Normal magic square and its some matrix properties

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Abstract: Some matrix properties on Eigen values, singular values, norm and condition number of normal magic square are presented in this paper. The properties are verified on a number of test examples and numerical results confirm that the properties are very effective for normal magic square. All the properties are checked using computational code in Matlab. In addition moment of inertia, surface plot and contour plot for different order normal magic squares are drawn

Keywords: Eigen value, singular value, norm, conditional number, normal magic square.

I. INTRODUCTION

In discrete mathematics, magic square of order n is an arrangement of n^2 numbers, usually distinct integers in a square, such that the n numbers in all rows, all columns, and both diagonals sum to the same constant.

Normal magic square of order n is a square matrix of order n in which all the elements are distinct and from 1 to n^2 and row sum, column sum and both diagonals sum are equal and it is equal to $\frac{n(n^2+1)}{2}(=M$, say) This number is called magic constant. Throughout the ages, near 4120 yrs. human civilisation became very much involved with magic squares. In various culture including Egyptian and Indian culture, one can find the impact of magic square. Magic squares were engraved on stone or metal and it was believed that magic squares had astrological and divinatory qualities and also it can prevent disease. There are many ways to construct magic squares. For examples, LUX method (John Horton Conway) and the Strachey method, genetic algorithm, group theory etc. Magic squares exist for all values of n, with only one exception: it is impossible to construct a magic square of order 2. It is unsolved problem in mathematics to find the number magic square of order greater than 5(see [3]). If $A=(a_{ij})_{n\times n}$ be square matrix over the field $\Re^{n\times n}$, then 1-norm, ∞-norm, 2-norm, Frobenius norm of the matrix A can be defined as $\|A\|_1 = \underset{1\le j\le n}{Max}\sum_{i=1}^{n}|a_{ij}|$ (the maximum absolute column sum of the matrix), $\|A\|_\infty = \underset{1\le i\le n}{Max}\sum_{j=1}^{n}|a_{ij}|$ (the maximum absolute row sum of the matrix), $\|A\|_2=\sqrt{\lambda_{\max}(A^TA)}=\sigma_{\max}(A)$ (largest singular value of the matrix), $\|A\|_F=\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2}=\sqrt{tr(A^TA)}=\sqrt{\sum_{i=1}^{n}\sigma_i^2}$ .The condition number of the matrix A can be defined as $cond(A)=\|A\|\|A^{-1}\|$. The goal of this paper is to investigate Eigen values, singular values, and norm and condition number of any normal magic square of order both even and odd. It is to be noted that rotation of any magic square has been considered as separate magic square.

II. RANK, INVERSE & EIGEN VALUE

(i) Even order normal magic square is singular matrix and odd order normal magic square is non singular matrix (see [1]).

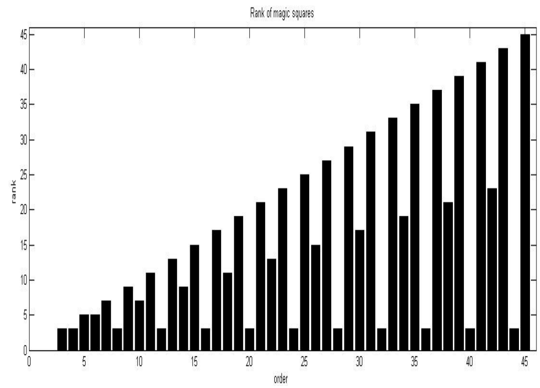

Figure1. *Rank of normal magic squares*

From the figure, it is clearly shown that odd order normal magic square of order n, has full rank n. Doubly even order normal magic squares, n = 4, 8, 12,*16*,….., have rank three no matter how large n is. They might be called *very singular*. Singly even order normal magic squares, n = 6, 10, 14… have rank $(\frac{n}{2}+2)$.

(ii) Inverse of odd order normal magic square is a magic square but the elements are not necessarily integer elements.

For example,

$A = magic(5) =$ $\quad\quad$ $inv(A) =$

$$\begin{pmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{pmatrix}, \ldots \begin{pmatrix} -0.0049 & 0.0512 & -0.0354 & 0.0012 & 0.0034 \\ 0.0431 & -0.0373 & -0.0046 & 0.0127 & 0.0015 \\ -0.0303 & 0.0031 & 0.0031 & 0.0031 & 0.0364 \\ 0.0047 & -0.0065 & 0.0108 & 0.0435 & -0.0370 \\ 0.0028 & 0.0050 & 0.0415 & -0.0450 & 0.0111 \end{pmatrix}$$

(iii) Magic constant of inverse of odd order normal magic square is $\frac{1}{M}$ where M is magic constant of the normal magic square.

For the above example, $M = 65, \frac{1}{M} = 0.0154$.

(iv) If $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ be the Eigen values of a normal magic square of order n, then $\lambda_1 + \lambda_2 + \lambda_3 + ... + \lambda_n = M$. i.e. sum of all eigen values of a normal magic square is the magic constant since sum of all eigen values of any matrix is the trace of the matrix. For normal magic square trace is M . For example, for the matrix A ,normal magic square of order 5,

$$eig(A) = \begin{pmatrix} 65.0000 \\ -21.2768 \\ -13.1263 \\ 21.2768 \\ 13.1263 \end{pmatrix}, sum(eig(A)) = 65.00 = M$$

(v) The largest eigen value of any normal magic square is the magic constant. It is clearly shown in the above example.

(vi) For odd order normal magic square, all the eigen values, except the largest eigen value(λ), are pair wise equal in magnitude but opposite in sign. It is shown in the above example. (vii) For even order normal magic square, except the largest eigen value (λ) and the eigen value 0, all other eigen values are pair wise equal in magnitude but opposite in sign.

For example,

$X = magic(6) =$ $\quad\quad$ $eig(X) =$

$$\begin{pmatrix} 35 & 1 & 6 & 26 & 19 & 24 \\ 3 & 32 & 7 & 21 & 23 & 25 \\ 31 & 9 & 2 & 22 & 27 & 20 \\ 8 & 28 & 33 & 17 & 10 & 15 \\ 30 & 5 & 34 & 12 & 14 & 16 \\ 4 & 36 & 29 & 13 & 18 & 11 \end{pmatrix}, \ldots \begin{Bmatrix} 111.0000 \\ 27.0000 \\ -27.0000 \\ 9.7980 \\ 0 \\ -9.7980 \end{Bmatrix}.$$

It is to be noted that this properties ((vi) & (vii)) do not hold for any normal magic square of same order. It holds for at least one of magic square of same order.

III. SINGULAR VALUE

(vii) All the singular values of any odd order normal magic square are positive numbers as odd order normal magic square is non-singular.

For example,

$Y = magic(7) =$ $\quad\quad$ $svd(Y) =$

$$\begin{pmatrix} 30 & 39 & 48 & 1 & 10 & 19 & 28 \\ 38 & 47 & 7 & 9 & 18 & 27 & 29 \\ 46 & 6 & 8 & 17 & 26 & 35 & 37 \\ 5 & 14 & 16 & 25 & 34 & 36 & 45 \\ 13 & 15 & 24 & 33 & 42 & 44 & 4 \\ 21 & 23 & 32 & 41 & 43 & 3 & 12 \\ 22 & 31 & 40 & 49 & 2 & 11 & 20 \end{pmatrix}, \ldots \begin{Bmatrix} 175.0000 \\ 57.4356 \\ 57.4356 \\ 31.5526 \\ 31.5526 \\ 24.6086 \\ 24.6086 \end{Bmatrix}$$

(viii) Largest singular value of normal magic square is its magic constant. For example, magic square of order 7, Y has largest Singular value 175 also it has magic constant (M) =175.

(ix) Zero is singular value of even order normal magic square as even order normal magic square is singular.

For example

$Q = magic(6) =$ $\quad\quad$ $svd(Q) =$

$$\begin{pmatrix} 35 & 1 & 6 & 26 & 19 & 24 \\ 3 & 32 & 7 & 21 & 23 & 25 \\ 31 & 9 & 2 & 22 & 27 & 20 \\ 8 & 28 & 33 & 17 & 10 & 15 \\ 30 & 5 & 34 & 12 & 14 & 16 \\ 4 & 36 & 29 & 13 & 18 & 11 \end{pmatrix}, \ldots \begin{Bmatrix} 111.0000 \\ 50.6802 \\ 34.3839 \\ 10.1449 \\ 5.5985 \\ 0.0000 \end{Bmatrix}$$

IV. NORM

(x) Since for any normal magic square, row sum, column sum and diagonal sum are equal, so 1-norm, 2-norm and infinity norm of that magic square are equal and it is equal to the magic constant. For the above magic square Q,

$$\|Q\|_1 = \|Q\|_2 = \|Q\|_\infty = 111 = M$$

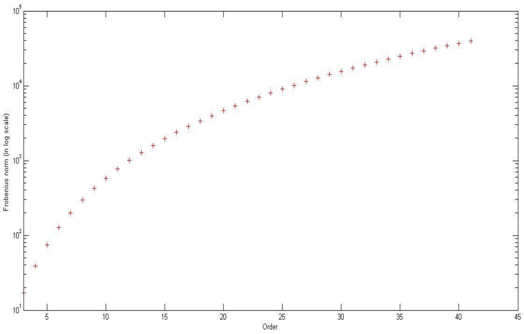

Figure2. Frobenius norm of different order normal magic squares

V. CONDITION NUMBER

(xi) Condition number with 2-norm of any normal magic square of odd order (say, n) lies in the interval $[n, n+0.47]$, except normal magic square of order 3. Condition number with 2 norm for normal magic square of order 3 is 4.3301. [Condition number with 2 norm is defined as the ratio of largest singular value to the smallest].

For example,

Order	Condition number
5	5.4618
9	9.1017
93	93.0081
103	103.0076

VI. MOMENT OF INERTIA

For any normal magic square of order n, moment of inertia (see [6]) is calculated by the formula $I_n = \dfrac{n^2(n^4-1)}{12}$.

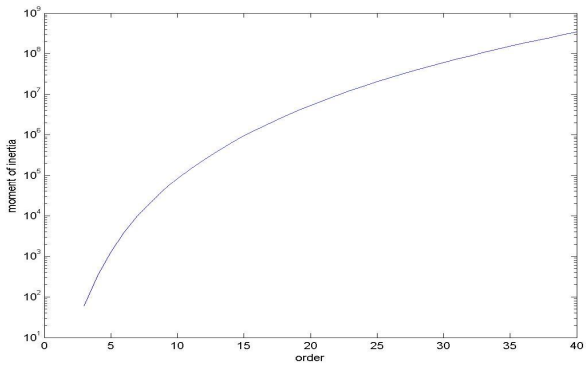

Figure 3. Moment of inertia of different order normal magic squares

VII. SURFACE & CONTOUR PLOTS

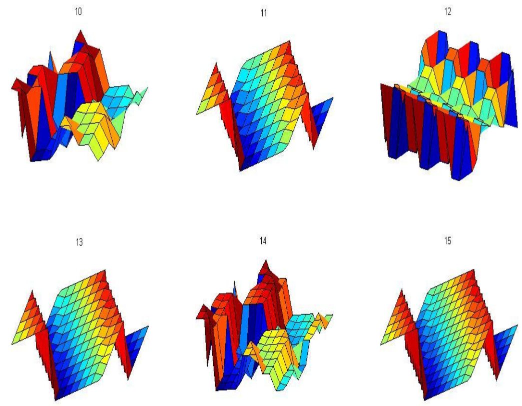

Figure4. Surface plots of different order normal magic squares

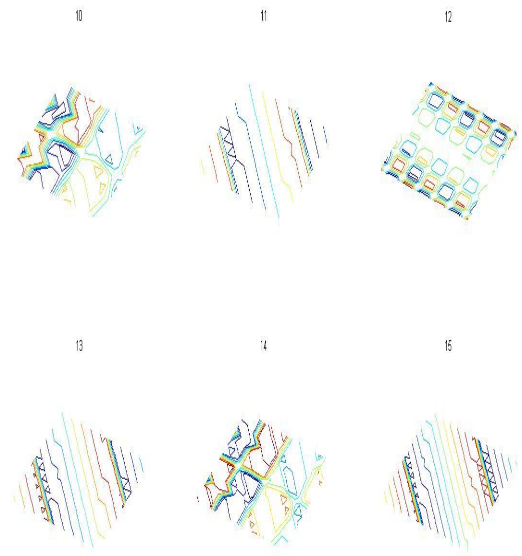

Figure5. Contour plots of different order normal magic squares

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