Properties of semigroups in (semi-)automata

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Abstract *There is a close relationship between automata and semigroups. Automata includes calculating machines, computers, telephone switch boards, elevators and so on. Algebraic automata theory make extensive use of semigroups. The aim of this paper is to summarize the results of semigroup theory which will be described through (semi-)automata. For different types of (semi-)automata, we identified different structures of semigroups and their properties.*

Keywords *semigroup, automata, (semi-)automata, monoid, regular, band*.

Introduction: Algebraic structures play a prominent role in mathematics with wide range of applications in science and engineering. A semigroup is one of the algebraic structure, a set with one binary operation satisfying the law of associativity. It is a generalization of the concept of group. Semigroups are called monoids if they have the identity property. The theory of semigroups is one of the relatively young branches of algebra [1].

During the past few decades, connection in the theory of semigroups and the theory of machines became of increasing importance both theories enriching each other [2]. In association with the study of machines and automata, other areas of applications such as formal languages and the software use the language of modern algebra in terms of Boolean algebra, semigroups and others. Also parts of other areas such as Biology, Psychology, Biochemistry and sociology make use of semigroups [1].

Our present work is to study some of those structures of semigroups in which we find applications in different fields.

Objective: To look into the applications of semigroups with (semi-)automata as a primary tool.

Justification: Automaton is an abstract model of computing device [3, 4]. By describing an abstract model of (semi-)automata which will be amenable to mathematical treatment we see that there is a close relationship between (semi-)automata and semigroup.

We now aim to establish a correspondence between (semi-)automata and monoids.

Semiautomata & Automata:

Semiautomata: A semi automaton is a triple S= (Z , A ,δ) consisting of two non empty sets Z and A and a function $\delta : Z \times A \rightarrow Z$.

Z is called the set of states, A is the set of input alphabet and δ the "next-state function" of S [1].

Automata: An automaton is a quintuple A = (Z , A ,B , δ , λ) where (Z , A , δ) is a semiautomaton, B is a non empty set called the output alphabet and $\lambda : Z \times A \rightarrow B$ is the "output function" [5]

If $z \in Z$ and $a \in A$ then we interpret $\delta(z, a) \in Z$ as the next state into which z is transformed by the input a. $\lambda(z, a) \in B$ is the output of z resulting from the input a. If automaton is in the stage z and receives input a, then it changes to state δ(z, a) with an output λ(z, a).

A (semi)-automaton is finite, if all the sets Z, A and B are finite, finite automata are also called "*Mealy automata".*

CONSTRUCTION OF A MONOID FROM A (SEMI-) AUTOMATA:

We consider the set of all finite sequences of elements of the set A including the empty sequence 'Λ'. In other words in our study of automata, we extend the input set A to the free monoid $\bar{A} = F_A$ with Λ as identity [1, 2].
We also extend δ from Z × A to Z × $\bar{A}$ by defining $z \in Z$ & $a_1, a_2, \ldots, a_r \in A$ such that

$\bar{\delta}(z, \Lambda) = z$, $\bar{\delta}(z, a_1) = \delta(z, a_1)$, $\bar{\delta}(z, a_1, a_2) = \delta(\bar{\delta}(z, a_1), a_2)$……………………

$\delta(z, a_1, a_2 \ldots a_r) = \delta(\bar{\delta}(z, a_1, a_2, \ldots a_{r-1}), a_r)$.

In this way we obtain functions $\bar{\delta}: Z \times \bar{A} \rightarrow Z$. Similarly we obtain functions $\bar{\lambda}: Z \times \bar{A} \rightarrow \bar{B}$.

The semiautomaton S = (Z , A ,δ) (the automaton A = (Z, A, B, δ, λ)) is thus generalized to the new semiautomaton $\bar{S}$ = (Z , $\bar{A}$, $\bar{\delta}$) (the automaton $\bar{A}$ = (Z , $\bar{A}$, $\bar{B}$, $\bar{\delta}$, $\bar{\lambda}$)) respectively. If the automaton is in state z and an input sequence $a_1a_2\ldots\ldots a_r \in \bar{A}$ operates then the states are changed from z = z_1 to z_2, z_3 and so on [6, 7].

The monoid of a (Semi-)automaton and the (Semi-)automaton of a monoid:

We now show that any (Semi-)automaton determines a certain monoid and conversely that any monoid gives rise to a certain (Semi-)automaton [1].

Let us consider the (Semi-)automaton $\bar{S}$= (Z, A, δ) as introduced above.

For $\alpha \in \bar{A}$, let $f_\alpha : Z \rightarrow Z;\ z \mapsto \bar{\delta}(z,\alpha)$.

Where ($\{f_\alpha / \ \alpha \in \bar{A}\}$,o) =: Ms is a monoid (submonoid of (z^z,o)) called the syntactic monoid of S. $\forall \alpha,\beta \in \bar{A}$ we have $f_\alpha o f_\beta = f_{\beta\alpha}$.

The monoid of an automaton is defined as the monoid of the underlying semiautomaton [8].

Definitions:

- In a semigroup (S,o) if every element is an idempotent (a^2 = a,$\forall a \in S$) then S is called a **Band**.
- A commutative (ab = ba,$\forall a,b \in S$) band is called semilattice.
- In a semigroup (S,o) an element **a**$\in$**S** is said to be regular if there exists **b** $\in$**S** such that **aba=a**.
- If every element in S is regular then S is called a regular semigroup.

Description by tables: Let A = $\{a_1, a_2, \ldots. a_n\}$, B = $\{b_1, b_2, \ldots b_m\}$ and Z = $\{z_1, z_2 \ldots\ldots z_k\}$

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Input table

δ	a_1	$a_2 \ldots\ldots$	$\ldots a_n$
z_1	$\delta(z_1, a_1)$	$\delta(z_1, a_2)$.....	$\delta(z_1, a_1)$
z_2	$\delta(z_2, a_1)$	$\delta(z_2, a_2)$.....	$\delta(z_2, a_n)$
.	.	.	.
.	.	.	.
z_k	$\delta(z_k. a_1)$	$\delta(z_k, a_2)$.....	$\delta(z_k, a_n)$

Output table

λ	a_1	$a_2 \ldots\ldots$	$\ldots a_n$
z_1	$\lambda(z_1, a_1)$	$\lambda(z_1, a_2)$	$\lambda(z_1, a_1)$
z_2	$\lambda(z_2, a_1)$	$\lambda(z_2, a_2)$.....	$\lambda(z_2, a_n)$
.	.	.	.
.	.	.	.
z_k	$\lambda(z_k. a_1)$	$\lambda(z_k, a_2)$.....	$\lambda(z_k, a_n)$

PROBLEMS:

a.) Identity Reset flip flop model (IR flip flop) (Semi-)automata:

Where A = {e,0,1} ; B = Z = {0,1}. We have input and output tables as

Input table

δ	e	0	1
0	0	0	1
1	1	0	1

Output table

λ	e	0	1
0	0	0	0
1	1	1	1

Let {e,0,1} = $\{a_1,a_2,a_3\}$. We have semiautomaton and monoid given by

*	f_Λ	fa_1	fa_2	fa_3
z_1	z_1	z_1	z_1	z_2
z_2	z_2	z_2	z_1	z_2

o	fa_1	fa_2	fa_3
fa_1	fa_1	fa_2	fa_3
fa_2	fa_2	fa_2	fa_3
fa_3	fa_3	fa_2	fa_3

CONCLUSION: *Thus the IR flip flop semiautomaton is a semigroup structure which is a monoid, band and hence regular (since a semigroup which is band is regular) and non commutative.*

b.) Parity check automaton where z ={ z_0,z_1 }; A=B={0,1}

A possible interpretation would be for the two states of a number storage with:

z_0 : Machine stores '0'
z_1 : Machine stores '1'

We have input and output tables as:

Input table

δ	0	1
z_0	z_0	z_1
z_1	z_1	z_0

Output table

λ	0	1
z_0	0	1
z_1	0	1

We now show that any semiautomaton determines a certain monoid. Let us define fa : S→S given by fa(z) = δ(z,a), for a ∈ A ; z ∈ S. We also define a binary operation 'o' on fa_0,fa_1 as follows:

First we construct the table for f_Λ, fa_0, fa_1 indicating their actions on the states z_0, z_1.

*	f_Λ	fa_0	fa_1
z_0	z_0	z_0	z_1
z_1	z_1	z_1	z_0

o	fa_0	fa_1
fa_0	fa_0	fa_1
fa_1	fa_1	fa_1

CONCLUSION: *The above parity check (semi-) automaton is thus elucidated to be a semigroup structure which is commutative, band and regular. Hence a semilattice.*

c.) Cafeteria Automaton:

We now consider Cafeteria automaton in students life where set of states and inputs are

Z_1: Student is angry
a_1: Cafeteria is closed

Z_2: Student is bored
a_2: Cafeteria offers junk food

Z_3: Student is happy
a_3: Cafeteria offers good food

Outputs:

b_1: Student shouts
b_2: Student is quiet.

Thus we have input and output tables as:

Input table

δ	a_1	a_2	a_3
z_1	z_1	z_1	z_3
z_2	z_2	z_1	z_3
z_3	z_3	z_2	z_3

Output table

λ	a_1	a_2	a_3
z_1	b_2	b_1	b_2
z_2	b_2	b_2	b_2
z_3	b_2	b_2	b_2

The semiautomata is rewritten and the monoid is

*	f_Λ	fa_1	fa_2	fa_3
z_1	z_1	z_1	z_1	z_3
z_2	z_2	z_2	z_1	z_3
z_3	z_3	z_3	z_2	z_3

o	fa_1	fa_2	fa_3
fa_1	fa_1	fa_2	fa_3
fa_2	fa_2	fa_2a_2	fa_3
fa_3	fa_3	fa_3a_2	fa_3

The above monoid does not satisfy closure property since there are two new elements fa_2a_2 & fa_3a_2.

Hence we reconstruct the above monoid as:

o	fa_1	fa_2	fa_3	fa_2a_2	fa_3a_2
fa_1	fa_1	fa_2	fa_3	fa_2a_2	fa_3a_2
fa_2	fa_2	fa_2a_2	fa_3	fa_2a_2	fa_3a_2
fa_3	fa_3	fa_3a_2	fa_3	fa_2a_2	fa_3a_2
fa_2a_2	fa_2a_2	fa_2a_2	fa_3	fa_2a_2	fa_3a_2
fa_3a_2	fa_3a_2	fa_3a_2	fa_3	fa_2a_2	fa_3a_2

CONCLUSION: *Thus we conclude that the given Cafeteria automaton is only a monoid (Not a band, not commutative).*

d.) Marriage Automaton:

We now consider a situation in a house hold, where set of states and inputs are

Z_1: Husband is angry
a_1: Wife is quite

Z_2: Husband is bored
a_2: Wife shouts

Z_3: Husband is happy
a_3: Wife cooks

Assuming that husband shouts only if he is angry and his wife shouts. Otherwise he is quiet.

We have the outputs as follows:

Outputs:
b_1: Husband shouts
b_2: Husband is quiet.

Thus we have input and output tables as:

Input table

δ	a_1	a_2	a_3
z_1	z_1	z_1	z_3
z_2	z_2	z_1	z_3
z_3	z_3	z_2	z_3

Output table

λ	a_1	a_2	a_3
z_1	b_2	b_1	b_2
z_2	b_2	b_2	b_2
z_3	b_2	b_2	b_2

The semiautomata is rewritten and the monoid is

The above monoid does not satisfy closure property since there are two new elements fa_2a_2 & fa_3a_2. Hence we reconstruct the above monoid as:

CONCLUSION: *Thus we conclude that the given marriage automaton is only a monoid (Not a band, not commutative).*

*	f_Λ	fa_1	fa_2	fa_3
z_1	z_1	z_1	z_1	z_3
z_2	z_2	z_2	z_1	z_3
z_3	z_3	z_3	z_2	z_3

o	fa_1	fa_2	fa_3
fa_1	fa_1	fa_2	fa_3
fa_2	fa_2	fa_2a_2	fa_3
fa_3	fa_3	fa_3a_2	fa_3

e.) Automaton with four set of states:

Let us consider another Automaton with Z =

o	fa_1	fa_2	fa_3	fa_2a_2	fa_3a_2
fa_1	fa_1	fa_2	fa_3	fa_2a_2	fa_3a_2
fa_2	fa_2	fa_2a_2	fa_3	fa_2a_2	fa_3a_2
fa_3	fa_3	fa_3a_2	fa_3	fa_2a_2	fa_3a_2
fa_2a_2	fa_2a_2	fa_2a_2	fa_3	fa_2a_2	fa_3a_2
fa_3a_2	fa_3a_2	fa_3a_2	fa_3	fa_2a_2	fa_3a_2

$\{z_1,z_2,z_3,z_4\}$; $A = \{a,b\}$; $B = \{0,1\}$ with input and output tables given by

Input table

δ	a	b
z_1	z_2	z_3
z_2	z_3	z_4
z_3	z_4	z_2
z_4	z_4	z_4

Output table

λ	a	b
z_1	0	0
z_2	1	0
z_3	0	1
z_4	0	0

We now reconstruct the semiautomaton as follows:

*	fa	fb	faa	fab	fba	fbb	faaa	id
z_1	z_2	z_3	z_3	z_2	z_4	z_4	z_4	z_1
z_2	z_3	z_4	z_4	z_4	z_3	z_2	z_4	z_2
z_3	z_4	z_2	z_4	z_4	z_4	z_4	z_4	z_3
z_4	z_4	z_4	z_4	z_4	z_4	z_4	z_4	z_4

CONCLUSION: *Thus we conclude that the given (Semi-) automaton is only a monoid (Not a band, not commutative).*

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