An Improved Predictor-Corrector Method for Delay Differential Equations of Fractional Order

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Abstract *This article provides an analysis for the delay fractional differential equations in Caputo sense by an introduction of an improved predictor-corrector formula. The delay term is expressed either as a constant or time varying. The implication of this new approach is used to improvise the algorithm. A vivid description of the convergence and detailed error analysis of the improved predictor-corrector method is clearly presented. The efficiency of the proposed method is highlighted with numerical examples.*

Keywords — *Fractional derivative, Delay differential equations, Predictor-Corrector algorithm.*

I. INTRODUCTION

In this present scenario, various branches of mathematics hold an important place in the fields like science, engineering and technology. Amongst them fractional differential equations (FDE) is found to be highly imperative. Simulations recently developed in the areas like viscoelasticity, rheology, diffusion process, etc. takes its expression in the form of fractional derivatives or fractional calculus [24]. It is evident that most of the non-linear FDE cannot be solved exactly because of its non-local nature; hence numerical can be used [16]. The Adams-Basforth- Moulton method is generally used to solve non-linear FDE in numerical approach which has been initiated by Diethelm et al. [9]. The chaotic behaviour of fractional order systems have been successfully determined by implying this algorithm. The FDE has a vivid description with a detailed error analysis, accuracy and the effective numerical approach [10].

The deliberate applications of Delay differential equations(DDE) is clearly observed in many practical systems such as automatic control, lasers, traffic models, metal cuttings, neuroscience and so on [5, 12]. Science and engineering takes DDE in its areas of applications with respect to time delay. Our goal of this paper is to improve the predictor-corrector method for delay FDE.

This paper is prepared as follows. In Section 2, we review basic concepts and give the algorithm of Adams-Bashforth-Moulton method. In Section 3, we derive the improved predictor-corrector schemes for delay FDE, in section 4; the detailed error analysis and convergence are also discussed. In Section 5, the suggest numerical method is exemplified.

II. BASIC CONCEPTS

Definition 2.1. A real function $f(t), t>0$, is said to be in the space $C_\mu, \mu \in R$ if there exist a real number $p>\mu$, such that $f(t)=t^{p} f_1(t)$, where $f_1(t) \in C(0,\infty)$, and it is said to be in the space C_μ^η if and only if $f^{(n)} \in C_\mu, n \in N$.

Definition 2.2. The Riemann-Lioville fractional integral operator I_a^α of order $\alpha \geq 0$, with $a \geq 0$ of a function $f \in C_\mu, \mu \geq -1$ is defind as

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t>0 \qquad (1)$$

$$I_a^0 f(t) = f(t) \text{ for } \alpha = 0. \qquad (2)$$

$\Gamma(z)$ is the well known Gamma function. Some of the properties of the operator I^α, which we will need here are in below:

For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma \geq -1$.

(a) $I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$

(b) $I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t)$

(c) $I^\alpha t^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$

Definition 2.3. The fractional derivative $({}^c D_a^\alpha)$ of $f(t)$ in the Caputo sense is defined as

$${}^c D_a^\alpha = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds \qquad (3)$$

for $n-1<\alpha<n, n \in N, t \geq a, f \in C_{-1}^n$.

The following are two basic properties of the Caputo's fractional derivatives [16];

(a) Let $f \in C_{-1}^{n}, n \in N$. Then ${}^{c}D_{a}^{\alpha}$; $0 \le \alpha \le n$, is well defined and ${}^{c}D_{a}^{\alpha} f \in C_{-1}$.

(b) Let $n-1 < \alpha \le n, n \in N$ and $f \in C_{\mu}^{n}, \mu \ge -1$. Then

$$I_{a}^{\alpha}({}^{c}D_{a}^{\alpha}) f(t) = f(t) - \sum_{k=0}^{n-1} f^{k}(a) \frac{(t-a)^{k}}{k!} \quad (4)$$

2.2 Adams-Bashforth-Moulton method:

The recall of the basic ideas of the one-step Adams-Bashforth-Moulton algorithm for the ordinary differential equation (ODE) enhances to introduce the new algorithm.

We consider IVP,

$$y'(t) = f(t, y(t)) \quad (5)$$

$$y(0) = y_0 \quad (6)$$

On considering $t \in [0,T]$, the uniform grid $t_j = jh$, $(j = 0,1,2..N)$ and $h = \frac{T}{N}$ is time step. Assume the calculated approximation $y_j \approx y(t_j)$, $(j = 0,1,2..N)$, then the equations (5)and (6) are equivalent to

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(s, y(s))ds \quad (7)$$

By applying trapezoidal quadrature formulae for replacing right-hand side integral in (7),

$$\int_{t_n}^{t_{n+1}} f(s, y(s))ds \approx \frac{h}{2}(f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))) \quad (8)$$

thus, we can get the approximation to y_{n+1} as follows:

$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] \quad (9)$$

In the above equation (9), in order to compute the value of y_{n+1}, it is required to have y_{n+1} on the right hand side. For this purpose, the Euler's method is used to calculate the value of y_{n+1}, which is denoted by $y_{n+1,p}$. Now consider the forward Euler's formula

$$y_{n+,p} = y_n + hf(t_n, y_n) \quad (10)$$

Equation (8) can be written as

$$y_{n+1,c} = y_n + \frac{h}{2}[f(t_{n,}, y_n) + f(t_{n+1}, y_{n+1}{}^{p})]. \quad (11)$$

The convergence of this algorithm is,

$$\max_{j=0,1,2...N} | y(t_j) - y_j | = O(h^2) \quad (12)$$

III. An Improved Predictor-Corrector Scheme for Delay FDE

We consider delay FDE defined by

$$D_t^{\alpha} y(t) = f(t, y(t), y(t-\tau)), \quad (13)$$
$$t \ge 0, m-1 \le \alpha \le m$$

$$y(t) = g(t), t \le 0 \quad (14)$$

where, the approximation to the delay term $y(t-\tau)$ which consist following two types.

Type I: (when τ is constant)

It is clearly evident that $t_{j-\tau}$ may not be a grid point t_n for any n, if τ is any positive constant. Suppose that $(m-\xi)h = \tau$ and $0 \le \xi < 1$. Taking $\xi = 0, y(t_n - \tau)$ can be approximated by

$$y(t_n - \tau) \approx \begin{cases} y_{n-m}, n > m, \\ g_n, n \le m, \end{cases} \quad (15)$$

It is also found that, $y(t_n - \tau)$ cannot be directly calculated when $0 < \xi < 1$.

Let ϖ_{n+1} be the approximation to $y(t_{n+1} - \tau)$ for the case $(m-1)h < \tau < mh$. On interpolating it by the two nearest points, that is,

$$\varpi_{n+1} = \xi y_{n+2-m} + (1-\xi) y_{n+1-m} \quad (16)$$

Equation (16) implies the implicit of the numerical equation if $m > 1$which can be directly determined. It is observed that if $m = 1$ and $\xi \ne 0$, that is, $\tau < h$ the first term on the right-hand side of the above equation is ξy_{n+1} . Further prediction is required in this case.

$$\varpi_{n+1} = \xi y_{n+1,p} + (1-\xi) y_n \quad (17)$$

Type II: (when τ is time varying)

If $T = \tau(t)$ the approximation seems to be tedious. Let $\varpi_{n+1} \approx y(t_{n+1} - \tau)$. In order to approximate the delay term, the linear interpolation of y_j at point $t = t_{n+1} - \tau(t_{n+1})$ is implicated. Let $\tau(t_{n+1}) = (m_{n+1} - \xi_{n+1})h$, where $m_{n+1} \in Z^{+}$ and $\xi_{n+1} \in [0,1)$, then

$$\varpi_{n+1} = \xi_{n+1} y_{n+2-m_{n+1}} + (1-\xi_{n+1}) y_{n+1-m_{n+1}} \quad (18)$$

In this case when τ is constant, for given h and τ, it can be inferred if $m = 1$ or $m > 1$ holds at the initial start of the program. But in this case, when τ is time varying, m is also time varying; it is inferred that at one moment it is equal to 1, and at another moment it may greater than 1. Further prediction is required if $m_{n+1}=1$ in the first term in the right-hand side of (18) and it is not needed if $m_{n+1}>1$. Hence in each step of the computational procedure, a condition $m_n=1$ or not is initially checked. This inference helps out for further prediction or not.

Now we derive the numerical algorithm for the delay FDE: (13)-(14).

We know that the delay IVP (13)-(14) is equivalent to Volterra integral equation [9]:

$$y(t)=\sum_{k=0}^{m-1} g(t)\frac{t^k}{k!}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s),y(s-\tau))ds \tag{19}$$

Now it suffices to compute the integral term in (19). The integrant on right-hand side of (19) is modified by the use of product trapezoidal quadrature formula, in which the nodes t_j, $(j=0,1,..n+1)$ are considered with respect to the weight function $(t_{n+1}-\cdot)^{\alpha-1}$. That is, we get the approximation

$$\int_0^{t_{n+1}} (t_{k+1}-z)^{\alpha-1} g(z)dz \approx \int_0^{t_{n+1}} (t_{n+1}-z)^{\alpha-1} \bar{g}_{n+1}(z)dz \tag{20}$$

Here $\tilde{g}_{n+1}(\cdot)$ is the piecewise linear interpolation for $g(.)$ with nodes and knots chosen at t_j, $(j=0,1,..n+1)$. The integral on the right-hand side of (20) can be written by the use of the standard technique of quadrature theory

$$\int_0^{t_{n+1}} (t_{n+1}-z)^{\alpha-1} \tilde{g}_{n+1}(z)dz=\frac{h^\alpha}{\alpha(\alpha+1)}\sum_{j=0}^{n+1} a_{j,n+1} g(t_j) \tag{21}$$

Where,

$$a_{j,n+1}=\begin{cases} n^{\alpha+1}-(n+1)^\alpha(n-\alpha),\ j=0 \\ (n+2-j)^{\alpha+1}-2(n-j+1)^{\alpha+1}+(n-j)^{\alpha+1}, 1\le j\le n, \\ 1,\ j=n+1 \end{cases} \tag{22}$$

Therefore, the numerical scheme for FDE (13)-(14) can be formulated as:

$$y_{n+1,c}=\begin{cases} g(0)+\dfrac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1},y_{n+1,p},\varpi_{n+1}) \\ +\dfrac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{j,n+1} f(t_j,y_j,\varpi_j), \end{cases} \tag{23}$$

$$\varpi_{n+1}=\begin{cases} \xi y_{n+2-m}+(1-\xi)y_{n+1-m}, m>1 \\ \xi y_{n+1,p}+(1-\xi)y_n, m=1 \end{cases} \tag{24}$$

$$y_{n+1,p}=g(0)+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n} b_{j,n+1} f(t_j,y_j,\varpi_j) \tag{25}$$

Where

$$b_{j,n+1}=\frac{h^\alpha}{\alpha}((n+1-j)^\alpha-(n-j)^\alpha). \tag{26}$$

Now we make some improvement for the scheme (23)-(25). We modify the approximation of (20) as,

$$\int_0^{t_{n+1}} (t_{k+1}-z)^{\alpha-1} g(z)dz \approx \int_0^{t_n} (t_{k+1}-z)^{\alpha-1} \tilde{g}_n(z)dz + \int_{t_n}^{t_{n+1}} (t_{k+1}-z)^{\alpha-1} g(t_n)dz \tag{27}$$

On choosing the nodes and knots at t_j, $(j=0,1,..n)$, $\tilde{g}_n$ gives the piecewise linear interpolation of g. The right-hand side of (20) gives

$$\int_0^{t_{n+1}} (t_{k+1}-z)^{\alpha+1} \tilde{g}_n(z)dz+\int_{t_n}^{t_{n+1}} (t_{n+1}-z)^{\alpha-1} g(t_n)dz$$

$$=\frac{h^\alpha}{\alpha(\alpha+1)}\sum_{j=0}^{n} \tilde{b}_{j,n+1} g(t_j), \tag{28}$$

Where

$$\tilde{b}_{j,n+1}=\begin{cases} \begin{cases} a_{j,n+1}, 0\le j\le n-1 \\ 2^{\alpha+1}-1, j=n \end{cases} \quad n>0 \\ b_{0,1}=\alpha+1, n=0 \end{cases}$$

Hence, this procedure for the predictor step can be improved as [7]

$$y_{n+1,p}=g(0)+\frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n} \tilde{b}_{j,n+1} f(t_j,y_j,\varpi_j). \tag{29}$$

The **new predictor corrector approach** (29) and (23) has the numerical accuracy $O(h^{\min\{2,1+2\alpha\}})$ (the detailed analysis is given in section 4). Obviously half of the computational cost can be reduced, for $0\le\alpha\le1$, if we modify (29) and (23) as

$$y_{n+1,p}=\begin{cases} g(0)+\dfrac{h^\alpha}{\Gamma(\alpha+1)} f(t_0,y_0,\varpi_0), n=0 \\ g(0)+\dfrac{h^\alpha}{\Gamma(\alpha+2)}(2^{\alpha+1}-1) f(t_n,y_n,\varpi_n) \\ +\dfrac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n-1} a_{j,n+1} f(t_j,y_j,\varpi_j), n\ge1 \end{cases}$$

(30)

And

$$y_{n+1,c}=\begin{cases} g(0)+\dfrac{h^{\alpha}}{\Gamma(\alpha+2)}(f(t_1,y_1,\varpi_1)+\alpha f(t_0,y_0,\varpi_0)), n=0 \\ g(0)+\dfrac{h^{\alpha}}{\Gamma(\alpha+2)}(f(t_{n+1},y_{n+1,p},\varpi_{n+1})+(2^{\alpha}-2)f(t_n,y_n,\varpi_n)) \\ +\dfrac{h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j=0}^{n-1}a_{j,n+1}f(t_j,y_j,\varpi_j), n\geq 1. \end{cases} \tag{31}$$

IV. Detailed Error Analysis

Let us consider the following lemma for giving proof of theorem.

Lemma 4.1. [10]

(a) Let $z\in C^1[0,1]$, then

$$\left|\int_0^{n+1}(t_{n+1}-t)^{\alpha-1}z(t)dt-\sum_{j=0}^{n}b_{j,n+1}z(t_j)\right|\leq\frac{1}{\alpha}\|z'\|_{\infty}t^{\alpha}{}_{n+1}h^1. \tag{32}$$

(b) Let $z\in C^2[0,1]$, then

$$\left|\int_0^{n+1}(t_{n+1}-t)^{\alpha-1}z(t)dt-\sum_{j=0}^{n}a_{j,n+1}z(t_j)\right|\leq C_{\alpha}^{Tr}\|z'\|_{\infty}t^{\alpha}{}_{n+1}h^2. \tag{33}$$

Here, we assume that $f(\cdot)$ in (13) satisfies the following Lipschitz conditions with respect to its variables as follows:

$$|f(t,y_1,v)-f(t,y_2,v)|\leq K_1|y_1-y_2|, \tag{34}$$

$$|f(t,y,v_1)-f(t,y,v_2)|\leq K_2|v_1-v_2|, \tag{35}$$

Where L_1, L_2 are positive constants.

Theorem 4.1. Assume that the solution y of IVP is such that

$$\left|\int_0^{n+1}(t_{n+1}-t)^{\alpha-1}D_t^{\alpha}y(t)dt-\sum_{j=0}^{n}b_{j,n+1}D_t^{\alpha}y(t_j)\right|\leq Ct_{n+1}{}^{\gamma_1}h^{\delta_2}. \tag{36}$$

$$\left|\int_0^{n+1}(t_{n+1}-t)^{\alpha-1}D_t^{\alpha}y(t)dt-\sum_{j=0}^{n}a_{j,n+1}D_t^{\alpha}y(t_j)\right|\leq Ct_{n+1}{}^{\gamma_2}h^{\delta_2}. \tag{37}$$

with some $\gamma_1,\gamma_2\geq 0$ and $\delta_1,\delta_2>0$. Then, for some suitable T > 0, we have

$$\max_{0\leq j\leq N}|y(t_j)-y_j|\leq Ch^q, \tag{38}$$

Where $q=\min\{\delta_1+\alpha,\delta_2\}, N=[T/h]$, and C is a positive constant.

Proof. The detailed proof is discussed in [27].

V. Illustrative Examples

Example 5.1. Consider a fractional order version of the DDE given in [26]

$$D_t^{0.99}y(t)=\frac{2y(t-3)}{1+y(t-3)^{9.65}}-y(t) \qquad y(t)=0.5, t\leq 0. \tag{39}$$

Assuming the step size as h=0.01 in this example and on approximation of the improved predictor-corrector method (30)-(31) for the above system,

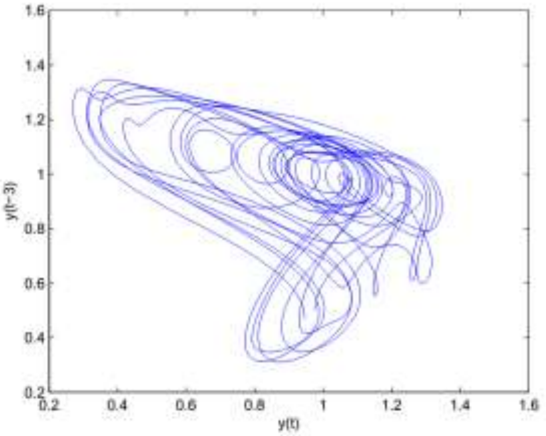

Fig.1. $y(t)$ versus $y(t-3)$

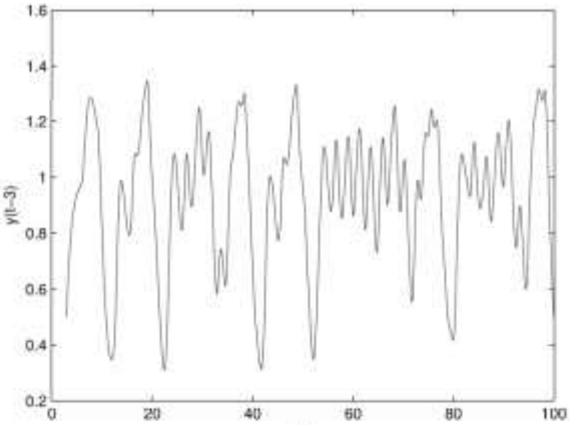

Fig.2. Solution of the equation

approximate solution is obtained which is depicted in Figures 1 and 2.

It may be analysed from these figures that the system shows chaotic behaviour.

Example 5.2. In this example we consider the fractional order version of the DDE

$$D_t^{0.97}y(t)=2y(1-y(t-1)), \qquad y(t)=0.5, -1\leq t\leq 0. \tag{40}$$

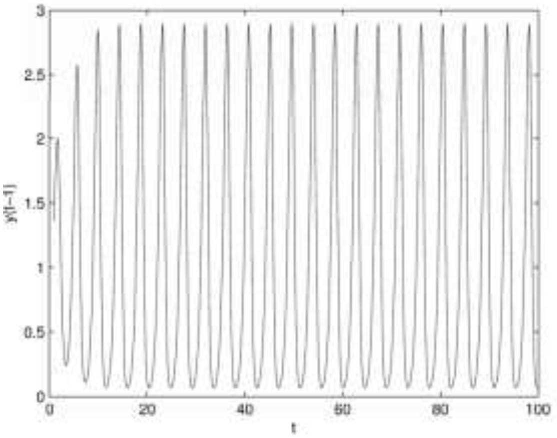

Fig. 3. Solution of the equation.

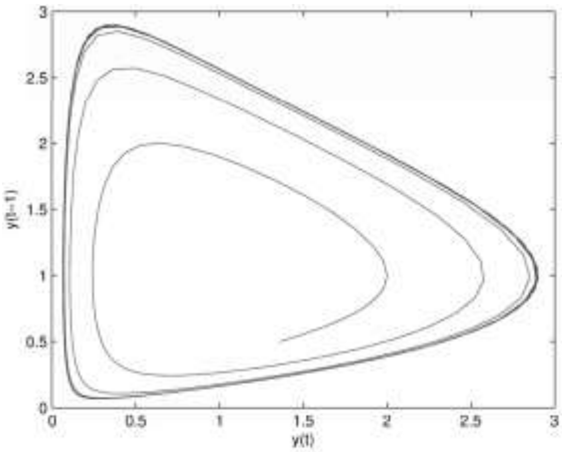

Fig.4. $y(t)$ versus $y(t-1)$.

Figure 3 shows the evolution of the system (40) for $\alpha = 0.97$. Plot of $y(t)$ versus $y(t-1)$ is drawn in Figure 4.

VI. CONCLUSION

The application of the improved predictor-corrector method for solving delay differential equations of fractional order is vividly described in this paper. The study of the detailed error analysis of the numerical examples with constant delay is an evidence for the efficiency of the proposed method.

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