over all minimum monophonic hull sets M in G.

Example 2.2. For the graph G given in Figure 2.1, $M = \{v_1, v_8\}$ is the unique minimum monophonic hull set of G so that $mh(G) = 2$ and $f_{mh}(G) = 0$. Also $S_1 = \{v_1, v_5, v_8\}$ and $S_1 = \{v_1, v_6, v_8\}$ are the only two h-sets of G such that $f_h(S_1)=1$, $f_h(S_2)=1$ so that $f_h(G) = 1$. For the graph G given in Figure 2.2, $M_1 = \{v_1, v_4\}$, $M_2 = \{v_1, v_6\}$, $M_3 = \{v_1, v_7\}$ and $M_4 = \{v_1, v_8\}$ are the only four mh-sets of G such that $f_{mh}(M_1) = 1$, $f_{mh}(M_2) = 1$, $f_{mh}(M_3) = 1$ and $f_{mh}(M_4) = 1$ so that $f_{mh}(G) = 1$. Also, $S = \{v_1, v_7\}$ is the unique minimum hull set of G so that $h(G) = 2$ and $f_h(G) = 0$.

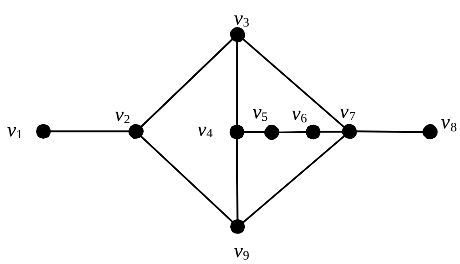

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Figure 2.1

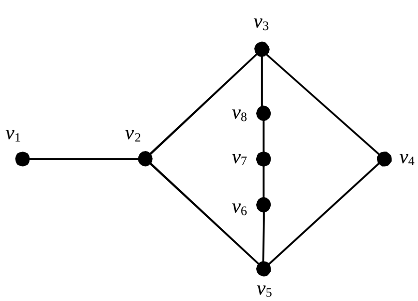

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Figure 2.2

The next theorem follows immediately from the definitions of the monophonic hull number of a connected graph G.

Theorem 2.3. For every connected graph G, $0 \le f_{mh}(G) \le mh(G)$.

The following theorems characterizes graphs for which the bounds in Theorem 2.3 are attained and also graphs for which $f_{mh}(G) = 1$.

Theorem 2.4. Let G be a connected graph. Then

$f_{mh}(G) = 0$ if and only if G has a unique mh-set.

$f_{mh}(G) = 1$ if and only if G has at least two mh-sets, one of which is a unique mh-set containing one of its elements, and

$f_{mh}(G) = mh(G)$ if and only if no mh-set of G is the unique mh-set containing any of its proper subsets.

Proof. (a) Let $f_{mh}(G) = 0$. Then, by definition, $f_{mh}(S) = 0$ for some minimum monophonic hull set S of G so that the empty set ϕ is the minimum forcing subset for S. Since the empty set ϕ is a subset of every set, it follows that S is the unique minimum monophonic hull set of G. The converse is clear.

(b) Let $f_{mh}(G) = 1$. Then by Theorem 2.4(a), G has at least two minimum monophonic hull sets. Also, since $f_{mh}(G) = 1$, there is a singleton subset T of a minimum monophonic hull set S of G such that T is not a subset of any other minimum monophonic hull set of G. Thus S is the unique minimum monophonic hull set containing one of its elements. The converse is clear.

(c) Let $f_{mh}(G) = m(G)$. Then $f_{mh}(S) = mh(G)$ for every minimum monophonic hull set S in G. Also, by Theorem 2.3, $mh(G) \ge 2$ and hence $f_{mh}(G) \ge 2$. Then by Theorem 2.4(a), G has at least two minimum monophonic hull sets and so the empty set ϕ is not a forcing subset for any minimum monophonic hull set of G. Since $f_{mh}(S) = mh(G)$, no proper subset of S is a forcing subset of S. Thus no minimum monophonic hull set of G is the unique minimum monophonic hull set containing any of its proper subsets. Conversely, the data implies that G contains more than one minimum monophonic hull set and no subset of any minimum monophonic hull set S other than S is a forcing subset for S. Hence it follows that $f_{mh}(G) = mh(G)$. ■

Definition 2.5. A vertex v of a graph G is said to be a *monophonic hull vertex* if v belongs to every mh-set of G.

Theorem 2.6. Let G be a connected graph and let $\Im$ be the set of relative complements of the minimum forcing subsets in their respective minimum monophonic hull sets in G. Then $\bigcap_{F \in \Im} F$ is the set of monophonic hull vertices of G.

Proof. Let W be the set of all monophonic hull vertices of G. We are to show that $W = \bigcap_{F \in \Im} F$. Let $v \in W$. Then v is a monophonic hull vertex of G that belongs to every minimum monophonic hull set S of G. Let $T \subseteq S$ be any minimum forcing subset for any minimum monophonic hull set S of G. We claim that $v \notin T$. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that S is the unique minimum monophonic hull set containing T' so that T' is a forcing subset for S with $|T'| < |T|$, which is a contradiction to T is a minimum forcing subset for S. Thus $v \notin T$ and so $v \in F$, where F is the relative complement of T in S. Hence $v \in \bigcap_{F \in \Im} F$ so that $W \subseteq \bigcap_{F \in \Im} F$.

Conversely, let $v \in \bigcap_{F \in \Im} F$. Then v belongs to the relative complement of T in S for every T and every S such that $T \subseteq S$, where T is a minimum forcing subset for S. Since F is the relative complement of T in S, we have $F \subseteq S$ and thus $v \in S$ for every S, which implies that v is a monophonic hull vertex of G. Thus $v \in W$ and so $\bigcap_{F \in \Im} F \subseteq W$. Hence $W = \bigcap_{F \in \Im} F$. ■

Corollary 2.7. Let G be a connected graph and S a minimum monophonic hull set of G. Then no monophonic hull vertex of G belongs to any minimum forcing set of S.

Proof. The proof is contained in the proof of the first part of Theorem 2.6. ■

Theorem 2.8. Let G be a connected graph and S be the set of all monophonic hull vertices of G. Then $f_{mh}(G) \leq mh(G) - |S|$.

Proof. Let M be any mh-set of G. Then $mh(G) = |M|$, $S \subseteq M$ and M is the unique mh-set containing $M - S$. Thus $f_{mh}(G) \leq |M - S| = |M| - |S| = mh(G) - |S|$. ■

Corollary 2.9. If G is a connected graph with k extreme vertices, then $f_{mh}(G) \leq mh(G) - k$.

Proof. This follows from Theorem 1.1(a) and Theorem 2.8. ■

Theorem 2.10. For any complete graph $G = K_p (p \geq 2)$ or any non-trivial tree $G = T$, $f_{mh}(G) = 0$.

Proof. For $G = K_p$, it follows from Theorem 1.1(a) that the set of all vertices of G is the unique monophonic hull set. Hence it follows from Theorem 2.4(*a*) that $f_{mh}(G) = 0$. For any non-trivial tree G, the monophonic hull number $mh(G)$ equals the number of end vertices in G. In fact, the set of all end vertices of G is the unique mh- set of G and so $f_{mh}(G) = 0$ by Theorem 2.4(a). ■

Theorem 2.11. For a complete bi-partite graph $G = K_{m,n} (2 \leq m \leq n)$, $S = \{u,v\}$ is a minimum monophonic hull set of G if and only if u and v are independent.

Proof. Let $S = \{u,v\}$, be a minimum monophonic hull set of G. Suppose that u and v are adjacent. Then uv is a chord for the path u-v and so $\{u, v\}$ is not a monophonic hull set of G, which is a contradiction. Conversely, let $S = \{u, v\}$, where u and v are independent. It is clear that S is a monophonic hull set of G. Since $|S| = 2$, S is a minimum monophonic hull set of G. ■

Theorem 2.12. For a complete bipartite graph $G = K_{m,n}$,

$$f_{mh}(G) = \begin{cases} 0\,; & m = 1, n \geq 2 \\ 1\,; & m = 2, n \geq 2\,. \\ 2\,; & 3 \leq m \leq n \end{cases}$$

Proof. If $m = 1$, $n \geq 2$, the result follows from Theorem 2.10. For $m = 2, n \geq 2$, let $U = \{u_1, u_2\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be the bipartite sets of G. Then $S = \{u_1, u_2\}$ is a mh-set of G. It is clear that S is the only mh-set containing u_1 so that $f_{mh}(G) = 1$. For $3 \leq m \leq n$, let $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be the bipartite sets of G. By Theorem 2.11, $mh(G) = 2$ and by Theorem 2.3, $0 \leq f_{mh}(G) \leq 2$. Suppose $0 \leq f_{mh}(G) \leq 1$. Since $mh(G) = 2$ and the mh-set of G is not unique, by Theorem 2.4 (b), $f_{mh}(G) = 1$. Let $S = \{u, v\}$ be a mh-set of G. Let us assume that $f_{mh}(S) = 1$. By Theorem 2.4 (b), S is the only mh-set containing u or v. Let us assume that S is the only mh-set containing u. Then $m = 2$, which is a contradiction to $m \geq 3$. Therefore $f_{mh}(G) = 2$. ■

Theorem 2.13. For any cycle $G = C_p (p \geq 4)$, $S = \{u,v\}$ is a minimum monophonic hull set of G if and only if u and v are independent.

Proof. Let $S = \{u,v\}$, be a minimum monophonic hull set of G. Suppose that u and v are adjacent. Then uv is a chord for the path u-v and so $\{u, v\}$ is not a monophonic hull set of G, which is a contradiction. Conversely, let $S = \{u, v\}$, where u and v are independent. It is clear that S is a monophonic hull set of G. Since $|S| = 2$, S is a minimum monophonic hull set of G. ■

Theorem 2.14. For any cycle $G = C_p (p \geq 5)$, $f_{mh}(G) = 2$.

Proof. By Theorem 2.13, $mh(G) = 2$ and by Theorem 2.3, $0 \leq f_{mh}(G) \leq 2$. Suppose $0 \leq f_{mh}(G) \leq 1$. Since $mh(G) = 2$ and the mh-set of G is not unique by Theorem 2.4 (b), $f_{mh}(G) = 1$. Let $S = \{u, v\}$, be a mh-set of G. Let us assume that $f_{mh}(S) = 1$. By Theorem 2.4 (b), S is the only mh- *set* containing u or v. Let us assume that S is the only mh-set containing u. By Theorem 2.13, u is adjacent to more than two vertices of G, which is a contradiction to G is a cycle. Therefore $f_{mh}(G) = 2$. ■

In view of Theorem 2.3, we have the following realization result.

Theorem 2.15. For every pair a, b of integers with $0 \leq a \leq b$ and $b \geq 2$, there exists a connected graph G such that $f_{mh}(G) = a$ and $mh(G) = b$.

Proof. If $a = 0$, let $G = K_b$. Then by Theorems1.1(b), $mh(G) = b$ and by Theorem 2.10, $f_{mh}(G) = 0$. For $a \geq 1$, let Q_i : $u_i, v_i, x_i, y_i, w_i, u_i$ $(1 \leq i \leq a)$ be a copy of cycle C_5. Let H be the graph obtained from Q_i by adding new vertex x and joining the edges and the edges xv_i, xw_i $(1 \leq i \leq a)$. Let G be the graph given in Figure 2.3 is obtained from H by adding new vertices $z_1, z_2, \ldots, z_{b-a}$ and joining the edges xz_i $(1 \leq i \leq b-a)$. Let $Z = \{z_1, z_2, \ldots, z_{b-a}\}$ be the set of end vertices of G. By Theorem 1.2(a), Z is a subset of every monophonic hull set of G. For $1 \leq i \leq a$, let $F_i = \{u_i, x_i, y_i\}$. We observe that every mh-set of G must contain at least one vertex from each F_i so that $mh(G) \geq b - a + a = b$. Now $M_1 = Z \cup \{x_1, x_2, x_3, \ldots, x_a\}$ is a monophonic hull set of G so that $mh(G) \leq b - a + a = b$. Thus $mh(G) = b$. Next we show that $f_{mh}(G) = a$. Since every mh-set contains Z, it follows from Theorem 2.8 that $f_{mh}(G) \leq mh(G) - |Z| = b - (b - a) = a$. Now, since $mh(G) = b$ and every mh-set of G contains Z, it is easily seen that every mh-set M is of the form $Z \cup \{d_1, d_2, d_3, \ldots d_a\}$, where $d_i \in F_i (1 \leq i \leq a)$. Let T be any proper subset of M with $|T| < a$. Then it is clear that there exists some j such that $T \cap F_j = \Phi$, which shows that $f_{mh}(G) = a$. ■

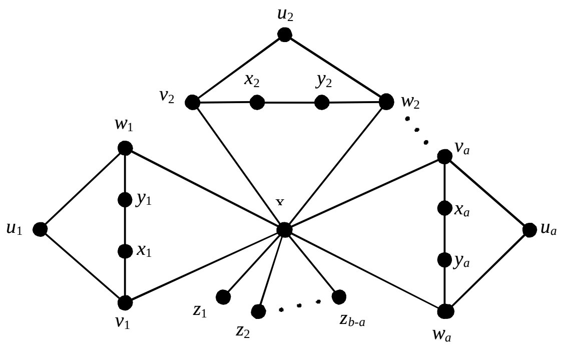

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Figure 2.3

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