Radon Measure on Measure Manifold

S. C. P. Halakatti[1] and Soubhagya Baddi[2]

[1]Department of Mathematics Karnatak University, Dharwad, India.

[2] Research Scholar, Department of Mathematics, Karnatak University, Dharwad, India.

Abstract: *In this paper, the Radon measure is assigned on compact measurable charts and atlases of measure manifold on which the different versions of measurable compactness properties like measurable Heine-Borel property, measurable countably compactness property, measurable Lindeloff property and measurable paracompactness property are studied. Further, we show that these properties remain invariant under measurable diffeomorphism and Radon measure-invariant function. We prove that the set of these functions form a group structure on the set* $\mathcal{M} = \{M_1, M_2,...., M_n\}$ *of Radon measure manifolds.*

Keywords: *Measurable Heine-Borel property, measurable countably compactness property, measurable Lindeloff property, measurable paracompactness properties, measurable diffeomorphism and Radon measure- invariant map.*

I. INTRODUCTION

Historically, Peter Ganssler [11] had studied compactness and sequentially compactness on measure space in terms of Borel subsets of topological spaces. Radon measure on first-countable space is studied by Grzegorz Plebanck [26]. David Fremlin [9] discussed some relationships between topological properties of a Hausdorff space and the properties of the Radon measures it carries, concentrating on ideas involving cardinal functions. Sherry L. Gale [12] had shown the relations between covering properties and measure compactness properties on topological spaces. Radon measure on a subset of a paracompact regular topological space which is a Radon space was studied by Baltasar Rodriguez-Salinas [2]. Gardner and Pfeffer [10] studied Radon measures μ in a wider class of topological spaces and proved that μ is localizable and locally determined. Coverable Radon measures in topological spaces with covering properties and their relationships was defined by Yoshihiro Kabokawa [20]. Also in [6], it is proved that a diffeomorphism preserves the Borel and the Lebesgue measurable sets. Relationship between concepts of topology and soft covering lower and soft covering upper approximations was studied by Naime Tozlu, Saziye Yuksel, Tugba Han Dizman [24]. Mohamed M. Osman [22], [23] have worked on fixed spin structure with its Dirac operator D on a compact Riemannian manifold (M, g) and have proved some results. The concept of Riemannian spin manifold is introduced along with few results. If the compact Riemannian manifold admits Radon measure, then one can measure the measurable spin of Riemannian spin manifold with killing vectors.

All these measure-compactness and covering properties are studied on topological spaces.

In our work [13], [14], S. C. P. Halakatti have approximated open subsets which cover the topological space into Borel subsets of measure space which are the smallest subsets. This study is extended on measure manifold which is a new class of differentiable manifolds. The extended topological properties which are well defined on measure manifold are studied. These properties are measurable and measure-invariant under measurable homeomorphism [15]. It is shown that some of the topological properties are Radon measurable and Radon measure-invariant on measure space and on measure manifold [16], [17], [18]. Further, it is shown that different aspects of measurable Heine-Borel property, measurable countably compact, measurable Lindeloff and measurable paracompactness on measure manifold remain invariant under measurable homeomorphism and measure-invariant transformation. Such measurable compactness properties are redesignated as eHBp, e-countably compact and e-Lindeloff [15].

Now, in this paper, the Radon measure is assigned on compact measurable charts and atlases of a measure manifold on which the different versions of measurable compactness properties like eHBp, e-countably compactness property, e-Lindeloff property and e-paracompactness property remain invariant under measurable diffeomorphism and Radon measure-invariant functions. We prove that the set of these functions form a group structure on the set $\mathcal{M} = \{M_1, M_2 ,...., M_n\}$ of Radon measure manifolds.

II. PRELIMINARIES

In this section, we consider some basic concepts:

Definition 2.1 [10]:

Let M_1 and M_2 be smooth manifolds with corresponding maximal atlases $\mathcal{A}_{M_1}$ and $\mathcal{A}_{M_2}$. We say that a map $F : M_1 \longrightarrow M_2$ is of class C^r at $p \in M_1$ if there exists a chart (V,ψ) from $\mathcal{A}_{M_2}$ with $F(p) \in V$, and a chart (U,φ) from $\mathcal{A}_{M_1}$ with $p \in U$, such that $F(U) \subset V$ and such that $\psi \circ F \circ \varphi^{-1} : \varphi(F^{-1}(V) \cap U) \longrightarrow \psi(V)$ is of class C^r. If F is of class C^r at every point p

$\in M_1$, then we say that F is of class C^r. Maps of class C^∞ are called smooth maps.

We know that a C^r map is continuous and a composition of C^r maps is a C^r map. Let $F: M_1 \longrightarrow M_2$ be a map and suppose that (U,φ) and (V,ψ) are admissible charts for M_1 and M_2 respectively. If $F^{-1}(V) \cap U \neq \emptyset$, then we have a composition $\psi \circ F \circ \varphi^{-1} : \varphi(F^{-1}(V) \cap U) \longrightarrow \psi(V)$. Maps of this form are called the local representative maps for f.

Definition 2.2: C^r diffeomorphism [10]:

Let M_1 and M_2 be smooth (or C^r) manifolds. A homeomorphism $F : M_1 \longrightarrow M_2$ such that F and F^{-1} are C^r differentiable with (r $\geq$ 1) is called a C^r diffeomorphism.

The set of all C^r diffeomorphisms of a manifold M onto itself is a group under the operation of composition.

Definition 2.3: Radon Measure on (M, τ, Σ, μ) [19]:

A Radon measure on a measure manifold (M, τ, Σ, μ) is a positive Borel measure

$$\mu : B \longrightarrow [0,\infty]$$

which is finite on compact Borel subsets and is inner regular in the sense that for every Borel charts (U,φ) $\subset$ (M, τ, Σ, μ) we have

(i) μ_R(U) = sup { μ_R(K) : K $\subseteq$ U, K $\in \mathcal{K}$}

where $\mathcal{K}$ denote the family of all compact Borel subsets.

μ_R is outer regular on a family $\mathcal{F}$ of Borel charts if for every Borel charts (U,φ) $\subset$ (M, τ, Σ, μ) we have,

(ii) μ_R(U) = inf { μ_R(O) : O $\supseteq$ U, O $\in \mathcal{O}$}

where $\mathcal{O}$ denote the family of all open Borel subsets.

Definition 2.4: Measurable Homeomorphism [16]:

Let $(M_1, \tau_1, \Sigma_1, \mu_1)$ and $(M_2, \tau_2, \Sigma_2, \mu_2)$ be two measure manifolds. Then the function $F : M_1 \longrightarrow M_2$ is called measurable homeomorphism if

(i) F is bijective and bi continuous

(ii) F and F^{-1} are measurable

Definition 2.5: Measure Invariant [16]:

Let $(M_1, \tau_1, \Sigma_1, \mu_1)$ and $(M_2, \tau_2, \Sigma_2, \mu_2)$ be measure manifolds and $F : (M_1, \tau_1, \Sigma_1, \mu_1) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_2)$ be a measurable homeomorphism. Then F is said to be measure preserving if for all measure charts $(U,\varphi) \in M_2$ we have that, $\mu_1(F^{-1}(U)) = \mu_2(U)$ where $F^{-1}(U) \in M_1$.

Definition 2.6: Measurable Heine-Borel property [17]:

A measure manifold endowed with extended topological property eHBp is called as the measurable Heine-Borel property.

Definition 2.7: Measurable Lindeloff property [17]:

A measure manifold endowed with extended topological property e-Lindeloff is called as the measurable Lindeloff .

Definition 2.8: Measurable countably compact property [17]:

A measure manifold endowed with extended topological property e-countably compact property is called as the measurable countably compact.

Definition 2.9: Paracompact space [5]:

A paracompact space (X,τ) is a Hausdorff space with the property that every open cover of X has an open finite refinement.

Inverse Function Theorem on Radon measure manifolds [20]:

Let $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be a C^∞ measurable homeomorphism and Radon measure - invariant map of Radon measure manifolds $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where μ_{R_1} and μ_{R_2} are Radon measures on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ respectively and suppose that F_{*p} : $T_p(M_1) \longrightarrow T_{f(p)}(M_2)$ is a linear isomorphism at some point p of M_1. Then there exists a Radon measure chart (U, φ) of p in M_1 such that the restriction of F to (U, φ) is a diffeomorphism onto a Radon measure chart (V, φ) of $F(p)$ in M_2. This implies for every function F which is measurable homeomorphism and Radon measure-invariant has a C^∞ map F^{-1}: $(M_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ which is also measurable homeomorphism and Radon measure-invariant.

III. MAIN RESULTS

In this paper, we conduct a case study on different versions of measurable compactness properties like measurable Heine-Borel property, measurable countably compactness property, measurable Lindeloff property and measurable paracompactness property by assigning a Radon measure on compact measurable atlases of measure manifold.

Let $(M_1, \tau_1, \Sigma_1, \mu_1)$ and $(M_2, \tau_2, \Sigma_2, \mu_2)$ be measure manifolds with corresponding (maximal) measure atlases $\mathcal{A}_{M_1}$ and $\mathcal{A}_{M_2}$. Let F: $(M_1, \tau_1, \Sigma_1, \mu_1) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_2)$ be a measurable homeomorphism and measure-invariant function. Let $(U_\alpha, \varphi_\alpha)$ and (V_β, ψ_β) be measure charts from $\mathcal{A}_{M_1}$ and $\mathcal{A}_{M_2}$ of M_1 and M_2 respectively with $p \in U_\alpha$ and $F(p) \in V_\beta$. Let $U = F^{-1}(V_\beta) \cap U_\alpha$, we have a measure chart $(U, \varphi_{\alpha/U})$ on M_1 with $p \in U$ and $F(U) \subset V_\beta$ and $F^{-1}(V) \cap U \neq \emptyset$ where (U, φ) is a measure chart on M_1.

Definition 3.1 Measurable diffeomorphism:

Let $(M_1, \tau_1, \Sigma_1, \mu_1)$ and $(M_2, \tau_2, \Sigma_2, \mu_2)$ be measure manifolds with corresponding (maximal) measure atlases $\mathcal{A}_{M_1}$and $\mathcal{A}_{M_2}$ with $(U\ , \varphi)$ and $(V\ , \psi)$ are measure charts on M_1 and M_2 respectively .We say that a C^r measurable homeomorphism

F: $(M_1, \tau_1, \Sigma_1, \mu_1) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_2)$ is said to be measurable diffeomorphism if ,

(i) $\psi \circ F \circ \varphi^{-1} : \varphi(F^{-1}(V) \cap U) \longrightarrow \psi(V)$ is of class C^r and measurable,

(ii) F and F^{-1} are C^r differentiable with r ≥ 1and are measurable.

If r = ∞ then F: $(M_1, \tau_1, \Sigma_1, \mu_1) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_2)$ is a C^∞ measurable diffeomorphism.

Now we conduct a case study on each of the following measurable compactness properties:

Case 3.1: Measurable Heine-Borel property on measure manifold with Radon measure

Let $(M\ \ , \tau, \Sigma, \mu)$ be a measure manifold and let $\mathcal{A} \in A^k(M)$ be a measure atlas such that $\mathcal{A} \subseteq \{\bigcup_{i=1}^{\infty}(U_i, \varphi_i)\}$ = $(M\ \ , \tau, \Sigma, \mu_R)$ where (U_i, φ_i) are measure charts of $(M\ , \tau, \Sigma, \mu)$.

For every collection $\{\bigcup_{i\in I}(U_i, \varphi_i)\}$ of measure charts (U_i, φ_i) that cover $\mathcal{A}$, if there exists a finite Borel sub cover $\{\bigcup_{j=1}^{n}(U_{i_j}, \varphi_i)\}$ of measure charts (U_{i_j}, φ_i) that cover $\mathcal{A}$ such that $\mathcal{A} \subseteq \{\bigcup_{j=1}^{n}(U_{i_j}, \varphi_i)\}$ satisfying the following measure conditions:

(i) $\mu(\mathcal{A}) > 0$,

(ii) $\mu(\mathcal{A}) \ \leq \ \mu(\bigcup_{j=1}^{n} U_{i_j}) \ = \ \sum_{j=1}^{n} \mu(U_{i_j})$ ....(σ-additivity)

then the measure atlas is compact in $(M\ , \tau, \Sigma, \mu_R)$ and is Radon measurable as follows:

Let $\mathcal{O} = \{\bigcup_{i=1}^{\infty}(U_i, \varphi_i)\}$ be a family of all measure charts of $(M\ , \tau, \Sigma, \mu)$.On a measure manifold endowed with measurable Heine-Borel property, a Borel measure μ : B ⟶ [0,∞] is finite. Inner and outer regularities of Borel subsets are equivalent properties on it. Then a Radon measure μ_R for a measurable atlas $\mathcal{A}$ is a positive Borel measure μ : B ⟶ [0,∞] which is finite on compact Borel sets and is inner regular in the sense that for every measurable atlas, we have

(i) $\mu_R(\mathcal{A})$ = sup { $\mu_R(U_i)$: $i \in I\ U_i \subseteq \ \mathcal{A}\ , \forall\ U_i\ \in\ \mathcal{O}$},

Also μ_R is outer regular on a family $\mathcal{O}$ of measurable charts if, for every measurable atlas $\mathcal{A} \in A^k(M)$ we have,

(ii) $\mu_R(\mathcal{A})$ = inf { μ_R(O): O⊇ $\mathcal{A}$, $\mathcal{O} \in M$}.

Now the measure atlas $\mathcal{A}$ of $(M\ , \tau, \Sigma, \mu_R)$ is Radon measurable and the manifold $(M\ , \tau, \Sigma, \mu_R)$ is a Radon measure manifold endowed with measurable Heine-Borel property.

Remark 3.1:

We observe that a measurable Heine-Borel property say P_1 on any compact Borel set $A_1 \subset (M, \tau, \Sigma, \mu_R)$ containing compact Radon measure atlases on (M, τ, Σ, μ_R) holds $\mu_R - a.e.$ on (M, τ, Σ, μ_R) if the set $A_1 = \{\forall\ \mathcal{A}\ \ \subset (M, \tau, \Sigma, \mu_R) :\ P_1(A_1)$ is true }has positive measure i.e. $\mu_R(A_1) > 0$.

Suppose P_1 does not hold $\mu_R - a.e.$ on the set $A_1 \subset (M\ , \tau, \Sigma, \mu_R)$ then $\mu_R(A_1) = 0$. Then A_1 is a dark region of (M, τ, Σ, μ_R).

Case 3.2: Measurable Lindeloff property on measure manifold with Radon measure

Let $(M\ \ , \tau, \Sigma, \mu)$ be a measure manifold and let $\mathcal{A} \in A^k(M)$ be a measure atlas such that $\mathcal{A} \subseteq \{\bigcup_{i=1}^{n}(U_i, \varphi_i)\}$ = $(M\ \ , \tau, \Sigma, \mu_R)$ where (U_i, φ_i) are measure charts of $(M\ , \tau, \Sigma, \mu)$.

For every collection $\{\bigcup_{i\in I}(U_i, \varphi_i)\}$ of measure charts (U_i, φ_i) that cover $\mathcal{A}$, if there exists a countable Borel sub cover $\{\bigcup_{j=1}^{\infty}(U_{i_j}, \varphi_i)\}$ of measure charts (U_{i_j}, φ_i) that cover $\mathcal{A}$ such that $\mathcal{A} \subseteq \{\bigcup_{j=1}^{\infty}(U_{i_j}, \varphi_i)\}$

i.e. $\mathcal{A} = \bigcup_{i\in I}(U_i, \varphi_i) \subseteq \{\bigcup_{j=1}^{\infty}(U_{i_j}, \varphi_i)\}$, satisfying the following measure conditions:

(i) $\mu(\mathcal{A}) > 0$,

(ii) $\mu(\mathcal{A}) \ \leq \ \mu(\bigcup_{j=1}^{\infty} U_{i_j}) \ = \ \sum_{j=1}^{\infty} \mu(U_{i_j})$ (....σ-additivity)

then $\mathcal{A}$ is compact Borel atlas of $(M\ , \tau, \Sigma, \mu_R)$ which is Radon measurable as follows:

Let $\mathcal{O} = \{\bigcup_{i\in I}(U_i, \varphi_i)\}$ be a family of all measure charts of $(M\ \ , \tau, \Sigma, \mu_R)$. On a measure manifold endowed with measurable Lindeloff property, a Borel measure μ : Σ ⟶ [0,∞] is finite. Inner and outer regularities of Borel subsets are equivalent properties on it. Then a Radon measure for a measurable atlas $\mathcal{A}$ is a positive Borel measure μ : Σ ⟶ [0,∞] which is finite on compact sets and is inner regular in the sense that for every measurable atlas, we have

(i) $\mu_R(\mathcal{A})$ = sup { $\mu_R(U_i)$: $i \in I\ U_i \subseteq \ \mathcal{A}\ , \forall\ U_i\ \in\ \mathcal{O}$},

Also μ_R is outer regular on a family $\mathcal{O}$ of measurable charts if, for every measurable atlas $\mathcal{A} \in A^k(M)$ we have,

(ii) $\mu_R(\mathcal{A})$ = inf { μ_R(O) : O ⊇ $\mathcal{A}$, $\mathcal{O} \in M$}.

Now the Borel atlas $\mathcal{A}$ of (M, τ, Σ, μ_R) is Radon measurable and the manifold (M, τ, Σ, μ_R) is a Radon measure manifold endowed with measurable Lindeloff property.

Then the manifold (M, τ, Σ, μ_R) is Radon measure manifold endowed with measurable Lindeloff property.

Remark 3.2:

We observe that a measurable Lindeloff property say P_2 on any compact Borel set $A_2 \subset (M, \tau, \Sigma, \mu_R)$ containing compact Radon measure atlases on (M, τ, Σ, μ_R) holds $\mu_R - a.e.$ on (M, τ, Σ, μ_R) if the set $A_2 = \{\forall \mathcal{A} \subset (M\ , \tau, \Sigma, \mu_R):\ P_2(A_2)$ is true $\}$has positive measure i.e. $\mu_R(A_2) > 0$.

Suppose P_2 does not hold $\mu_R - a.e.$ on the set $A_2 \subset (M\ , \tau, \Sigma, \mu_R)$ then $\mu_R(A_2) = 0$. Then A_2 is a dark region of $(M\ , \tau, \Sigma, \mu_R)$.

Case 3.3: Measurable countably compact property on measure manifold with Radon measure

Let $(M\ , \tau, \Sigma, \mu)$ be a measure manifold and let $\mathcal{A} \in A^k(M)$ be a measure atlas such that $\mathcal{A} \subseteq \{\bigcup_{i=1}^{\infty}(U_i, \varphi_i)\} = (M\ , \tau, \Sigma, \mu_R)$ where (U_i, φ_i) are measure charts of $(M\ , \tau, \Sigma, \mu)$.

For every countable collection $\{\bigcup_{i=1}^{\infty}(U_i, \varphi_i)\}$ of measure charts (U_i, φ_i) that cover $\mathcal{A}$ such that $\mathcal{A} \subseteq \{\bigcup_{i=1}^{\infty}(U_i, \varphi_i)\}$

i.e. $\mathcal{A} = \bigcup_{i \in I}(U_i, \varphi_i) \subseteq \{\bigcup_{j=1}^{\infty}(U_{i_j}, \varphi_i)\}$, if there exists a finite Borel sub cover $\{\bigcup_{j=1}^{n}(U_{i_j}, \varphi_i)\}$ of measure charts (U_{i_j}, φ_i) that cover $\mathcal{A}$

i.e. $\mathcal{A} \subseteq \{\bigcup_{j=1}^{n}(U_{i_j}, \varphi_i)\}$, satisfying the following measure conditions:

(i) $\mu(\mathcal{A}) > 0$,

(ii) $\mu(\mathcal{A}) \leq \mu(\bigcup_{j=1}^{n} U_{i_j}) = \sum_{j=1}^{n} \mu(U_{i_j})$ ....(σ-additivity)

then the Borel atlas $\mathcal{A}$ is compact in $(M\ , \tau, \Sigma, \mu_R)$ and is Radon measurable as follows:

Let $\mathcal{O} = \{\bigcup_{i=1}^{\infty}(U_i, \varphi_i)\}$ be a family of all measure charts of $(M\ , \tau, \Sigma, \mu_R)$. On a measure manifold endowed with measurable countably compact property, a Borel measure μ : Σ ⟶ [0,∞] is finite. Inner and outer regularities of Borel subsets are equivalent properties on it. Then a Radon measure for a measurable atlas $\mathcal{A}$ is a positive Borel measure μ : B ⟶ [0,∞]

which is finite on compact sets and is inner regular in the sense that for every measurable atlas, we have

(i) $\mu_R(\mathcal{A})$ = sup { $\mu_R(U_i)$: $i \in I\ U_i \subseteq \mathcal{A}\ , \forall\ U_i \in \mathcal{O}$},

Also μ_R is outer regular on a family $\mathcal{O}$ of measurable charts if, for every measurable atlas $\mathcal{A} \in A^k(M)$ we have,

(ii) $\mu_R(\mathcal{A})$ = inf { μ_R(O): O⊇ $\mathcal{A}$, $\mathcal{O} \in M$}.

Now the Borel atlas $\mathcal{A}$ of $(M\ , \tau, \Sigma, \mu_R)$ is Radon measurable and the manifold $(M\ , \tau, \Sigma, \mu_R)$ is a Radon measure manifold endowed with measurable countably compact property.

Then the manifold $(M\ , \tau, \Sigma, \mu_R)$ is a Radon measure manifold endowed with measurable countably compact.

Remark 3.3:

We observe that a measurable countably compact property say P_3 on any compact Borel set $A_3 \subset (M, \tau, \Sigma, \mu_R)$ containing compact Radon measure atlases on (M, τ, Σ, μ_R) holds $\mu_R - a.e.$ on (M, τ, Σ, μ_R) if the set $A_3 = \{\forall \mathcal{A} \ \subset (M\ , \tau, \Sigma, \mu_R):$ $P_3(A_3)$ is true $\}$has positive measure i.e. $\mu_R(A_3) > 0$.

Suppose P_3 does not hold $\mu_R - a.e.$ on the set $A_3 \subset (M\ , \tau, \Sigma, \mu_R)$ then $\mu_R(A_3) = 0$. Then A_3 is a dark region of $(M\ , \tau, \Sigma, \mu_R)$.

Case 3.4: Measurable paracompact property on measure manifold with Radon measure

Let $(M\ , \tau, \Sigma, \mu)$ be a measure manifold and let $\mathcal{A} \in A^k(M)$ be a measure atlas such that $\mathcal{A} \subseteq \{\bigcup_{i=1}^{\infty}(U_i, \varphi_i)\} = (M\ , \tau, \Sigma, \mu_R)$ where (U_i, φ_i) are measure charts of $(M\ , \tau, \Sigma, \mu)$.

For every collection $\{\bigcup_{i \in I}(U_i, \varphi_i)\}$ of measure charts (U_i, φ_i) that cover $\mathcal{A}$, such that $\mathcal{A} \subseteq \{\bigcup_{i \in I}(U_i, \varphi_i)\}$, if there exists an open locally finite refinement $\{\bigcup_{j=1}^{n}(U_{i_j}, \varphi_i)\}$ of measure charts (U_{i_j}, φ_i) that cover $\mathcal{A}$ such that $\mathcal{A} \subseteq \{\bigcup_{j=1}^{n}(U_{i_j}, \varphi_i)\}$ where $\mu(U_{i_j}) > 0$ and $U_{i_j} \subset U_i$ satisfying the following measure conditions:

(i) $\mu(\mathcal{A}) > 0$,

(ii) $\mu(\mathcal{A}) \leq \mu(\bigcup_{j=1}^{n} U_{i_j}) = \sum_{j=1}^{n} \mu(U_{i_j})$ ....(σ-additivity)

then $\mathcal{A}$ is compact Borel atlas of $(M\ , \tau, \Sigma, \mu_R)$ which is measurable and is Radon measurable as follows:

Let $\mathcal{O} = \{\bigcup_{i \in I}(U_i, \varphi_i)\}$ be a family of all measure charts of $(M\ , \tau, \Sigma, \mu_R)$. On a measure manifold endowed with measurable paracompact property, a Borel measure μ : Σ ⟶ [0,∞] is finite. Inner and outer regularities of Borel subsets are equivalent properties on it. Then a Radon measure for a measurable atlas $\mathcal{A}$ is a positive Borel measure μ : Σ ⟶ [0,∞] which is finite on compact sets and is inner regular in the sense that for every measurable atlas, we have

(i) $\mu_R(\mathcal{A})$ = sup { $\mu_R(U_i)$: $i \in I\ U_i \subseteq \mathcal{A}\ , \forall\ U_i \in \mathcal{O}$},

Also μ_R is outer regular on a family $\mathcal{O}$ of measurable charts if, for every measurable atlas $\mathcal{A} \in A^k(M)$ we have,

(ii) $\mu_R(\mathcal{A})$ = inf { μ_R(O): O⊇ $\mathcal{A}$, $\mathcal{O} \in M$}.

Now the Borel atlas $\mathcal{A}$ of $(M\ , \tau, \Sigma, \mu_R)$ is Radon measurable and the manifold $(M\ , \tau, \Sigma, \mu_R)$ is a Radon measure manifold endowed with measurable paracompact property.

Then the manifold $(M\ , \tau, \Sigma, \mu_R)$ is Radon measure manifold endowed with measurable Lindeloff property.

Remark 3.4:

We observe that a measurable paracompact property say P_4 on any compact Borel set $A_4 \subset (M, \tau, \Sigma, \mu_R)$ containing compact Radon measure atlases on (M, τ, Σ, μ_R) holds $\mu_R - a.e.$ on (M, τ, Σ, μ_R) if the set $A_4 = \{ \forall \mathcal{A} \subset (M, \tau, \Sigma, \mu_R) : P_4(A_4)$ is true $\}$has positive measure i.e. $\mu_R(A_4) > 0$.

Suppose P_4 does not hold $\mu_R - a.e.$ on the set $A_4 \subset (M, \tau, \Sigma, \mu_R)$ then $\mu_R(A_4) = 0$. Then 4 is a dark region of (M, τ, Σ, μ_R).

By using definition 3.1 and Inverse Function Theorem on Radon measure manifolds [20], S. C. P. Halakatti proved the following results and we carry a study on it.

Theorem 3.1:

Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds endowed with measurable Heine-Borel property. If $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is a measurable diffeomorphism and Radon measure-invariant map and if the measurable Heine-Borel property holds $\mu_{R_2} - a.e.$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ then it holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$.

Proof: Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds endowed with measurable Heine-Borel property.

Let P_1 be a measurable Heine-Borel property which holds $\mu_{R_2} - a.e.$ on any set A_1 of compact Borel atlases in $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $A_1 = \{ \forall \mathcal{A} \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2}) : P_1(A_1)$ is true $\}$ has positive measure i.e. $\mu_{R_2}(A_1) > 0$.

We show that P_1 holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$.

Since $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is a measurable diffeomorphism and Radon measure-invariant map from $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ to $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ there exists

$F^{-1}: (M_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (M_1, \tau_1, \Sigma_1, \mu_{R_1})$. If P_1 holds $\mu_{R_2} - a.e.$ on any set A_1 of compact Borel atlas $\mathcal{A} \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where the set $A_1 = \{ \forall \mathcal{A} \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2}) : P_1(\mathcal{A})$ is true $\}$ has positive measure i.e. $\mu_{R_2}(A_1) > 0$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ then the property P_1 also holds $\mu_{R_1} - a.e.$ on any compact Borel set $F^{-1}(A_1)$ of compact Borel atlas $F^{-1}(\mathcal{A}) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ where the set $F^{-1}(A_1) = \{ \forall F^{-1}(\mathcal{A}) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1}) : P_1(F^{-1}(\mathcal{A}))$ is true $\}$ has positive measure i.e. $\mu_{R_1}(F^{-1}(A_1)) > 0$ in $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$.

Therefore, if measurable Heine-Borel property holds $\mu_{R_2} - a.e.$ $\forall A_1 \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ then the property also holds $\mu_{R_1} - a.e.$ on $F^{-1}(A_1) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ under measurable diffeomorphism and Radon measure invariant map $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$. ∎

Theorem 3.2:

Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds endowed with measurable Lindeloff property. If $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is a measurable diffeomorphism and Radon measure-invariant map and if the measurable Lindeloff property holds $\mu_{R_2} - a.e.$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ then it holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$.

Proof: Proof is similar to Theorem 3.1.

Theorem 3.3:

Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds endowed with measurable countably compact property. If $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is a measurable diffeomorphism and Radon measure-invariant map and if the measurable countably compact property holds $\mu_{R_2} - a.e.$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ then it also holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$.

Proof: We proceed as in Theorem 3.1.

Theorem 3.4:

Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ and $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ be Radon measure manifolds endowed with measurable paracompact property. If $F : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ is a measurable diffeomorphism and Radon measure-invariant map and if the measurable paracompact property holds $\mu_{R_2} - a.e.$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ then it also holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$.

Proof: We proceed as in Theorem 3.1.

Theorem 3.5:

Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ be Radon measure manifolds endowed with measurable Heine-Borel property. Let $F_1 : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $F_2 : (M_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ be measurable diffeomorphism and Radon measure-invariant maps. Then if measurable Heine-Borel property holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then it also holds $\mu_{R_3} - a.e.$ on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ under the composition mapping $F_2 \circ F_1 : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$.

Proof: Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ be Radon measure manifolds endowed with measurable Heine-Borel property. Let $F_1 : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $F_2 : (M_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ be measurable diffeomorphism and Radon measure-invariant maps.

We show that, if measurable Heine-Borel property say P_1 holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_1 also holds $\mu_{R_3} - a.e.$ on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ under the composition mapping $F_2 \circ F_1 : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$.

From theorem 3.1, if measurable Heine-Borel property P_1 holds $\mu_{R_1} - a.e.$ on any compact Borel set $A_1 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ where $A_1 = \{ \forall \mathcal{A} \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1}) : P_1(\mathcal{A})$ is true $\}$ has positive measure i.e. $\mu_{R_1}(A_1) > 0$, then P_1 also holds $\mu_{R_2} - a.e.$ on any compact Borel set $F_1(A_1) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $F_1(A_1) = \{ \forall F_1(\mathcal{A}) \subset (M_2, \tau_2, 2, \mu_{R_2}) : P_1(F_1(\mathcal{A}))$ is true $\}$ has positive measure i.e. $\mu_{R_2}(F_1(A_1)) > 0$ under the measurable diffeomorphism and Radon measure-invariant map $F_1 : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$. Similarly, if the property P_1 holds $\mu_{R_2} - a.e.$ on any compact Borel set $F_1(A_1) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ where $F_1(A_1) = \{ \forall F_1(\mathcal{A}) \subset (M_2, \tau_2, \Sigma_2, \mu_{R_2}) : P_1(F_1(\mathcal{A}))$ is true $\}$ has positive measure i.e. $\mu_{R_2}(F_1(\ A_1)) > 0$, then the property P_1 also holds $\mu_{R_3} - a.e.$ on any compact Borel set $F_2(F_1 A_1)) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ where $F_2(F_1(A_1) = \{ \forall\ F_2(F_1\ (\mathcal{A})) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3}) : P_1(F_2(F_1(\mathcal{A}))$ is true $\}$ has positive measure i.e. $\mu_{R_3}(F_2(F_1(A_1))) > 0$ under measurable diffeomorphism and Radon measure-invariant map $F_2 : (M_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$. Since $F_2(F_1(A_1) = (F_2 \circ F_1)(A_1)$, we have $(F_2 \circ F_1)(A_1) = \{ \forall (F_2 \circ F_1)(\mathcal{A}) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3}) :$ P$(F_2 \circ F_1(\mathcal{A}))$ is true$\}$ has positive measure i.e. $\mu_{R_3}(F_2 \circ F_1(A_1)) > 0$.

Since F_1 and F_2 are measurable diffeomorphisms and Radon measure-invariant maps, $F_2 \circ F_1 : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ is also measurable diffeomorphism and Radon measure-invariant map.

Also, for every $F_2 \circ F_1 : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ there exists an inverse map $(F_2 \circ F_1)^{-1} : (M_3, \tau_3, \Sigma_3, \mu_{R_3}) \longrightarrow (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ such that if P_1 holds $\mu_{R_3} - a.e.$ on any compact Borel set $F_2 \circ F_1(A_1) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ where $F_2 \circ F_1(A_1) = \{ \forall F_2 \circ F_1(\mathcal{A}) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3}) : P_1(F_2 \circ F_1(\mathcal{A}))$ is true $\}$ has positive measure i.e. $\mu_{R_3}(F_2 \circ F_1(A_1)) > 0$, then P_1 also holds $\mu_{R_1} - a.e.$ on any compact Borel set $(F_2 \circ F_1)^{-1}(A_1) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ where $(F_2 \circ F_1)^{-1}(A_1) = \{ \forall (F_2 \circ F_1)^{-1}(\mathcal{A})) \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1}) : P_1((F_2 \circ F_1)^{-1}(\mathcal{A})))$ is true $\}$ has positive measure i.e. $\mu_{R_1}((F_2 \circ F_1)^{-1}(A_1)\ ) > 0$.

Therefore, if P_1 holds $\mu_{R_1} - a.e.$ $\forall$ $A_1 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_1 also holds $\mu_{R_3} - a.e.$ $\forall F_2 \circ F_1(A_1) \subset (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ under measurable diffeomorphism and Radon measure-invariant map $F_2 \circ F_1 : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$.

Hence, the composition map $F_2 \circ F_1$ preserves measurable Heine-Borel property.

■

Theorem 3.6:

Let $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$, $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ be Radon measure manifolds endowed with measurable Lindeloff property. Let $F_1 : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_2, \tau_2, \Sigma_2, \mu_{R_2})$ and $F_2 : (M_2, \tau_2, \Sigma_2, \mu_{R_2}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$ be measurable diffeomorphism and Radon measure-invariant maps. Then if measurable Heine-Borel property say P_1 holds $\mu_{R_1} - a.e.$ on $A_1 \subset (M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then P_1 also holds $\mu_{R_3} - a.e.$ on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ under the composition mapping $F_2 \circ F_1 : (M_1, \tau_1, \Sigma_1, \mu_{R_1}) \longrightarrow (M_3, \tau_3, \Sigma_3, \mu_{R_3})$.

Proof: Proof is similar to theorem 3.5.

Similarly, we can show that the other measurable compactness properties remain invariant under the composition of measurable diffeomorphism and Radon measure-invariant maps.

In the above results, we have shown that, if the property P holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then it holds $\mu_{R_2} - a.e.$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ under measurable diffeomorphism and Radon measure-invariant function F_1 and if P holds $\mu_{R_2} - a.e.$ on $(M_2, \tau_2, \Sigma_2, \mu_{R_2})$ then it holds $\mu_{R_3} - a.e.$ on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ under measurable diffeomorphism and Radon measure-invariant function F_2. If the property P holds $\mu_{R_1} - a.e.$ on $(M_1, \tau_1, \Sigma_1, \mu_{R_1})$ then it also holds $\mu_{R_3} - a.e.$ on $(M_3, \tau_3, \Sigma_3, \mu_{R_3})$ under the composition $F_2 \circ F_1$.

By continuing this process, one can show that any $(M_i, \tau_i, \Sigma_i, \mu_{R_i})$ can be related to any $(M_j, \tau_j, \Sigma_j, \mu_{R_j})$ by composition of two measurable diffeomorphisms and Radon measure-invariant functions $F_i \circ F_j$.

Thus, if there exists $F_1 : M_1 \longrightarrow M_2$, $F_2 : M_2 \longrightarrow M_3$,...., $F_n : M_{n-1} \longrightarrow M_n$, then the composition of such functions $F_1, F_2, \ldots, F_n$ form a group structure on the non-empty set of Radon measure manifolds $(M_1, M_2, \ldots, M_n)$ with measurable Heine-Borel property.

That is, if G = $\{ F_1, F_2, \ldots, F_n \}$ is a non-empty set of $F_1, F_2, \ldots, F_n$ and if $\mathcal{M} = \{ M_1, M_2, \ldots, M_n \}$ is a set of Radon measure manifolds with measurable Heine-Borel property then G forms a group structure on $\mathcal{M}$.

The following result is introduced and proved by S. C. P. Halakatti and we conduct a study on it.

Theorem 3.7:

Let $\mathcal{M} = \{ \mathrm{M}_1, \mathrm{M}_2, \ldots, \mathrm{M}_n \}$ be a non-empty set of Radon measure manifolds and G = $\{ \mathrm{F}_1, \mathrm{F}_2, \ldots, \mathrm{F}_n \}$ be the set of measurable diffeomorphisms and Radon measure-invariant functions on $\mathcal{M}$. Then G forms a group structure on $\mathcal{M}$ under the composition of measurable diffeomorphisms and Radon measure-invariant functions F_i and F_j , $1 \leq i, j \leq n$.

Proof: Let $\mathcal{M}$ = { $M_1, M_2, \ldots, M_n$} be the set of Radon measure manifolds with measurable Heine-Borel property and let G = { $F_1, F_2, \ldots, F_n$ } be the set of measurable diffeomorphisms and Radon measure-invariant functions on $\mathcal{M}$.

In theorem 3.5, it is shown that if $F_1: M_1 \longrightarrow M_2$ and $F_2: M_2 \longrightarrow M_3$ are measurable diffeomorphisms and Radon measure-invariant functions, then $F_2 \circ F_1: M_1 \longrightarrow M_3$ and $F_1 \circ F_2: M_3 \longrightarrow M_1$ are also measurable diffeomorphisms and Radon measure-invariant functions.

We show that G forms a group structure on $\mathcal{M}$.

(i) If $F_i, F_j \in$ G then $F_i \circ F_j \in$ G [By theorem 3.5]

(ii) If $F_i, F_j, F_l \in$ G then $(F_i \circ F_j) \circ F_l = F_i \circ (F_j \circ F_l)$

Let $F_i \circ F_j = F_R$ and $F_j \circ F_l = F_S$

Then, $F_R \circ F_l = F_i \circ F_S$ [By theorem 3.5]

(iii) For every $F_i \in$ G there exists an inverse map $F_i^{-1} \in$ G such that

$F_i \circ F_i^{-1} = F_i^{-1} \circ F_i = id.$ [By theorem 3.1, since both F_i and F_i^{-1} are measurable diffeomorphisms and Radon measure-invariant functions]

(iv) For any $F_i \in$ G there exists an identity map *id*: $F_i \longrightarrow F_i \in$ G such that

$id \circ F_i = F_i \circ$ id $= F_i$ holds , where *id* ∈ G. [By theorem 3.5]

Therefore, G = { $F_1, F_2, \ldots F_i, id, \ldots, F_n$ } forms a group structure on $\mathcal{M}$ under the composition of measurable diffeomorphisms and Radon measure-invariant functions.

Therefore, (G, ∘) forms a group under the composition of measurable diffeomorphisms and Radon measure-invariant functions $F_1, F_2, \ldots, F_n$.

■

IV. CONCLUSION

Measurable compactness properties like measurable Heine-Borel property, measurable countably compact, measurable Lindeloff and measurable paracompactness on Radon measure manifolds also remain invariant under the composition of finite number of measurable diffeomorphisms and Radon measure-invariant functions. Further, we have seen that the set of all such measurable diffeomorphisms which are Radon measure-invariant functions form a group structure on $\mathcal{M}$. This leads us to study group structures and geometrical structures on $\mathcal{M}$ which generate a Network structure ($\mathcal{M}$, G,∘) on $\mathcal{M}$.

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