Subclasses of Analytic Functions Associated with Hyper geometric Functions

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ABSTRACT: *In the present work we find the necessary and sufficient conditions for functions $zF(a,b,c;z)$ to belong to the classes $T_\lambda(A,B)$ and $C_\lambda(A,B)$ and also consequences of these results are studied. We also consider an integral operator related to the hypergeometric function and discuss the convolution properties.*

Key words and Phrases: *Starlike, Convex, Hyper geometric function, Integral operator, Janowski class.*

1. INTRODUCTION

Let A denote the family of analytic functions defined in the open unit disc $U = \{z: |z| < 1\}$ which are of the form

(1.1) $f(z) = z + \sum_{n=0}^{\infty} a_n z^n.$

Let T denote the subclasses of A in U, consisting of analytic functions whose non-zero coefficients from the second term onwards are negative. That is, an analytic function $f \in T$ if it has a Taylor expansion of the form

(1.2) $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$

which are univalent in the open unit disc U.

For $-1 \le A < B \le 1$, $P_1(A,B)$[3] denotes the class of analytic functions in U which are of the form, $\frac{1+Aw(z)}{1+Bw(z)}$ where w is a bounded analytic function satisfying the conditions $w(0) = 0$ *and* $|w(z)| < 1$.

A function $f \in T$ is said to be in the class $T_\lambda(A,B)$ if

$$\frac{zf'(z)}{\lambda zf'(z)+(1-\lambda)f(z)} \prec \frac{1+Aw(z)}{1+Bw(z)}, -1 \le A < B \le 1, \quad z \in U.$$

Let $C_\lambda(A,B)$ denote the class of functions $f \in T$ such that $zf' \in T_\lambda(A,B)$.

For $\lambda = 0$ we get the well- known classes $T^*(A,B)$ and $C(A,B)$ studied by Ganesan [3].

For parametric values $A = 2\alpha - 1$ *and* $B = 1$ we get the classes $T(A,B)$and $C(A,B)$studied by Mostafa [5].

Let $F(a,b,c;z)$ be the hypergeometric function defined by

(1.3) $F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n},$

where, $c \neq 0, -1, -2, \ldots$ and $(a)_n$is the Pochhamer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

$$= \begin{cases} 1, & for\ n = 0 \\ a(a+1)(a+2)\ldots(a+n-1), & for\ n \in \mathbb{N}. \end{cases}$$

It is known that

(1.4) $F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \Re(c-a-b) > 0$

and the function $F(a,b,c;1)$ converges if $\Re(c-a-b) > 0$.

In the next section we obtain the characterization properties for the classes $T_\lambda(A,B)$ and $C_\lambda(A,B)$.

2. MAIN RESULTS

Theorem 2.1. A function $f \in T_\lambda(A,B)$ if and only if

(2.1) $\sum_{n=2}^{\infty}\{n(1+B)-(1+A)[\lambda(n-1)+1]\}a_n \leq B-A.$

Proof: Suppose $f \in T_\lambda(A,B)$. Then

$$\Re\left\{\frac{zf'(z)}{\lambda zf'(z)+(1-\lambda)f(z)}\right\} > \frac{1+A}{1+B}$$

$$\Re\left\{\frac{z-\sum_{n=2}^{\infty}na_nz^n}{z-\sum_{n=2}^{\infty}[\lambda(n-1)+1]a_nz^n}\right\} > \frac{1+A}{1+B}.$$

Letting z→1, then we get,

$$\left[1-\sum_{n=2}^{\infty}na_n\right](1+B) > (1+A)\left[1-\sum_{n=2}^{\infty}[\lambda(n-1)+1]a_n\right].$$

Hence

$$\sum_{n=2}^{\infty}\{n(1+B)-(1+A)[\lambda(n-1)+1]\}a_n \leq B-A.$$

Conversely, if (2.1) holds, it sufficient to show that $|w(z)|<1$. From (1.3), we have

$$|w(z)| = \left|\frac{\sum_{n=2}^{\infty}[(\lambda-1)(n-1)]\,a_nz^n}{(B-A)-\sum_{n=2}^{\infty}[nB-A(\lambda-1-n\lambda)]a_nz^n}\right|$$

$$\leq \frac{\sum_{n=2}^{\infty}[(\lambda-1)(n-1)]\,a_n}{(B-A)-\sum_{n=2}^{\infty}[nB-A(\lambda-1-n\lambda)]a_n}.$$

The last expression is bounded by 1 if

$$\sum_{n=2}^{\infty}[(\lambda-1)(n-1)]\,a_n \leq (B-A)-\sum_{n=2}^{\infty}[nB-A(\lambda-1-n\lambda)]a_n$$

which is equivalent to (2.1). Hence the proof.

Analogous to the above Theorem we get the following result.

Theorem2.2. A function $f \in C_\lambda(A,B)$ if and only if

(2.2) $\sum_{n=2}^{\infty} n\{n(1+B)-(1+A)[\lambda(n-1)+1]\}a_n \leq B-A.$

For $A=2\alpha-1, B=1$ we get the lemma 2.1 in [5].

Theorem 2.3: (i) If $a,b>-1, c>0$ *and* $ab<0$, then $zF(a,b,c;z)$ is in

$T_\lambda(A,B)$ if and only if

(2.3)

$$c > a+b+1-\frac{ab[(1+B)-\lambda(1+A)]}{B-A}.$$

(ii) If $a,b>0$ *and* $c>a+b+1$, then $F_1(a,b,c;z)=z[2-F(a,b,c;z)]$ is in $T_\lambda(A,B)$ if and only if

(2.4) $$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{[(1+B)-\lambda(1+A)]ab}{(B-A)(c-a-b-1)}\right] \leq 2.$$

Proof: (i) Since

$$zF(a,b,c;z) = z-\frac{ab}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n$$

$$= z-|ab/c|\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n.$$

From Theorem 2.1 we have to show that

(2.6) $\sum_{n=2}^{\infty}\{n(1+B)-(1+A)[\lambda(n-1)+1]\}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \left|\frac{c}{ab}\right| \leq (B-A).$

The left hand side of (2.6) diverges if $c>a+b+1$.

$$\sum_{n=2}^{\infty}\{n(1+B)-(1+A)[\lambda(n-1)+1]\}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$=[(1+B)-(1+A)\lambda]\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}+(B-A)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$=[(1+B)-(1+A)\lambda]\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}+\frac{(B-A)c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$=[(1+B)-(1+A)\lambda]\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}+\frac{(B-A)c}{ab}\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right].$$

Hence, (2.6) is equivalent to

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left[[(1+B)-(1+A)\lambda]+\frac{(B-A)(c-a-b-1}{ab}\right]\le(B-A)\left[\left|\frac{c}{ab}\right|+\frac{c}{ab}\right]=0.$$

Thus, (2.7) is valid if and only if

$$[(1+B)-\lambda(1+A)]+\frac{(B-A)(c-a-b-1)}{ab}\le 0.$$

Or equivalently

$$c\ \ge a+b+1-\frac{[(1+B)-(1+A)\lambda]ab}{B-A}.$$

(ii) Since

$$F_1(a,b,c;z)=z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n.$$

From Theorem 2.1, we need only to show that

$$\sum_{n=2}^{\infty}\{n(1+B)-(1+A)[\lambda(n-1)+1]\}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\le B-A.$$

Now

(2.8)

$$\sum_{n=2}^{\infty}\{n(1+B)-(1+A)[\lambda(n-1)+1]\}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

$$=[(1+B)-(1+A)\lambda]\sum_{n=1}^{\infty}n\frac{(a)_n(b)_n}{(c)_n(1)_n}+(B-A)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$=[(1+B)-(1+A)\lambda]\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n-1}}+(B-A)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}.$$

The last expression (2.8) may be expressed as,

$$[(1+B)-(1+A)\lambda]\frac{ab}{c}\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}}+(B-A)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$=[(1+B)-(1+A)\lambda]\frac{ab}{c}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+(B-A)\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\right]$$

$$=[(1+B)-(1+A)\lambda]\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}+(B-A)\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right]$$

$$=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[B-A+\frac{ab[(1+B)-\lambda(1+A)]}{(c-a-b-1)}\right]-(B-A).$$

This last expression is bounded above by $B-A$ if and only if (2.4) holds.

Theorem2.4: (i) If $a,b > -1, ab < 0$ and $c > a + b + 2$, then $zF(a,b,c;z)$ is in $C_\lambda(A,B)$ if and only if

(2.9) $[(1+B)-(1+A)\lambda](a)_2(b)_2 + [3(1+B) - 2\lambda(1+A) - (1+A)]ab(c-b-a-2) + [(1+B)-(1+A)](c-b-a-2)_2 \geq 0.$

(ii) If $a,b > 0$ and $c > a+b+2$, then $F_1(a,b,c;z) = z[2 - F(a,b,c;z)]$ is in $C_\lambda(A,B)$

(2.10)

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left\{1+\frac{[(1+B)-\lambda(1+A)](a)_2(b)_2}{[(1+B)-(1+A)](c-a-b-2)_2}+\left(\frac{[3(1+B)-2\lambda(1+A)-(1+A)]}{(1+B)-(1+A)}\right)\left(\frac{ab}{c-a-b-1}\right)\right\}\leq 2.$$

Proof. (i) Since $zF(a,b,c;z)$ has the form (2.5), from Theorem 2.2 that our conclusion is equivalent to

(2.11)

$$\sum_{n=2}^{\infty} n\{n(1+B)-(1+A)[\lambda(n-1)+1]\}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{(n-1)}} \leq \left|\frac{c}{ab}\right| B - A.$$

Note that for $c > a+b+2$, the left side of (2.11), converges. Writing

$$(n+2)\{(n+2)[(1+B)-\lambda(1+A)] - (1+A)(1-\lambda)\} = (n+1)^2[(1+B)-\lambda(1+A)] + (n+1)[2(1+B)-(1+A)-\lambda(1+A)] + [(1+B)-(1+A)],$$

we see that

$$\sum_{n=0}^{\infty}(n+2)\{(n+2)[(1+B)-\lambda(1+A)] - (1+A)(1-\lambda)\}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$= [(1+B)-\lambda(1+A)]\sum_{n=0}^{\infty}(n+1)^2\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + [2(1+B)-(1+A)-\lambda(1+A)]\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + (B-A)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$= [(1+B)-\lambda(1+A)]\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + [2(1+B)-(1+A)-\lambda(1+A)]\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (B-A)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$= [(1+B)-\lambda(1+A)]\sum_{n=0}^{\infty}n\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + [3(1+B)-(1+A)-2\lambda(1+A)]\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (B-A)\sum_{n=0}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_n}$$

$$= \frac{[(1+B)-\lambda(1+A)](a+1)(b+1)}{(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n} + [3(1+B)-(1+A)-2\lambda(1+A)]\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (B-A)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$= \frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\Bigg[[(\mathbf{1}+\boldsymbol{B}) - \boldsymbol{\lambda}(\mathbf{1}+\boldsymbol{A})](\boldsymbol{a}+\mathbf{1})(\boldsymbol{b}+\mathbf{1}) + [\mathbf{3}(\mathbf{1}+\boldsymbol{B}) - (\mathbf{1}+\boldsymbol{A}) - \mathbf{2}\boldsymbol{\lambda}(\mathbf{1}+\boldsymbol{A})](\boldsymbol{c}-\boldsymbol{a}-\boldsymbol{b}-\mathbf{2}) + \frac{(\boldsymbol{B}-\boldsymbol{A})}{\boldsymbol{ab}}(\boldsymbol{c}-\boldsymbol{a}-\boldsymbol{b}-\mathbf{2})_{\mathbf{2}}\Bigg] - \frac{(\boldsymbol{B}-\boldsymbol{A})\boldsymbol{c}}{\boldsymbol{ab}}.$$

This last expression is bounded above by $\left|\frac{c}{ab}\right|(B-A)$ if and only if

$$[(\mathbf{1}+\boldsymbol{B}) - \boldsymbol{\lambda}(\mathbf{1}+\boldsymbol{A})](\boldsymbol{a}+\mathbf{1})(\boldsymbol{b}+\mathbf{1}) + [\mathbf{3}(\mathbf{1}+\boldsymbol{B}) - (\mathbf{1}+\boldsymbol{A}) - \mathbf{2}\boldsymbol{\lambda}(\mathbf{1}+\boldsymbol{A})](\boldsymbol{c}-\boldsymbol{a}-\boldsymbol{b}-\mathbf{2}) + \frac{(\boldsymbol{B}-\boldsymbol{A})}{\boldsymbol{ab}}(\boldsymbol{c}-\boldsymbol{a}-\boldsymbol{b}-\mathbf{2})_{\mathbf{2}} \le 0$$

which is equivalent to (2.9).

(ii) From Theorem 2.2 it is enough to show that

$$\sum_{n=2}^{\infty} n\{[\boldsymbol{n}(\mathbf{1}+\boldsymbol{B}) - (\mathbf{1}+\boldsymbol{A})[\boldsymbol{\lambda}(\boldsymbol{n}-\mathbf{1}) + \mathbf{1}]]\}\frac{(\boldsymbol{a})_{\boldsymbol{n}-\mathbf{1}}(\boldsymbol{b})_{\boldsymbol{n}-\mathbf{1}}}{(\boldsymbol{c})_{\boldsymbol{n}-\mathbf{1}}(\mathbf{1})_{\boldsymbol{n}-\mathbf{1}}} \le (\boldsymbol{B}-\boldsymbol{A}).$$

Now

(2.12) $\sum_{n=2}^{\infty} n\{[\boldsymbol{n}(\mathbf{1}+\boldsymbol{B}) - (\mathbf{1}+\boldsymbol{A})[\boldsymbol{\lambda}(\boldsymbol{n}-\mathbf{1}) + \mathbf{1}]]\}\frac{(\boldsymbol{a})_{\boldsymbol{n}-\mathbf{1}}(\boldsymbol{b})_{\boldsymbol{n}-\mathbf{1}}}{(\boldsymbol{c})_{\boldsymbol{n}-\mathbf{1}}(\mathbf{1})_{\boldsymbol{n}-\mathbf{1}}}$

$$= \sum_{\boldsymbol{n}=\mathbf{0}}^{\infty}(\boldsymbol{n}+\mathbf{2})\{(\boldsymbol{n}+\mathbf{2})[(\mathbf{1}+\boldsymbol{B}) - \boldsymbol{\lambda}(\mathbf{1}+\boldsymbol{A})] - (\mathbf{1}+\boldsymbol{A})(\mathbf{1}-\boldsymbol{\lambda})\}\frac{(\boldsymbol{a})_{\boldsymbol{n}+\mathbf{1}}(\boldsymbol{b})_{\boldsymbol{n}+\mathbf{1}}}{(\boldsymbol{c})_{\boldsymbol{n}+\mathbf{1}}(\mathbf{1})_{\boldsymbol{n}+\mathbf{1}}}$$

$$= [(\mathbf{1}+\boldsymbol{B}) - \boldsymbol{\lambda}(\mathbf{1}+\boldsymbol{A})]\sum_{\boldsymbol{n}=\mathbf{0}}^{\infty}(\boldsymbol{n}+\mathbf{1})^{\mathbf{2}}\frac{(\boldsymbol{a})_{\boldsymbol{n}+\mathbf{1}}(\boldsymbol{b})_{\boldsymbol{n}+\mathbf{1}}}{(\boldsymbol{c})_{\boldsymbol{n}+\mathbf{1}}(\mathbf{1})_{\boldsymbol{n}+\mathbf{1}}} - (\mathbf{1}+\boldsymbol{A})(\mathbf{1}-\boldsymbol{\lambda})\sum_{\boldsymbol{n}=\mathbf{0}}^{\infty}(\boldsymbol{n}+\mathbf{2})\frac{(\boldsymbol{a})_{\boldsymbol{n}+\mathbf{1}}(\boldsymbol{b})_{\boldsymbol{n}+\mathbf{1}}}{(\boldsymbol{c})_{\boldsymbol{n}+\mathbf{1}}(\mathbf{1})_{\boldsymbol{n}+\mathbf{1}}}.$$

Writing $(n+2) = (n+1) + 1$, we have

(2.13) $\sum_{n=0}^{\infty}(n+2)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} = \sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} + \sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$

$$= \sum_{n=0}^{\infty}\frac{(\boldsymbol{a})_{\boldsymbol{n}+\mathbf{1}}(\boldsymbol{b})_{\boldsymbol{n}+\mathbf{1}}}{(\boldsymbol{c})_{\boldsymbol{n}+\mathbf{1}}(\mathbf{1})_{\boldsymbol{n}}} + \sum_{n=0}^{\infty}\frac{(\boldsymbol{a})_{\boldsymbol{n}+\mathbf{1}}(\boldsymbol{b})_{\boldsymbol{n}+\mathbf{1}}}{(\boldsymbol{c})_{\boldsymbol{n}+\mathbf{1}}(\mathbf{1})_{\boldsymbol{n}+\mathbf{1}}}$$

and

(2.14) $\sum_{n=0}^{\infty}(n+2)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$

$$= \sum_{n=0}^{\infty}(n+1)\frac{(\boldsymbol{a})_{\boldsymbol{n}+\mathbf{1}}(\boldsymbol{b})_{\boldsymbol{n}+\mathbf{1}}}{(\boldsymbol{c})_{\boldsymbol{n}+\mathbf{1}}(\mathbf{1})_{\boldsymbol{n}}} + 2\sum_{n=0}^{\infty}\frac{(\boldsymbol{a})_{\boldsymbol{n}+\mathbf{1}}(\boldsymbol{b})_{\boldsymbol{n}+\mathbf{1}}}{(\boldsymbol{c})_{\boldsymbol{n}+\mathbf{1}}(\mathbf{1})_{\boldsymbol{n}}} + \sum_{n=0}^{\infty}\frac{(\boldsymbol{a})_{\boldsymbol{n}+\mathbf{1}}(\boldsymbol{b})_{\boldsymbol{n}+\mathbf{1}}}{(\boldsymbol{c})_{\boldsymbol{n}+\mathbf{1}}(\mathbf{1})_{\boldsymbol{n}+\mathbf{1}}}$$

$$= \sum_{n=1}^{\infty}\frac{(\boldsymbol{a})_{\boldsymbol{n}+\mathbf{1}}(\boldsymbol{b})_{\boldsymbol{n}+\mathbf{1}}}{(\boldsymbol{c})_{\boldsymbol{n}+\mathbf{1}}(\mathbf{1})_{\boldsymbol{n}-\mathbf{1}}} + \mathbf{3}\sum_{n=0}^{\infty}\frac{(\boldsymbol{a})_{\boldsymbol{n}+\mathbf{1}}(\boldsymbol{b})_{\boldsymbol{n}+\mathbf{1}}}{(\boldsymbol{c})_{\boldsymbol{n}+\mathbf{1}}(\mathbf{1})_{\boldsymbol{n}}} + \sum_{n=1}^{\infty}\frac{(\boldsymbol{a})_{\boldsymbol{n}}(\boldsymbol{b})_{\boldsymbol{n}}}{(\boldsymbol{c})_{\boldsymbol{n}}(\mathbf{1})_{\boldsymbol{n}}}.$$

Substituting (2.13) and (2.14) into the right hand side of (2.12), we get

(2.15)

$$[(\mathbf{1}+\boldsymbol{B}) - \boldsymbol{\lambda}(\mathbf{1}+\boldsymbol{A})]\sum_{n=0}^{\infty}\frac{(\boldsymbol{a})_{\boldsymbol{n}+\mathbf{2}}(\boldsymbol{b})_{\boldsymbol{n}+\mathbf{2}}}{(\boldsymbol{c})_{\boldsymbol{n}+\mathbf{2}}(\mathbf{1})_{\boldsymbol{n}}} + [\mathbf{3}(\mathbf{1}+\boldsymbol{B}) - (\mathbf{1}+\boldsymbol{A}) - \mathbf{2}\boldsymbol{\lambda}(\mathbf{1}+\boldsymbol{A})]\sum_{n=0}^{\infty}\frac{(\boldsymbol{a})_{\boldsymbol{n}+\mathbf{1}}(\boldsymbol{b})_{\boldsymbol{n}+\mathbf{1}}}{(\boldsymbol{c})_{\boldsymbol{n}+\mathbf{1}}(\mathbf{1})_{\boldsymbol{n}}} + (B-A)\sum_{n=1}^{\infty}\frac{(\boldsymbol{a})_{\boldsymbol{n}}(\boldsymbol{b})_{\boldsymbol{n}}}{(\boldsymbol{c})_{\boldsymbol{n}}(\mathbf{1})_{\boldsymbol{n}}}.$$

Since $(a)_{n+k} = (a)_k(a+k)$, we may write (2.15) as

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{[(1+B)-\lambda(1+A)](a)_2(b)_2}{(c-a-b-2)_2} + \frac{[3(1+B)-(1+A)-2\lambda(1+A)]ab}{(c-a-b-1)} + (B-A) - (B-A)\right].$$

On Simplification, we see that the last expression is bounded above by $(B-A)$ if and only if (2.10) holds.

In the following theorem, we obtain similar results in connection with a particular integral operator $G(a,b,c;z)$ acting on $F(a,b,c;z)$ as follows:

(2.16) $$G(a,\ b,\ c;\ z) = \int_0^z F(a,b,c;t)dt.$$

Theorem2.5: Let $a,b > -1, ab < 0$ and $c > max\{0, a+b\}$. Then $G(a,b,c;z)$ defined by (2.16) is in $T_\lambda(A,B)$ if and only if

2.17 $$\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(a-1)\Gamma(c-b)}\left[\frac{[(1+B)-\lambda(1+A)]}{ab} - \frac{(1+A)(1-\lambda)(c-a-b)}{(a-1)_2(b-1)_2}\right] + \frac{(1+A)(1-\lambda)(c-1)_2}{(a-1)_2(b-1)_2} \le 0.$$

Proof: Since

$$G(a,b,c;z) = z - \left|\frac{ab}{c}\right|\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}z^n,$$

From Theorem 2.1 it is enough to show that

$$\sum_{n=2}^{\infty}\{n(1+B)-(1+A)[\lambda(n-1)+n]\}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n} \le \left|\frac{c}{ab}\right|(B-A).$$

Now

$$\sum_{n=2}^{\infty}\{n[(1+B)-\lambda(1+A)] - (1+A)(1-\lambda)\}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}$$

$$=[(1+B)-\lambda(1+A)]\sum_{n=0}^{\infty}(n+2)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}} - (1+A)(1-\lambda)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}}$$

$$=[(1+B)-\lambda(1+A)]\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} - (1+A)(1-\lambda)\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n+1}}$$

$$=[(1+B)-\lambda(1+A)]\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_n} - (1+A)(1-\lambda)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}$$

$$=[(1+B)-\lambda(1+A)]\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} - (1+A)(1-\lambda)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}$$

$$=[(1+B)-\lambda(1+A)]\frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\right] - (1+A)(1-\lambda)\frac{(c-1)_2}{(a-1)_2(b-1)_2}\sum_{n=2}^{\infty}\frac{(a-1)_n(b-1)_n}{(c-1)_n(1)_n}$$

$$\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{[(1+B)-\lambda(1+A)]}{ab} - \frac{(1+A)(1-\lambda)(c-a-b)}{(a-1)_2(b-1)_2}\right] + \frac{(1+A)(1-\lambda)(c-1)_2}{(a-1)_2(b-1)_2} - \frac{(B-A)c}{ab},$$

Which is bounded above by $\left|\frac{c}{ab}\right|(B-A)$ if and only if (2.17) holds.

Analogously we obtain the following result.

Theorem 2.6: Let $a,b > -1, ab < 0$ and $c > a+b+2$. Then $G(a,b,c;z)$ defined by (2.16) is in $C_\lambda(A,B)$ if and only if

$$c > a+b+1 - \frac{[(1+B)-(1+A)\lambda]ab}{B-A}.$$

In the next theorem we investigate the properties of the convolution of functions from the class $T_\lambda(A,B)$.

Theorem 2.7: If f is of the form (1.2) and $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ belongs to the $T_\lambda(A,B)$ then $h(z) = (f\cdot g) = z - \sum_{n=2}^{\infty} a_n b_n z^n$ will be an element of $T_\lambda(A_1,B_1)$ with

$-1 \le A_1 < B_1 \le 1$, $where\ B_1 \ge \frac{A_1+\delta}{1-\delta}$, $A_1 \le 1-2\delta$, where

$$\boldsymbol{\delta} = \frac{(\mathbf{1}+\lambda)(\boldsymbol{B}-\boldsymbol{A})^{\mathbf{2}}}{[\mathbf{2}(\mathbf{1}+\boldsymbol{B})-(\mathbf{1}+\boldsymbol{A})(\mathbf{1}+\lambda)]-(\mathbf{1}+\lambda)(\boldsymbol{B}-\boldsymbol{A})^{\mathbf{2}}}.$$

Proof: From (2.1), we have

(2.18) $\sum_{n=2}^{\infty}\frac{\{n(1+B)-(1+A)[\lambda(n-1)+1]\}}{B-A}\,a_n \le 1$

And

(2.19) $\sum_{n=2}^{\infty}\frac{\{n(1+B)-(1+A)[\lambda(n-1)+1]\}}{B-A}\,b_n \le 1.$

In view of Cauchy-Schwarz inequality, from (2.18) and (2.19) we obtain

(2.20 $\sum_{n=2}^{\infty} u\sqrt{a_n b_n} \le 1$

$where\ u = \frac{n(1+B)-(1+A)[\lambda(n-1)+1]}{B-A}.$

We need to find $A_1\ and\ B_1$ such that $h(z) = f\cdot g \in T_\lambda(A_1, B_1)$, or equivalently

(2.21) $\sum_{n=2}^{\infty} u_1 a_n b_n \le 1$

$where\ u = \frac{n(1+B_1)-(1+A_1)[\lambda(n-1)+1]}{B_1-A_1}.$

The inequality (2.21) will be true if $u_1 a_n b_n \le u\sqrt{a_n b_n}$ or

(2.22) $\sqrt{a_n b_n} \le \frac{u}{u_1}.$

But from (2.20) we get $\sqrt{a_n b_n} \le \frac{1}{u}$. Thus (2.22) will be true if $u_1 \le u^2$, that is

$$\frac{n(1+B_1)-(1+A_1)[\lambda(n-1)+1]}{B_1-A_1} \le u^2.$$

This yield,

(2.23) $A_1 \le \frac{u^2 B_1+[\lambda(n-1)+1]-n(1+B_1)}{u^2-[\lambda(n-1)+1]}.$

It is easy to verify $u^2 > 1$ for $n \ge 2$. Now on simple computation of (2.23) yields

(2.24) $\frac{B_1-A_1}{1+B_1} \ge \frac{n-[\lambda(n-1)+1]}{u^2-[\lambda(n-1)+1]}, for\ n \ge 2.$

The right hand member decreases as n increases and is maximum for n=2 therefore

(2.25) $\frac{B_1-A_1}{1+B_1} \ge \frac{(\mathbf{1}+\lambda)(\boldsymbol{B}-\boldsymbol{A})^{\mathbf{2}}}{[\mathbf{2}(\mathbf{1}+\boldsymbol{B})-(\mathbf{1}+\boldsymbol{A})(\mathbf{1}+\lambda)]-(\mathbf{1}+\lambda)(\boldsymbol{B}-\boldsymbol{A})^{\mathbf{2}}} = \boldsymbol{\delta}.$

Obviously, $\boldsymbol{\delta} < 1$ and fixing A_1 *in* (2.25) we get

(2.26) $B_1 \ge \frac{\boldsymbol{\delta}+A_1}{1-\boldsymbol{\delta}}.$

For parametric value $\lambda = 0$ we get Theorem (3.2.1) studied by Ganesan[3].

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