Homomorphism and Anti Homomorphism of L-Fuzzy Quotient ℓ-Group

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Abstract

This paper contains some definitions and results of L-fuzzy quotient ℓ -group and its generalized characteristics.

Keywords

Fuzzy set, L -fuzzy set, L-fuzzy sub ℓ --group, L-fuzzy quotient ℓ -group , homomorphism and anti homomorphism L-fuzzy quotient ℓ-group

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1.Introduction

L. A. Zadeh[9] introduced the notion of a fuzzy subset of a set S as a function from X into I = [0, 1]. Rosenfeld[1] applied this concept in group theory and semi group theory, and developed the theory of fuzzy subgroups and fuzzy subsemigroupoids respectively J.A.Goguen [3] replaced the valuations set [0, 1],by means of a complete lattice in an attempt to make a generalized study of fuzzy set theory by studying L-fuzzy sets.. In fact it seems in order to obtain a complete analogy of crisp mathematics in terms of fuzzy mathematics, it is necessary to replace the valuation set by a system having more rich algebraic structure. These concepts ℓ -groups play a major role in mathematics and fuzzy mathematics. G.S.V Satya Saibaba [7] introduced the concept of L- fuzzy sub ℓ -group and L-fuzzy ℓ -ideal of ℓ -group. In this paper, we initiate the study of homomorphism and anti homomorphism of L-fuzzy quotient ℓ -groups .

2. Preliminaries

This section contains some definitions and results to be used in the sequel.

2.1 Definition [2,3]

A lattice ordered group (ℓ -group) is a system G= (G, *, $\leq$) where

i (G, *) is a group

ii (G , $\leq$) is a lattice

iii the inclusion is invariant under all translations

$x \to a + x + b$ i.e. $x \leq y \Rightarrow a + x + b \leq a + y + b$, for all a, b$\in$G.

2.2 Definition [8,9]

Let X be a non-empty set. A fuzzy subset A of X is a function $A : X \to [0, 1]$.

2.3 Definition [1]

An L-fuzzy subset A of G is called an L-fuzzy subgroup (ALFS) of G if for every x,y $\in$ G,

i $A(xy) \geq A(x) \wedge A(y)$

ii $A(x^{-1}) = A(x)$.

2.4 Definition [7]

An L-fuzzy subset A of G is said to be an L-fuzzy sub ℓ - group(LFS ℓ) of G if for any x , y $\in$ G

i $A(xy) \geq A(x) \wedge A(y)$

ii $A(x^{-1}) = A(x)$

iii $A(x \vee y) \geq A(x) \wedge A(y)$

iv $A(x \wedge y) \geq A(x) \wedge A(y)$.

2.5 Definition [5]

Let G and G′ are any two groups. Then the function f: G $\to$ G′ is said to be a homomorphism if f (xy) = f (x) f (y) for all x, y in G.

2.6 Definition:[4,6,8]

Let A be an L-fuzzy normal sub ℓ -group of G with identity e. Let K = { x $\in$ G / A(x) = A(e) }. Consider the map $\overline{A} = G/K \to L$ defined by $\overline{A}(xK) = \vee A(xk)$ for all k $\in$K and x $\in$ G. Then , the L-fuzzy sub ℓ -group $\overline{A}$ of G/K is called an L-fuzzy quotient ℓ -group of A by K.

Remarks:

i. $\overline{A}$ is not an L-fuzzy normal quotient ℓ -group of G/K.

Since, $\overline{A}(xKyK) \neq \overline{A}(yKxK)$.

ii. Consider the map, $\overline{A} : G/K \to L$ defined by $\overline{A}(xK) = A(x)$ for all k $\in$ K and x $\in$ G. Then, $\overline{A}$ is an L-fuzzy normal quotient ℓ -group of G/K.

3. Properties of an L-fuzzy quotient ℓ-group $\overline{A}$ determined by A and K under ℓ-homomorphism and ℓ-anti homomorphism

In this section, we discuss some of the properties of an L-fuzzy quotient ℓ-group of an ℓ-group G/K determined by A and K under ℓ-homomorphism and ℓ-anti homomorphism.

3.1 Theorem:

Let G and G′ be any two ℓ-groups. Let f: G → G′ be an ℓ-homomorphism and onto. Let $\overline{A} : G/K \to L$ be an L-fuzzy quotient ℓ-group of G/K. Then $f(\overline{A})$ is an L-fuzzy quotient ℓ-group of G'/K, if $\overline{A}$ has sup property and $\overline{A}$ is f- invariant and $\overline{f(A)} = f(\overline{A})$.

Proof:

Let $\overline{A}$ be an L-fuzzy quotient ℓ-group of G/K.

i $f(\overline{A})(f(x)f(y)K)$

$= (f(\overline{A}))(f(xy)K) = \overline{A}(xyK)$

$\geq \overline{A}(xK) \wedge \overline{A}(yK)$

$= (f(\overline{A}))(f(x)K) \wedge (f(\overline{A})(f(y)K)$

$f(\overline{A})(f(x)f(y)K)$

$\geq (f(\overline{A}))(f(x)K) \wedge (f(\overline{A})(f(y)K)$.

ii $f(\overline{A})([f(x)]^{-1}K)$

$= f(\overline{A})[f(x^{-1})K]$

$= \overline{A}(x^{-1}K)$

$= \overline{A}(xK)$

$= f(\overline{A})[f(x)K]$

$f(\overline{A})([f(x)]^{-1}K) = f(\overline{A})[f(x)K]$.

$f(\overline{A})((f(x)\vee f(y))K)$

iii. $= (f(\overline{A}))(f(x\vee y)K)$

$= \overline{A}((x\vee y)K)$

$\geq \overline{A}(xK) \wedge \overline{A}(yK)$

$= (f(\overline{A}))(f(x)K) \wedge (f(\overline{A})(f(y)K)$

$f(\overline{A})((f(x)\vee f(y))K)$

$\geq (f(\overline{A}))(f(x)K) \wedge (f(\overline{A}))(f(y)K)$.

iv. $f(\overline{A})((f(x)\wedge f(y))K)$

$= (f(\overline{A}))(f(x\wedge y)K)$

$= \overline{A}((x\vee y)K)$

$\geq \overline{A}(xK) \wedge \overline{A}(yK)$

$= (f(\overline{A}))(f(x)K) \wedge (f(\overline{A})(f(y)K)$

$f(\overline{A})((f(x)\wedge f(y))K)$

$\geq (f(\overline{A}))(f(x)K) \wedge (f(\overline{A}))(f(y)K)$.

Hence $f(\overline{A})$ is an L- fuzzy quotient ℓ-group of G'/K.

Also,

$\overline{f(A)}(yK) = \vee f(A)(yK)$, for all $k \in K$ and $y \in G'$.

$= \vee f(A)(f(x)K)$, f is onto and $x \in G$.

$= \vee A(xK)$

$= \overline{A}(xK)$

$= f(\overline{A})(f(x)K)$

$= f(\overline{A})(yK)$.

Hence, $\overline{f(A)}(yK) = f(\overline{A})(yK)$.

3.2 Theorem:

Let G and G′ be any two ℓ-groups. Let f: G→ G′ be an ℓ-homomorphism. Let $\overline{B} : G' \to L$ be an L-fuzzy quotient ℓ-group of G'/K. Then $f^{-1}(\overline{B})$ is an L-fuzzy quotient ℓ-group of G/K and $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

Proof:

Let $\overline{B}$ be an L-fuzzy quotient ℓ-group of G'/K.

i. $f^{-1}(\overline{B})(xyK) = \overline{B}(f(xy)K)$

$= \overline{B}(f(x)f(y)K)$

$\geq \overline{B}(f(x)K) \wedge \overline{B}(f(y)K)$

$\geq f^{-1}(\overline{B})(xK) \wedge f^{-1}(\overline{B})(yK)$

$f^{-1}(\overline{B})(xyK)$

$\geq f^{-1}(\overline{B})(xK) \wedge f^{-1}(\overline{B})(yK)$.

ii. $f^{-1}(\overline{B})(x^{-1}K) = \overline{B}(f(x^{-1})K)$

$= \overline{B}((f(x))^{-1}K)$

$= \overline{B}(f(x)K)$

$= f^{-1}(\overline{B})(xK)$

$f^{-1}(\overline{B})(x^{-1}K) = f^{-1}(\overline{B})(xK)$.

iii. $f^{-1}(\overline{B})(x\vee yK) = \overline{B}(f(x\vee y)K)$

$= \overline{B}((f(x)\vee f(y))K)$

$\geq \overline{B}(f(x)K) \wedge \overline{B}(f(y)K)$

$\geq f^{-1}(\overline{B})(xK) \wedge f^{-1}(\overline{B})(yK)$

$f^{-1}(\overline{B})(x \vee yK) \geq f^{-1}(\overline{B})(xK) \wedge f^{-1}(\overline{B})(yK)$.

iv. $f^{-1}(\overline{B})(x \wedge yK) = \overline{B}(f(x \wedge y)K) = \overline{B}((f(x) \wedge f(y))K)$

$\geq \overline{B}(f(x)K) \wedge \overline{B}(f(y)K)$

$\geq f^{-1}(\overline{B})(xK) \wedge f^{-1}(\overline{B})(yK)$

$f^{-1}(\overline{B})(x \wedge yK) \geq f^{-1}(\overline{B})(xK) \wedge f^{-1}(\overline{B})(yK)$.

Hence, $f^{-1}(\overline{B})$ is an L-fuzzy quotient ℓ-group of G/K.

Also, $\overline{f^{-1}(B)}(xK) = \vee f^{-1}(B)(xK)$, for all $k \in K$ and $x \in G$.

$= \vee B(f(x)K)$

$= \overline{B}(f(x)K)$

$= f^{-1}(\overline{B})(xK)$.

Hence, $\overline{f^{-1}(B)}(xK) = f^{-1}(\overline{B})(xK)$.

3.3 Theorem:

Let G and G′ be any two ℓ-groups. Let f: $G \to G'$ be an ℓ-anti homomorphism and onto. Let $\overline{A} : G/K \to L$ be an L-fuzzy quotient ℓ-group of G/K. Then $f(\overline{A})$ is an L-fuzzy quotient ℓ-group of G'/K, if $\overline{A}$ has sup property and $\overline{A}$ is f- invariant and $f(\overline{A}) = \overline{f(A)}$.

Proof:

Let $\overline{A}$ be an L-fuzzy quotient ℓ-group of G/K.

i $f(\overline{A})(f(x)f(y)K)$

$= (f(\overline{A}))(f(yx)K)$

$= \overline{A}(yxK)$

$\geq \overline{A}(yK) \wedge \overline{A}(xK)$

$\geq \overline{A}(xK) \wedge \overline{A}(yK)$

$= (f(\overline{A}))(f(x)K) \wedge (f(\overline{A})(f(y)K)$

$f(\overline{A})(f(x)f(y)K)$

$\geq (f(\overline{A}))(f(x)K) \wedge (f(\overline{A})(f(y)K)$.

ii $f(\overline{A})([f(x)]^{-1}K) = f(\overline{A})[f(x^{-1})K]$

$= \overline{A}(x^{-1}K)$

$= \overline{A}(xK)$

$= f(\overline{A})[f(x)K]$

$f(\overline{A})([f(x)]^{-1}K)$

$= f(\overline{A})[f(x)K]$.

iii. $f(\overline{A})((f(x) \vee f(y))K) = (f(\overline{A}))(f(y \vee x)K)$

$= \overline{A}((y \vee x)K)$

$\geq \overline{A}(yK) \wedge \overline{A}(xK)$

$\geq \overline{A}(xK) \wedge \overline{A}(yK)$

$= (f(\overline{A}))(f(x)K) \wedge (f(\overline{A})(f(y)K)$

$f(\overline{A})(f(x) \vee f(y)K) \geq (f(\overline{A}))(f(x)K) \wedge (f(\overline{A}))(f(y)K)$.

iv. $f(\overline{A})((f(x) \wedge f(y))K)$

$= (f(\overline{A}))(f(y \wedge x)K)$

$= \overline{A}((y \wedge x)K)$

$\geq \overline{A}(yK) \wedge \overline{A}(xK)$

$\geq \overline{A}(xK) \wedge \overline{A}(yK)$

$= (f(\overline{A}))(f(x)K) \wedge (f(\overline{A})(f(y)K)$

$f(\overline{A})(f(x) \wedge f(y)K)$

$\geq (f(\overline{A}))(f(x)K) \wedge (f(\overline{A}))(f(y)K)$.

Hence, $f(\overline{A})$ is an L-fuzzy quotient ℓ-group of G'/K.

Also, $\overline{f(A)}(yK) = \vee f(A)(yK)$, for all $k \in K$ and $y \in G'$.

$= \vee f(A)(f(x)K)$, f is onto and $x \in G$.

$= \vee A(xK)$

$= \overline{A}(xK)$

$= f(\overline{A})(f(x)K)$

$= f(\overline{A})(yK)$.

Hence, $\overline{f(A)}(yK) = f(\overline{A})(yK)$.

3.4 Theorem:

Let G and G′ be any two ℓ-groups. Let f: $G \to G'$ be an ℓ-anti homomorphism. Let $\overline{B} : G' \to L$ be an L-fuzzy quotient ℓ-group of G'/K. Then $f^{-1}(\overline{B})$ is an L-fuzzy quotient ℓ-group of G/K and $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

Proof:

Let $\overline{B}$ be an L-fuzzy quotient ℓ-group of G'/K.

i. $f^{-1}(\overline{B})(xyK) = \overline{B}(f(xy)K)$

$= \overline{B}(f(y)f(x)K)$

$\geq \overline{B}(f(y)K) \wedge \overline{B}(f(x)K)$

$\geq \overline{B}(f(x)K) \wedge \overline{B}(f(y)K)$

$\geq f^{-1}(\overline{B})(xK) \wedge f^{-1}(\overline{B})(yK)$

$f^{-1}(\overline{B})(xyK) \geq f^{-1}(\overline{B})(xK) \wedge f^{-1}(\overline{B})(yK)$.

ii. $f^{-1}(\overline{B})(x^{-1}K) = \overline{B}(f(x^{-1})K)$

$= \overline{B}((f(x))^{-1}K)$

$= \overline{B}(f(x)K)$

$= f^{-1}(\overline{B})(xK)$

$f^{-1}(\overline{B})(x^{-1}K) = f^{-1}(\overline{B})(xK)$.

iii. $f^{-1}(\overline{B})(x \vee yK) = \overline{B}(f(x \vee y)K)$

$= \overline{B}((f(y) \vee f(x))K)$

$\geq \overline{B}(f(y)K) \wedge \overline{B}(f(x)K)$

$\geq \overline{B}(f(x)K) \wedge \overline{B}(f(y)K)$

$\geq f^{-1}(\overline{B})(xK) \wedge f^{-1}(\overline{B})(yK)$

$f^{-1}(\overline{B})(x \vee yK) \geq f^{-1}(\overline{B})(xK) \wedge f^{-1}(\overline{B})(yK)$.

iv. $f^{-1}(\overline{B})(x \wedge yK)$

$= \overline{B}(f(x \wedge y)K)$

$= \overline{B}((f(y) \wedge f(x))K)$

$\geq \overline{B}(f(y)K) \wedge \overline{B}(f(x)K)$

$\geq \overline{B}(f(x)K) \wedge \overline{B}(f(y)K)$

$\geq f^{-1}(\overline{B})(xK) \wedge f^{-1}(\overline{B})(yK)$

$f^{-1}(\overline{B})(x \vee yK) \geq f^{-1}(\overline{B})(xK) \wedge f^{-1}(\overline{B})(yK)$.

Hence, $f^{-1}(\overline{B})$ is an L-fuzzy quotient ℓ-group of G/K.

Also, $\overline{f^{-1}(B)}(xK) = \vee f^{-1}(B)(xK)$, for all $k \in K$ and $x \in G$.

$= \vee B(f(x)K)$

$= \overline{B}(f(x)K)$

$= f^{-1}(\overline{B})(xK)$.

Hence, $\overline{f^{-1}(B)}(xK) = f^{-1}(\overline{B})(xK)$.

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