

# New parametric Measures of Entropy

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**Abstract-** Two new measures of parametric entropy have been obtained which are generalizations of Shannon's Kapur's, Bose Einstein, Fermi-Dirac, and Havrda-Charvat's measures of Entropy. We have also examined its concavity property and some special cases.

**Key words-** Measure of Entropy

**Subject classification-94A**

## Introduction

Shannon (1948) [6] called the expression  $-\sum_{i=1}^n p_i \ln p_i$  as a measure of entropy, Where  $\sum_{i=1}^n p_i = 1, p_i \geq 0, i = 1 \dots n(1)$  For any probability distribution  $p = (p_1, p_2, \dots, p_n)$  a measure of entropy needs to satisfy for the following conditions.

- i) It should be a continuous function of  $p_1, p_2, \dots, p_n$ .
- ii) It should be permutationally symmetric function of  $p_1, p_2, \dots, p_n$ .
- iii) It should be maximum subject to  $\sum_{i=1}^n p_i = 1$  i.e. when  $p_1 = p_2 = p_3 \dots = p_n = \frac{1}{n}$
- iv) It should be always non-negative and its maximum value zero should occur for the n degenerate distribution.

$$D_1 = (1,0, \dots, 0), D_2 = (0,1,0 \dots 0), \dots \dots D_n = (0,0,0 \dots 1)$$

Then Shannon [6] obtained the function

$$H_0(P) = -\sum_{i=1}^n p_i \ln p_i(2)$$

For measuring the uncertainty or Entropy of the probability distribution.

After this foundation of development of information theory and its great application and introduction of other generalized measure of one parametric entropy by A.Renyi(1961)[5],Kapur [4], introduced generalized one parametric entropy in 1967.In 1994,Kapur [4] defined two-two parametric measure of entropy and corresponding directed divergence.

In this paper we have introduced some two and three parametric measures of entropy.

## New two parametric Measures of entropy

Consider the measure

$$H_b^a(P) = -\sum_{i=1}^n p_i \ln p_i + \frac{a}{b} \sum_{i=1}^n \left(1 + \frac{b}{a} p_i\right) \ln \left(1 + \frac{b}{a} p_i\right) - \frac{a}{b} \left(1 + \frac{b}{a}\right) \ln \left(1 + \frac{b}{a}\right);$$

$$a > 0, b \geq -1, a \neq b(3)$$

## Special cases-

### Case I-

When  $a \rightarrow 1$  we get the Kapur's [2] generalized parametric measure of entropy.

$$H_b(P) = -\sum_{i=1}^n p_i \ln p_i + \frac{1}{b} \sum_{i=1}^n (1 + b p_i) \ln(1 + b p_i) - \frac{1}{b} (1 + b) \ln(1 + b); \quad b \geq -1(4)$$

It is easily verified that,  $H_b(P) \rightarrow H_0(P)$  as  $b \rightarrow 0$ ,  $H_b(P)$  also includes the following two cases

### Case II-

When  $a \rightarrow 1$  and  $b \rightarrow 1$  we get, Bose – Einstein Entropy

$$H_1(P) = -\sum_{i=1}^n p_i \ln p_i + \sum_{i=1}^n (1 + p_i) \ln(1 + p_i) - 2 \ln 2 \quad (5)$$

### Case III-

When  $a \rightarrow 1, b \rightarrow -1$  we get, Fermi – Dirac Entropy

$$H_{-1}(P) = -\sum_{i=1}^n p_i \ln p_i - \sum_{i=1}^n (1 - p_i) \ln(1 - p_i) \quad (6)$$

The Measure of parametric Entropy due to Havrda-Charvat [1] (1967) is

$$H^\alpha(P) = \frac{1}{1 - \alpha} \left( \sum_{i=1}^n p_i^\alpha - 1 \right) \alpha \neq 1, \alpha > 0$$

It is easily verified that,  $H^\alpha(P) \rightarrow H_0(P)$  as  $\alpha \rightarrow 1(7)$

## New three parametric Measure of Entropy

Now we combine the two generalizations and get three parametric measure of entropy combining (2) and (3) we get,

$$H_{\frac{b}{a}}^{\alpha}(P) = \frac{1}{1-\alpha} \left[ \sum_{i=1}^n p_i^{\alpha} - \sum_{i=1}^n p_i \right] - \frac{a}{b} \cdot \frac{1}{1-\alpha} \left[ \left( \sum_{i=1}^n \left(1 + \frac{b}{a} p_i\right)^{\alpha} \right) - \sum_{i=1}^n \left(1 + \frac{b}{a} p_i\right) \right] + \frac{a}{b} \cdot \frac{1}{1-\alpha} \left[ \left(1 + \frac{b}{a}\right)^{\alpha} - \left(1 + \frac{b}{a}\right) \right]$$

Where  $a > 0, b \geq 1, a \neq b, \alpha \neq 1, \alpha > 0$  (8)

**Special cases-**

Case I: - This approaches to Kapur's measure [4], When  $a \rightarrow 1, \alpha \rightarrow 1$

Case II: - This approaches to Havrda-Chavat [1] measure, When  $\frac{b}{a} \rightarrow 0$

Case III: - This approaches to Fermi Dirac Entropy, When  $\alpha \rightarrow 1$  and  $\frac{b}{a} \rightarrow -1$

Case IV: - This approaches to Bose-Einstein Entropy, When  $\alpha \rightarrow 1, \frac{b}{a} \rightarrow 1, a \rightarrow 1$

Thus (8) includes all the five measures of Entropy as special but that does not clear that (8) is itself a measure of Entropy.

**Concavity-**

To examine whether (8) is concave or not we note that since both

$$\left. \begin{aligned} & \frac{1}{1-\alpha} \left( \sum_{i=1}^n p_i^{\alpha} - \sum_{i=1}^n p_i \right) \text{ and} \\ & \frac{1}{1-\alpha} \left( \sum_{i=1}^n \left(1 + \frac{b}{a} p_i\right)^{\alpha} - \sum_{i=1}^n \left(1 + \frac{b}{a} p_i\right) \right) \end{aligned} \right\} (9)$$

Is a concave function,

(8) Will be concave function if  $\frac{b}{a} < 0$  since the sum of two concave function and difference may or may not be concave,

To examine its concavity we consider the function

$$\begin{aligned} \phi(x) &= \frac{x^{\alpha}-x}{1-\alpha} - \frac{a}{b} \frac{(1+\frac{b}{a}x)^{\alpha} - (1+\frac{b}{a}x)}{1-\alpha} \quad (10) \\ \phi'(x) &= \frac{1}{1-\alpha} \left\{ (\alpha x^{\alpha-1} - 1) - \frac{b}{a} \left[ \alpha \left(1 + \frac{b}{a}x\right)^{\alpha-1} \cdot \frac{b}{a} - \frac{b}{a} \right] \right\} \\ \phi''(x) &= -\alpha x^{\alpha-2} + \frac{b}{a} \alpha \left(1 + \frac{b}{a}x\right)^{\alpha-2} \\ &= -\alpha \left[ x^{\alpha-2} - \frac{b}{a} \left(1 + \frac{b}{a}x\right)^{\alpha-2} \right] \quad (11) \end{aligned}$$

From (8) and (10)

$$H_{\frac{b}{a}}^{\alpha}(P) = \sum \phi(p_i) - \phi(1) \quad (12)$$

Therefore,  $H_{\frac{b}{a}}^{\alpha}(P)$  will be concave if,

$$x^{\alpha-2} \geq \frac{b}{a} \left(1 + \frac{b}{a}x\right)^{\alpha-2} \quad \text{When } 0 \leq x \leq 1 \quad (13)$$

Now we consider the three cases,

**Case I:** if  $\alpha > 2$  this requires

$$x \geq \left(\frac{b}{a}\right)^{\frac{1}{\alpha-2}} \left(1 + \frac{b}{a}x\right)^{\frac{\alpha-2}{\alpha-2}} \quad \text{Or}$$

$$\begin{aligned} x &\geq \left(\frac{b}{a}\right)^{\frac{1}{\alpha-2}} \left(1 + \frac{b}{a}x\right) \quad (14) \\ x &\geq \left(\frac{b}{a}\right)^{\frac{1}{\alpha-2}} + \left(\frac{b}{a}\right)^{\frac{\alpha-1}{\alpha-2}} x \end{aligned}$$

$$x \left[1 - \left(\frac{b}{a}\right)^{\frac{\alpha-1}{\alpha-2}}\right] > \left(\frac{b}{a}\right)^{\frac{1}{\alpha-2}} \quad (15)$$

This condition is not satisfied if  $\frac{b}{a} \geq 1$  or  $\frac{b}{a} < 1$  it requires

$$x \geq \frac{\left(\frac{b}{a}\right)^{\frac{1}{\alpha-2}}}{1 - \left(\frac{b}{a}\right)^{\frac{\alpha-1}{\alpha-2}}} \quad (16)$$

The R.H.S of (16) has a defined positive value and (16) will not be satisfied for the values of probability which values are less than this value.

Thus when  $\alpha > 2$ , (8) is not concave function except when  $\frac{b}{a} = 0$

**Case II** - if  $\alpha = 2$  (13) becomes

$$1 \geq \frac{b}{a} \quad (17)$$

So that when  $\alpha > 2$ ,  $H_{\frac{b}{a}}^{\alpha}(P)$  is concave whenever  $\frac{b}{a} \leq 1$  So that

$$\begin{aligned} H_{\frac{b}{a}}^2(P) &= \left(1 - \sum_{i=1}^n p_i^2\right) + \sum_{i=1}^n p_i \left(1 + \frac{b}{a} p_i\right) - \left(1 + \frac{b}{a}\right) \\ &= \left(1 - \frac{b}{a}\right) \left(1 - \sum_{i=1}^n p_i^2\right) \frac{b}{a} \leq 1 \quad (18) \end{aligned}$$

Is a measure of Entropy.

It is the positive multiple of Havrda Charvat [1] entropy of order 2

**Case III-** if  $\alpha < 2$  in this case (13)

$$\begin{aligned} \frac{1}{x^{\alpha-2}} &\geq \frac{b}{a} \frac{1}{\left(1 + \frac{b}{a}x\right)^{2-\alpha}} \quad \text{Or} \quad \frac{\left(1 + \frac{b}{a}x\right)^{2-\alpha}}{x^{2-\alpha}} \geq \left(\frac{b}{a}\right) \quad \text{or} \\ \frac{\left(1 + \frac{b}{a}x\right)}{x} &\geq \left(\frac{b}{a}\right)^{\frac{1}{2-\alpha}} \quad \text{Or} \quad \left(\frac{b}{a}\right)^{\frac{1}{(2-\alpha)}} - \left(\frac{b}{a}\right) \leq \frac{1}{x} \quad (19) \end{aligned}$$

Since  $\frac{1}{x}$  lies between 1 and  $\infty$  (19) will be satisfied if,

$$\left(\frac{b}{a}\right)^{\frac{1}{(2-\alpha)}} - \left(\frac{b}{a}\right) \leq 1 \quad \text{or} \quad \left(\frac{b}{a}\right) \left[\left(\frac{b}{a}\right)^{\frac{\alpha-1}{2-\alpha}} - 1\right] \leq 1 \quad (20)$$

We now consider two sub cases

**(a) Sub case** - When  $0 < \alpha < 1$ ,

$$\text{If } \frac{b}{a} > 1 \left(\frac{b}{a}\right)^{\frac{\alpha-1}{2-\alpha}} < 1,$$

And LHS of (20) is negative

If  $\left(\frac{b}{a}\right) < 1, \left(\frac{b}{a}\right)^{\frac{1}{2-\alpha}} < 1$  and  $\left(\frac{b}{a}\right)^{\frac{1}{2-\alpha}} - \frac{b}{a} < 1$  (20) is satisfied again thus when  $0 < \alpha < 1, H_{\frac{b}{a}}^{\alpha}(P)$  is a

concave function for all  $b \geq -1, a \neq b$  and  $a \rightarrow 1$

When  $\alpha = 1$ , it reduces to Kapur's measure [2] which is known to be concave.

**(b) Sub case:** -When  $1 < \alpha < 2$ , In this case, Let

$$\begin{aligned} f\left(\frac{b}{a}\right) &= \left(\frac{b}{a}\right)^{\frac{1}{2-\alpha}} - \left(\frac{b}{a}\right) - 1 = \frac{b}{a} \left[\left(\frac{b}{a}\right)^{\frac{\alpha-1}{2-\alpha}} - 1\right] - 1 \\ f(0) &= 1, f(1) = -1, f(\infty) = \infty \quad (22) \end{aligned}$$

$$f'\left(\frac{b}{a}\right) = \frac{1}{2-\alpha} \left(\frac{b}{a}\right)^{\frac{\alpha-1}{2-\alpha}} - 1 \quad (23)$$

$$f''\left(\frac{b}{a}\right) = \frac{\alpha-1}{2-\alpha} \left(\frac{b}{a}\right)^{\frac{2\alpha-3}{2-\alpha}} > 0 \quad (24)$$

So that  $f(b/a)$  is a convex function and its graphs is given below.

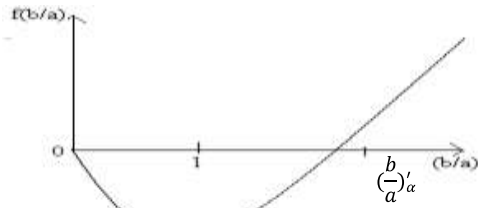


Fig.1

Thus for every value of  $\alpha$  between 1 and 2,  $f(b/a) < 0$  when  $(b/a) < (b/a)_\alpha$  Where  $(b/a)_\alpha > 1$  (25)

Thus the condition of concavity does not satisfy for all values of  $(b/a)$  when  $(b/a) < 0$  critical value  $(b/a)_\alpha$  for this value it is greater than unity and depends on  $\alpha$

$$\begin{aligned} \text{It is seen that, } \left(\frac{b}{a}\right)_\alpha &= \infty && \text{When } \alpha = 1 \\ &= \frac{1+\sqrt{5}}{2} && \text{When } \alpha = \frac{3}{2} \\ &= 1 && \text{When } \alpha = 2 \end{aligned} \quad (26)$$

$$\left(\frac{b}{a}\right)_\alpha^{\frac{1}{2-\alpha}} - \left(\frac{b}{a}\right)_\alpha - 1 = 0 \Rightarrow \left[\left(\frac{1}{2-\alpha}\right)\left(\frac{b}{a}\right)_\alpha^{\frac{\alpha-1}{2-\alpha}} - 1\right] db_\alpha' + dba\alpha' d\alpha + baa'12 - \alpha(12-\alpha)2lnbaa' = 0 \quad (27)$$

$$\text{If } a \rightarrow 1 \text{ we get, } \left[\left(\frac{1}{2-\alpha}\right)\left(b\right)_\alpha^{\frac{\alpha-1}{2-\alpha}} - 1\right] \frac{db_\alpha'}{d\alpha} + b_\alpha^{\frac{1}{2-\alpha}} \left(\frac{1}{2-\alpha}\right)^2 \ln(b)_\alpha = 0 \quad (28)$$

$$\text{Since, } b_\alpha' > 1, 1 < \alpha < 2, \text{ gives } \frac{db_\alpha'}{d\alpha} < 0 \quad (29)$$

So that  $b_\alpha'$  decreases with  $\alpha$  and  $\alpha$  goes for 1 to 2  $b_\alpha'$  decrease from  $\alpha$  to 1

Its graph is shown in figure 2

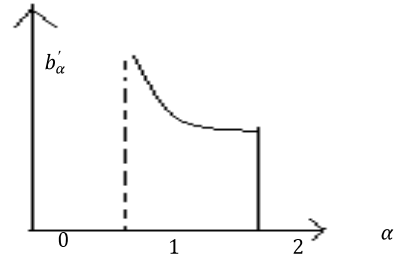


Fig.2

### Conclusion

$H_b^\alpha(P)$  is concave function of  $b$  for all values of  $b \geq -1$   $a \rightarrow 1, a \neq b$  if  $\alpha \leq 1$ . If  $\alpha$  lies between 1 and 2  $H_b^\alpha(P)$  in a concave function of  $b$  for and all values of  $b$  lying between -1 and  $b'_\alpha$ , where  $b'_\alpha$  is always  $\geq 1$  and decreases from  $\infty$  to unity as  $\alpha$  goes from 1 to 2. In any case  $H_b^\alpha(P)$  is always concave for all values of  $\alpha$  between 1 and 2 when  $b$  lies between -1 and 1 when  $\alpha > 0, H_b^\alpha(P)$  is not concave function of  $b$  for any value of  $b$  other than zero.

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