Magic and Bimagic Labeling for Disconnected Graphs

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***Abstract*— An edge magic total labeling of a graph $G(V,E)$ with *p* vertices and *q* edges is a bijection *f* from the set of vertices and edges to $1,2,...,p+q$ such that for every edge *uv* in *E*, $f(u)+f(uv)+f(v)$ is a constant *k*. If t here exist two constants k_1 and k_2 such that the above sum is either k_1 or k_2, it is said to be an edge bimagic total labeling. A total edge magic (edge bimagic) graph is called a super edge magic (super edge bimagic) if $f(V(G))=\{1,2,..., p\}$. A total edge magic (edge bimagic) graph is called a superior edge magic (superior edge bimagic) if $f(E(G))=\{1,2,..., q\}$.In this paper we give magic and bimagic labelings for some class of disconnected graphs**

AMS Subject Classification: 05C78
Keyword- **graph, labeling, magic labeling, bimagic labeling, function.**

I. Introduction

A labeling of a graph *G* is an assignment *f* of labels to either the vertices or the edges or both subject to certain conditions. Labeled graphs are becoming an increasingly useful family of Mathematical Models from a broad range of applications. Graph labeling was first introduced in the late 1960's. A useful survey on graph labeling by J.A. Gallian (2011) can be found in [1]. All graphs considered here are finite, simple and undirected. We follow the notation and terminology of [2]. In most applications labels are positive (or nonnegative) integers, though in general real numbers could be used. A (*p*, *q*)-graph $G=(V,E)$ with *p* vertices and *q* edges is called total edge magic if there is a bijection $f:V\cup E\rightarrow\{1,2,...,p+q\}$ such that there exists a constant *k* for any edge *uv* in *E*, $f(u)+f(uv)+f(v)=k$. The original concept of total edge-magic graph is due to Kotzig and Rosa[3]. They called it magic graph. A total edge-magic graph is called a super edge-magic if $f(V(G))=\{1,2,..., p\}$. Wallis [4] called super edge-magic as strongly edge-magic.

It becomes interesting when we arrive with magic type labeling summing to exactly two distinct constants say k_1 or k_2. Edge bimagic totally labeling was introduced by J.Baskar Babujee [5] and studied in [6] as (1,1) edge bimagic labeling. A graph $G(p,q)$ with *p* vertices and *q* edges is called total edge bimagic if there exists a bijection $f:V\cup E\rightarrow\{1,2,...,p+q\}$. such that for any edge $uv\in E$, we have two constants k_1 and k_2 with $f(u)+f(v)+f(uv)=k_1$ or k_2. A total edge-bimagic graph is called *super edge-bimagic* if $f(V(G))=\{1,2,..., p\}$. *S*uper edge-bimagic labeling for path, star- $K_{1,n}$, $K_{1,n,n}$ are proved in [7]. *S*uper edge-bimagic labeling for cycles, Wheel graph, Fan graph, Gear graph, Maximal Planar class-Pl_n: $n\geq 5$, $K_{1,m}\cup K_{1,n}(m,n\geq 1), P_n\cup P_{n+1}(n\geq 2)$, $C_3\cup K_{1,n}(n\geq 1), P_n+N_2(n\geq 3), P_2\cup mK_1+N_2)(m\geq 1)$, (3, *n*)-kite graph $(n\geq 2)$, are proved in [8, 9, 10].], In [11], defined the superior edge magic as "A graph *G* = (*V*,*E*) with *p* vertices and *q* edges has a superior edge magic total labeling if there is a bijection function *f*: $V\cup E\rightarrow\{1, 2,\ldots, p+q\}$ such that *f*(*u*)+*f*(*v*)+*f*(*uv*) are all constant for all $uv\in E(G)$, where $f(E(G))=\{1,2,\ldots,q\}$".In this paper we exhibit magic and bimagic labelings for some class of disconnected graphs.

II. Main Results

In this section we will give magic and bimagic labelling for disconnected

Algorithm 2.1

Input : The number of vertices and edges of the graph nP_3.
Output : Superior edge bimagic labeling for nP_3.
begin

$V = \{ v_i^j : 1\leq i\leq 3; 1\leq j\leq n\}$ and

$E = \{ v_i^j v_{i+1}^j : 1\leq i\leq 2; 1\leq j\leq n\}$.

for j = 1 to n
{
if ($j\equiv 1 \pmod 2$)
{

$$f(v_1^j) = 2n+\left(\frac{n}{2}\right)+\left(\frac{j+1}{2}\right);$$

$$f(v_3^j) = 3n+\left(\frac{j+1}{2}\right);$$

$$f(v_1^j v_2^j) = n + \left(\frac{j+1}{2}\right);$$
$$f(v_2^j v_3^j) = \left(\frac{n}{2}\right) + \left(\frac{j+1}{2}\right);$$

}

else

{

$$f(v_1^j) = 3n + \left(\frac{n}{2}\right) + \left(\frac{j}{2}\right);$$
$$f(v_3^j) = 2n + \left(\frac{j}{2}\right);$$
$$f(v_1^j v_2^j) = \left(\frac{j}{2}\right);$$
$$f(v_2^j v_3^j) = n + \left(\frac{n}{2}\right) + \left(\frac{j}{2}\right);$$

}

$f(v_2^j) = 5n - j + 1;$

}

end.

Theorem 2.2

The graph nP_3 has superior edge bimagic labeling when n is even.

Proof

Consider the graph nP_3 (n is even) with vertex set V = { v_i^j : $1 \le i \le 3$; $1 \le j \le n$} and edge set E = { $v_i^j v_{i+1}^j$: $1 \le i \le 2$; $1 \le j \le n$}.The bijective function $f : V \cup E \to \{1, 2, \ldots, 5n\}$ is defined in above algorithm 2.1. Now we have to prove that the graph nP_3 have two distinct constants k_1 and k_2.

For the edges $v_i^j v_{i+1}^j \in E$ for $1 \le i \le 2$ and $j \equiv 1 \pmod 2$.

$$f(v_1^j) + f(v_1^j v_2^j) + f(v_2^j)$$
$$= 2n + \frac{n}{2} + \frac{j+1}{2} + n + \frac{j+1}{2} + 5n - j + 1$$
$$= 8n + \frac{n}{2} + 2 = k_1$$

Similarly,

$$f(v_2^j) + f(v_2^j v_3^j) + f(v_3^j)$$
$$= 5n\text{-}j\text{+}1\text{+}\frac{n}{2} + \frac{j+1}{2} + 3n + \frac{j+1}{2}$$
$$= 8n + \frac{n}{2} + 2 = k_1$$

Now for the edges $v_i^j v_{i+1}^j \in E$ for $1 \le i \le 2$ and $j \equiv 0 \pmod 2$

$$f(v_1^j) + f(v_1^j v_2^j) + f(v_2^j)$$
$$= 3n + \frac{n}{2} + \frac{j}{2} + \frac{j}{2} + 5n - j + 1$$
$$= 8n + \frac{n}{2} + 1 = k_2$$

Similarly,

$$f(v_2^j) + f(v_2^j v_3^j) + f(v_3^j)$$
$$= 5n\text{-}j\text{+}1\text{+}n + \frac{n}{2} + \frac{j}{2} \text{+}2n\text{+} \frac{j}{2}$$
$$= 8n + \frac{n}{2} + 1 = k_2$$

we have two constants k_1 and k_2 for the edges of nP_3.

Hence the graph nP_3,n is even has superior edge bimagic labeling.

Algorithm 2.3

Input : The number of vertices and edges of the graph nP_3.

Output : Superior edge magic labeling of the graph nP_3.

begin

V = { v_i^j : $1 \le i \le 3$; $1 \le j \le n$} and

E = { $v_i^j v_{i+1}^j$: $1 \le i \le 2$; $1 \le j \le n$}.

for j = 1 to n

{

if ($j \equiv 1 \pmod 2$)

{

$$f(v_1^j) = 2n + \left(\frac{n-1}{2}\right) + \left(\frac{j+1}{2}\right);$$
$$f(v_3^j) = 3n + \left(\frac{j+1}{2}\right);$$
$$f(v_1^j v_2^j) = n + \left(\frac{j+1}{2}\right);$$
$$f(v_2^j v_3^j) = \left(\frac{n-1}{2}\right) + \left(\frac{j+1}{2}\right);$$

}

else

{

$$f(v_1^j) = 3n + 1 + \left(\frac{n-1}{2}\right) + \left(\frac{j}{2}\right);$$
$$f(v_3^j) = 2n + \left(\frac{j}{2}\right);$$
$$f(v_1^j v_2^j) = \frac{j}{2};$$
$$f(v_2^j v_3^j) = n + 1 + \left(\frac{n-1}{2}\right) + \left(\frac{j}{2}\right);$$

}

$f(v_2^j) = 5n - j + 1;$

}

end.

Theorem 2.4

The graph nP_3 has superior edge magic labeling when n is odd.

Proof

Consider the graph nP_3 (n is odd) with vertex set V = { v_i^j : $1 \le i \le 3$; $1 \le j \le n$} and edge set E = { $v_i^j v_{i+1}^j$: $1 \le i \le 2$; $1 \le j \le n$}. The bijective function $f : V \cup E \to \{1, 2, \ldots, 5n\}$ is defined in given algorithm. Now we have to prove that the graph nP_3 have only one constant k_1.

for the edges $v_i^j v_{i+1}^j \in E$ for $1 \le i \le 2$ and $j \equiv 1 \pmod 2$

$$f(v_1^j) + f(v_1^j v_2^j) + f(v_2^j)$$

$$= 2n + \frac{n-1}{2} + \frac{j+1}{2} + n + \frac{j+1}{2} + 5n - j + 1$$

$$= 8n + \frac{n-1}{2} + 2 = k_1$$

Similarly,

$$f(v_2^j) + f(v_2^j v_3^j) + f(v_3^j)$$

$$= 5n - j + 1 + \frac{n-1}{2} + \frac{j+1}{2} + 3n + \frac{j+1}{2}$$

$$= 8n + \frac{n-1}{2} + 2 = k_1$$

Now for the edges $v_i^j v_{i+1}^j \in E$ for $1 \le i \le 2$ and $j \equiv 0 \pmod 2$

$$f(v_1^j) + f(v_1^j v_2^j) + f(v_2^j)$$

$$= 3n + 1 + \frac{n-1}{2} + \frac{j}{2} + 5n - j + 1 + \frac{j}{2}$$

$$= 8n + \frac{n-1}{2} + 2 = k_1$$

similarly,

$$f(v_2^j) + f(v_2^j v_3^j) + f(v_3^j)$$

$$= 5n - j + 1 + n + 1 + \frac{n-1}{2} + \frac{j}{2} + 2n + \frac{j}{2}$$

$$= 8n + \frac{n-1}{2} + 2 = k_1$$

Thus we have only one constant k_1 for the edges of nP_3 (n is odd).

Hence the graph nP_3 has superior edge magic labeling when n is odd.

Algorithm 2.5

Input : The number of vertices and edges of the graph $K_{1,n} \cup K_{1,m}$.

Output : Edge bimagic total labeling for $K_{1,n} \cup K_{1,m}$.

begin

$V = \{ v_i^j : 1 \le i \le 2; 1 \le j \le n\}$ and

$E = \{ v_i^j v_{i+1}^j : 1 \le i \le 2; 1 \le j \le n\}$.

for j = 1 to n

{

if ($j \equiv 1 \pmod 2$) then

$f(v_1^j) = 1;$

else $f(v_1^j) = 2;$

$f(v_2^j) = (n+2) + (m+1) - j;$

$f(v_1^j v_2^j) = (n+1) + (m+1) + j;$

}

end.

Theorem 2.6

The graph $K_{1,n} \cup K_{1,m}$, ($n \le m$) has edge bimagic total labeling.

Proof

Consider the graph $K_{1,n} \cup K_{1,m}$ with vertex set $V = \{ v_i^j : 1 \le i \le 2; 1 \le j \le n\}$ and edge set $E = \{ v_i^j v_{i+1}^j : 1 \le i \le 2; 1 \le j \le n\}$.

The bijective function $f : V \cup E \to \{1, 2, \ldots, 2n+2m+2\}$ is defined in given algorithm.

Now, we have to prove that the graph $K_{1,n} \cup K_{1,m}$ have two constant k_1 and k_2.

For the edges $v_i^j v_{i+1}^j \in E$

for $1 \le i \le 2$, $1 \le j \le n$ and $j \equiv 0 \pmod 2$

$$f(v_1^j) + f(v_2^j) + f(v_1^j v_2^j)$$

$$= 1 + n + 2 + m + 1 - j + n + 1 + m + 1 + j$$

$$= 2n + 2m + 6 = k_1$$

Now, for the edges $v_i^j v_{i+1}^j \in E$

for $1 \le i \le 2$, $1 \le j \le n$ and $j \equiv 0 \pmod 2$

$$f(v_1^j) + f(v_2^j) + f(v_1^j v_2^j)$$

$$= 2 + n + 2 + m + 1 - j + n + 1 + m + 1 + j$$

$$= 2n + 2m + 7 = k_2$$

we have two constants k_1, k_2 for the graph $K_{1,n} \cup K_{1,m}$.

Hence the graph $K_{1,n} \cup K_{1,m}$, ($n \le m$) has edge bimagic total labeling.

Algorithm 2.7

Input : The number of vertices and edges of the graph $nP_2 \cup K_{1,n+1}$.

Output : Edge magic labeling for $nP_2 \cup K_{1,n+1}$.

begin

$V = \{ v_1, v_2^j, u_1^j, u_2^j : 1 \le j \le n\}$ and

$E = \{ v_2^j u_2^j, v_1 u_1^j : 1 \le j \le n\} \cup \{u_1^j v_1 : 1 \le j \le n+1\}$.

for j = 1 to n

{

if ($j \equiv 1 \pmod 2$)

{

$f(v_2^j) = \frac{j+1}{2};$

$f(u_2^j) = 2n + 2 + \frac{n}{2} + \frac{j+1}{2};$

}

else

{

$f(v_2^j) = \frac{n}{2} + 1 + \frac{j}{2};$

$f(u_2^j) = 2n + 2 + \frac{j}{2};$

}

$f(v_2^j u_2^j) = 4n + 3 - j;$

}

for j = 1 to n+1
{
$$f(v_1) = \frac{n}{2} + 1;$$
$$f(v_1u_1^j) = 3(n+1) + n + j - 1;$$
$$f(u_1^j) = 2(n+1) + 1 - j;$$
}

end.

Theorem 2.8
The graph $nP_2 \cup K_{1,n+1}$ has super edge magic labeling when n is even.
Proof
Consider the graph $nP_2 \cup K_{1,n+1}$ (n is even) with vertex set V = $\{ v_1, v_2^j, u_1^j, u_2^j \ : 1 \le j \le n\}$ and edge set
E = $\{ v_2^ju_2^j, v_1u_1^j \ : 1 \le j \le n\} \cup \{u_1^jv_1 : 1 \le j \le n+1\}$ ..
The bijective function $f : V \cup E \to \{1, 2, \ldots, 5n+3\}$ is defined in given algorithm.
Now, we have to prove that the graph $nP_2 \cup K_{1,n+1}$ has only one constant k_1.
For the edges $v_2^ju_2^j \in E$
for $1 \le j \le n$ and $j \equiv 1 \pmod 2$
$$f(v_2^j) + f(u_2^j) + f(v_2^ju_2^j)$$
$$= \frac{j+1}{2} + 2n + 2 + \frac{n}{2} + \frac{j+1}{2} + 4n + 3 - j$$
$$= \frac{n}{2} + 6n + 6 = k_1$$
Now, for the edges $v_2^ju_2^j \in E$
for $1 \le j \le n$ and $j \equiv 0 \pmod 2$
$$f(v_2^j) + f(u_2^j) + f(v_2^ju_2^j)$$
$$= 2n + 2 + \frac{j}{2} + \frac{n}{2} + 1 + \frac{j}{2} + 4n + 3 - j$$
$$= \frac{n}{2} + 6n + 6 = k_1$$
Now, for the edges $v_1u_1^j \in E$
for $1 \le j \le n+1$
$$f(v_1) + f(u_1^j) + f(v_1u_1^j)$$
$$= \frac{n}{2} + 1 + 2(n+1) + 1 - j + 3(n+1) + n + j - 1$$
$$= \frac{n}{2} + 1 + 2n + 2 + 1 - j + 3n + 3 + n + j - 1$$
$$= \frac{n}{2} + 6n + 6 = k_1$$
we have only one constant k_1 for the graph $nP_2 \cup K_{1,n+1}$. Thus graph $nP_2 \cup K_{1,n+1}$ has super edge magic labeling when n is even.

Algorithm 2.9
Input : The number of vertices and edges of the graph $nP_2 \cup K_{1,n+1}$.
Output : Edge bimagic labeling for $nP_2 \cup K_{1,n+1}$.
begin
V = $\{ v_1, v_2^j, u_1^j, u_2^j \ : 1 \le j \le n\}$ and
E = $\{ v_2^ju_2^j \ : 1 \le j \le n\} \cup \{u_1^jv_1 : 1 \le j \le n+1\}$.
for j = 1 to n
{
if ($j \equiv 1 \pmod 2$)
{
$$f(v_2^j) = \frac{j+1}{2};$$
$$f(u_2^j) = 2n + 1 + \frac{n+1}{2} + \frac{j+1}{2};$$
}
else
{
$$f(v_2^j) = \frac{n+1}{2} + \frac{j}{2};$$
$$f(u_2^j) = 2n + 2 + \frac{j}{2};$$
}
$$f(v_2^ju_2^j) = 4n + 3 - j;$$
}
for j = 1 to n+1
{
$$f(v_1) = n + 1;$$
$$f(v_1u_1^j) = 3(n+1) + n + j - 1;$$
$$f(u_1^j) = 2(n+1) + 1 - j;$$
}
end

Theorem 2.10
The graph $nP_2 \cup K_{1,n+1}$ has super edge bimagic labeling when n is odd.
Proof
Consider the graph $nP_2 \cup K_{1,n+1}$ (n is odd) with vertex set V = $\{ v_1, v_2^j, u_1^j, u_2^j \ : 1 \le j \le n\}$ and edge set
E = $\{ v_2^ju_2^j, u_1^jv_1 \ : 1 \le j \le n\} \cup \{u_1^jv_1 : 1 \le j \le n+1\}$.
The bijective function $f : V \cup E \to \{1, 2, \ldots, 5n+3\}$ is defined in the given algorithm.
Now, we have to prove that the graph $nP_2 \cup K_{1,n+1}$ has two distinct constants k_1 and k_2.

For the edges $v_2^j u_2^j \in E$

for $1 \le j \le n$ and $j \equiv 1 \pmod 2$

$f(v_2^j) + f(u_2^j) + f(v_2^j u_2^j)$

$$= \frac{j+1}{2} + 2n + 1 + \frac{n+1}{2} + \frac{j+1}{2} + 4n + 3 - j$$

$$= \frac{n+1}{2} + 6n + 5 = k_1$$

Now, for the edges $v_2^j u_2^j \in E$

for $1 \le j \le n$ and $j \equiv 0 \pmod 2$

$f(v_2^j) + f(u_2^j) + f(v_2^j u_2^j)$

$$= 2n + 2 + \frac{j}{2} + \frac{n+1}{2} + \frac{j}{2} + 4n + 3 - j$$

$$= \frac{n+1}{2} + 6n + 5 = k_1$$

Now, for the edges $v_1 u_1^j \in E$

for $1 \le j \le n+1$

$f(v_1) + f(u_1^j) + f(v_1 u_1^j)$

$$= n + 1 + 2(n+1) + 1 - j + 3(n+1) + n + j - 1$$

$$= n + 1 + 2n + 2 + 1 - j + 3n + 3 + n + j - 1$$

$$= 7n + 6 = k_2$$

We have two constants k_1 and k_2 for the edges of $nP_2 \cup K_{1,n+1}$. Thus the graph $nP_2 \cup K_{1,n+1}$ has super edge bimagic labeling when n is odd.

Theorem 2.11

If G has superior edge-magic labeling then $G \cup mK_{1,n} \cup (m-1)K_1$ admits edge bimagic total labeling.

Proof

Let G(p, q) be a superior edge magic graph with bijective function$f : V \cup E \to \{1, 2, \ldots, p+q\}$ such that $f(u) + f(uv) + f(v) = k_1$.Now we define the new graph $G' = G \cup mK_{1,n} \cup (m-1)K_1$ with vertex set

$V'\ V \cup \{ u_1^j v_i^j : 1 \le i \le n; 1 \le j \le m\} \cup \{w_j : 1 \le j \le m-1\}$ and set edge set $E' = E \cup \{ u_1^j v_i^j : 1 \le i \le n; 1 \le j \le m\}$

Now the bijective function $g : V' \cup E' \to \{1, 2, \ldots, p+q, p+q+1, \ldots, m(n+1)+\{m-1)\}$ is defined as follows. for s

$g(u) = f(u)$ for all $u \in V(G)$

$g(u_1^j) = p + q + j;\ 1 \le j \le m.$

$g(v_i^j) = p + q + m + (j-1)n + i;$

$g(w_j) = p + q + 2mn + m + 2 - nj - (j-1);$

and $g(uv) = f(uv)$ for all $uv \in E(G)$

$g(u_1^j v_i^j) = p + q + 2mn + m + 2 - (j-1)n - (i + j - 2);$

Since the graph G is superior edge magic with common count k_1. Implies that $g(u) + g(uv) + g(v) = k_1$ for all $uv \in E(G)$. Now we have to prove that the remaining edges in the set $\{ u_1^j v_i^j : 1 \le i \le n; 1 \le j \le m\}$ have the common count k_2.

For the edge $u_1^j v_i^j \in E'$

$g(u_1^j) + g(u_1^j v_i^j) + g(v_i^j)$

$$= p + q + j + p + q + 2mn + m + 2 - (j-1)n$$
$$- (i + j - 2) + p + q + m + (j-1)n + i$$
$$= 3(p+q) + 2m(n+1) + 4$$
$$= k_2$$

Thus the new graph G' have two constants k_1 and k_2. Hence the graph $G \cup mK_{1,n} \cup (m-1)K_1$ has edge bimagic total labeling.

III. Conclusion

Theorem 2.11 shows that $G \cup mK_{1,n} \cup (m-1)K_1$ admit edge bimagic total labeling if G has superior edge-magic labeling. Further investigation can be done to obtain the conditions at which $\cup G_i$'s admits edge bimagic total labeling.

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