Improved Ujević method for finding zeros of linear and nonlinear equations

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Abstract—In this paper, we suggest a new predictor- corrector method for solving nonlinear equations by combining Halley's method and Ujević method. The method is verified on a number of test examples and numerical results show that the proposed method is very effective with respect to Ujević method and Newton's method for finding zeros of nonlinear equations.

***Keywords*- Halley's method, Ujević method, Numerical examples, Nonlinear equations, Newton's method**

I. INTRODUCTION

One of the oldest numerical computation problems is of finding the values of x which satisfy the equation $f(x)=0$. Several numerical methods have been developed to compute the zeros of linear or nonlinear equation $f(x)=0$ including Newton's method. These methods have been developed using Taylor's interpolating polynomials and quadrature formulae (see [1-10]). Ujević (see [6]) suggests a new method which is a predictor-corrector type method. In this method, Newton's method acts as predictor method. These methods are extremely useful for finding simple zeros of the equation $f(x)=0$.

Inspired by the research going on this direction, we suggest and consider a new predictor-corrector method by combining Halley's method and Ujević method. Our Numerical experience shows that our algorithm [MNUM] gives better results than the algorithm of Ujevic´ [NU] and Newton method [NM].

One figure is given to show the efficiency of our algorithm.

II. NEWTON'S METHOD[NM]

Newton's method with weighting factor $\alpha \in (0,1]$ is as follows

<u>*Algorithm*</u> For given x_0, compute $x_1, x_2, x_3, \ldots$ such that

$$x_{k+1} = x_k - \alpha \left\{ \frac{f(x_k)}{f'(x_k)} \right\}$$

III. UJEVIĆ METHOD [6, NUM]

This method is derived using special type of quadrature formula. This method uses Newton's method with weighting factor $\alpha \in (0,1]$. The algorithm for this method is as follows

<u>*Algorithm*</u> For given x_0, compute $x_1, x_2, x_3, \ldots$ such that $z_k = x_k - \alpha \left\{ \frac{f(x_k)}{f'(x_k)} \right\}$

$$x_{k+1} = x_k + 4(z_k - x_k) \left\{ \frac{f(x_k)}{3f(x_k) - 2f(z_k)} \right\}$$

Analysis of this algorithm is given in [6].

IV. MODIFIED UJEVIC METHOD [MNUM]

If $f(x)$ is single function of C^2-class, then Halley's(see [9,10]) one step iterative formula for finding zeros of $f(x)=0$ is

$$x_{k+1} = x_k - \frac{2f(x_k)f'(x_k)}{2\{f'(x_k)\}^2 - f(x_k)f''(x_k)}, k = 0,1,2,3,\ldots..$$

This method is third order convergent. Convergence analysis Can be found in [9, 10]. In Ujevic method, Newton's method acts as predictor method. For modification of Ujević method**,** we use Halley's method (see [9, 10]) as predictor formula.

Then algorithm of modified Ujević method is as follows

<u>*Algorithm*</u> For given x_0, compute $x_1, x_2, x_3, \ldots$ such that $z_k = x_k - \frac{2\alpha f(x_k)f'(x_k)}{2\{f'(x_k)\}^2 - f(x_k)f''(x_k)}$,

$$x_{k+1} = x_k + 4(z_k - x_k) \left\{ \frac{f(x_k)}{3f(x_k) - 2f(z_k)} \right\}$$

V. NUMERICAL EXPERIMENTS

In all of our examples, the maximum number of iteration is $n = 1000$ and the examples are tested with precision $\varepsilon = 1\times10^{-10}$. We have checked the algorithm given above for 10 different values of α including 0.5. The following stopping criteria are used for our computer programs

(i) $|x_{k+1} - x_k| < \varepsilon$

(ii) $|f(x_{k+1})| < \varepsilon$

Example 1. Let $f(x) = x^3 - 3x + 2$ and $x_0 = 0.5$. Then number of iterations which we have got from the methods NM, NUM, and MNUM for different values of α is given in Table 1.

Table 1

NM	NUM	MNUM	α
250	74	59	0.1
134	42	34	0.2
89	31	24	0.3
67	24	20	0.4
53	20	15	0.5
43	18	13	0.6
36	15	11	0.7
30	13	10	0.8
26	11	8	0.9
22	10	6	1.0

Here we can see modified Ujević method gives better accuracy then Ujević method and Newton's method for all values of α .Here exact root of $f(x) = 0$ is 1

Example 2. Let $f(x) = 3x - \cos x - 1$ and $x_0 = 0.5$. Then number of iterations which we have got from the methods NM, NUM, and MNUM for different values of α is given in Table 2.

Table2.

NM	NUM	MNUM	α
137	37	34	0.1
69	19	18	0.2
44	13	13	0.3
32	8	8	0.4
24	4	3	0.5
19	8	8	0.6
14	11	11	0.7
14	13	13	0.8
14	14	14	0.9
15	17	15	1.0

As we can see modified Ujevic method gives better accuracy then Ujevic method and Newton's method for $\alpha = 0.1, 0.2, 0.5, 1.0$.

Here, the root is 0.6071016481 correct to 10 decimal places.

Example 3. Let $f(x) = x - \cos x$ and $x_0 = 0.7$ .Then number of iterations which we have got from the methods NM, NUM, and MNUM for different values of α is given in Table 3. Here the root is 0.7390851332, correct to 10 decimal places.

Table 3

NM	NUM	MNUM	α
72	30	30	0.1
57	16	14	0.2
38	11	10	0.3
27	7	7	0.4
21	4	3	0.5
17	7	6	0.6
13	9	9	0.7
15	11	9	0.8
13	13	12	0.9
12	13	12	1.0

Here we can see that modified Ujević method(MNUM) gives better accuracy then Ujević Method(NUM) and Newton's method (NM) for $\alpha = 0.2, 0.3, 0.5, 0.6, 0.9$

Example 4. Let $f(x) = 11x^{11} - 1$ and $x_0 = 0.5$. Then number of iterations which we have got from the methods NM, NUM, and MNUM for different values of α is given in Table 4.

Table 4.

NM	NUM	MNUM	α
175	34	38	0.1
115	152	20	0.2
88	D	13	0.3
72	D	9	0.4
61	D	6	0.5
53	40S	8	0.6
46	35S	10	0.7
41	45S	9	0.8
36	84S	14	0.9
31	23S	16	1.0

Here D means divergent, S - The method stuck after these numbers of iterations

In this example, Ujević method gets stuck for some values of α on a specified root which is not accurate on given precession.

Here also modified Ujević method gives better accuracy then Ujevic method and Newton's method for all values of α. Here Ujevic method (NUM) converges for $\alpha = 0.1, 0.2$ and diverges for $\alpha = 0.3, 0.4, 0.5$

Here, the root is 0.80413309759, correct to 10 decimal places.

Example 5. Let $f(x) = xe^{x} - 1$ and $x_0 = 0.5$.

Then number of iterations which we have got from the methods NM, NUM, and MNUM for different values of α is given in Table 5.

Table 5.

NM	NUM	MNUM	α
117	32	30	0.1
59	17	14	0.2
38	11	11	0.3
29	10	7	0.4
21	5	4	0.5
17	8	6	0.6
13	9	9	0.7
12	11	11	0.8
9	13	8	0.9
4	15	4	1.0

Here we can see modified Ujević method gives better accuracy then Ujević method and Newton's method for all values of α. Here the root is 0.5671433045, correct to 10 decimal places.

Example 6. Let $f(x) = \sin x + \cos x - 1$ and $x_0 = 1.0$.

Then number of iterations which we have got from the methods NM, NUM, and MNUM for different values of α is given in Table 6.

Table 6.

NM	NUM	MNUM	α
127	43	36	0.1
63	27	19	0.2
40	12	12	0.3
28	9	9	0.4
22	5	4	0.5
19	8	8	0.6
15	11	9	0.7
12	12	12	0.8
17	14	14	0.9
6	16	4	1.0

As we can see modified Ujević method gives better accuracy then Ujević method and Newton's method for all values of α.

In Table 7, the CPU time (in second) of two steps methods, NUM and MNUM for weighting factor $\alpha = 0.5$ are compared. All results show here are obtained using MATLAB on a computer having Pentium(R) dual- core CPU at 5.00 GHz

Table 7.

Equation	Initial Approx. (x_0)	NUM	MNUM
$x^3 - 3x + 2 = 0$	0.5	0.078000500	0.031200200
$3x - \cos x - 1 = 0$	0.0	0.093600600	0.015600099
$x - \cos x = 0$	0.7	0.015600099	0.154000100
$11x^{11} - 1 = 0$	0.5	-*	0.025600099
$xe^{x} - 1 = 0$	0.5	0.062400399	0.015600100
$\cos x + \sin x - 1 = 0$	1.0	0.062400400	0.015600099

* The method diverges

From these examples, it is clear that the proposed predictor corrector method [MNUM] performs better than Newton's method [NM] and Ujević method [NUM] for different values of α.

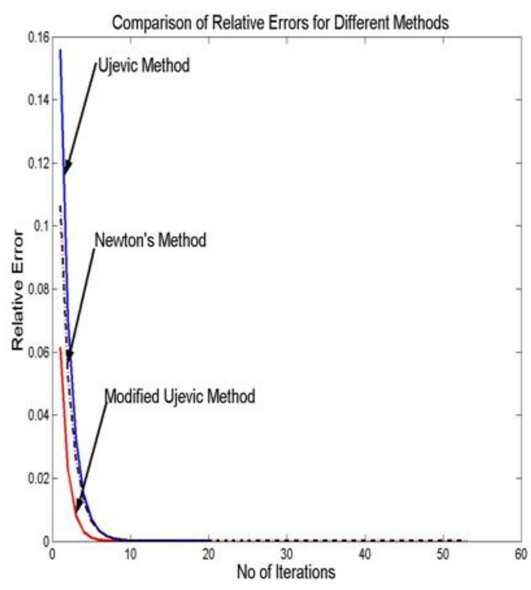

Fig: Comparison of relative errors for different methods **for** $x^3 - 3x + 2 = 0$ using $x_0 = 0.5$ and $\alpha = 0.5$

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