PURE RICKART MODULES AND THEIR GENERALIZATION

Ghaleb Ahmed
Department of Mathematics, College of Education for Pure Science/ Ibn-Al-Haitham
University of Baghdad, Iraq

ABSTRACT

Let R be a commutative ring with identity and M be an R-module. In this paper we introduce Pure Rickart modules and Pure π-Rickart modules as a generalization of Rickart modules and π-Rickart modules respectively. Also, Pure Rickart modules and Pure π-Rickart modules can be viewed as a generalization of PF-rings and GPF-rings respectively. Furthermore, Pure π-Rickart modules is a generalization of Pure Rickart modules. An R-module M is called Pure Rickart if for every $f \in \text{End}_R(M)$, $r_M(f)$ $=\text{Ker } f$ is a pure (in sense of Anderson and Fuller) submodule of M. An R-module M is called Pure π- Rickart if for every $f \in \text{End}_R(M)$, there exist a positive integer n such that $r_M(f^n) = \text{Ker} f^n$ is a pure (in sense of Anderson and Fuller) submodule of M. We show that several results of Rickart modules and π-Rickart modules can be extended to Pure Rickart modules and Pure π-Rickart modules for this general settings. Many results about these concepts are introduced and some relationships between these modules and other related modules are studied.

Key words: Pure Rickart modules, Pure π-Rickart modules, regular modules, Pure submodules.

1. INTRODUCTION

Throughout this paper R denotes a commutative ring with identity. For a right R-module M, $S = \text{End}_R(M)$ will denote the endomorphism ring of M ; thus M can be viewed as a left S- right R-bimodule. For $f \in \text{End}_R(M)$, the right annihilator of each element $f \in \text{End}_R$ in M is $r_M(Sf) = r_M(f) = \text{Ker } f = \{m \in M \mid f(m) = 0\}$. Following Rizvi and Roman [9], an R-module M is called *Rickart* if for every $f \in \text{End}_R(M)$, Ker f is a direct summand of M . The ring R is called *Rickart* if R is a Rickart as R-module, that is, the annihilator of any element is generated by an idempotent. It is obvious that the module R is Rickart as R-module if and only if the ring R is principally projective ring, that is a ring with the property that every principal ideal of

R is projective. An R-module M is called *π-Rickart* if for every $f \in \mathrm{End}_R(M)$, there exists a positive integer n such that $\mathrm{Ker}\, f^{\,n}$ is a direct summand of M [11].

The main goal of this research is to introduce and study the concept Pure Rickart modules as a generalization of Rickart modules as well as that of PF-rings . An R-module M is called Pure Rickart if for every $f \in \mathrm{End}_R(M)$, the kernel $r_M(f)$ =Ker f is a pure (in sense of Anderson and Fuller) submodule of M. Following Anderson and Fuller, a submodule N of an R-module M is called *pure* if $N \cap MI = NI$ for every ideal I of R [4]. An R-module M is called *regular* if every submodule of M is a pure submodule [12]. According this, our definition is different to the concept of purely Rickart modules ,which is introduced in [3]. An R-module M is called purely Rickart if for every $f \in \mathrm{End}_R(M)$, the kernel $r_M(f)$ =Ker f is a pure submodule (in sence of P.M.Cohn), where a submodule N of an R-module M is called *pure* (in sence of P.M.Cohn) if the sequence $0 \longrightarrow E \otimes N \longrightarrow E \otimes M$ is exact for all R-modules E [5]. It can easily see that the purity in the second definition implies to the first but not conversely. Clearly every purely Rickart module is Pure Rickart. However the converse is true in the projective modules because the concepts of purity conisides in this class of modules. Furthermore, we introduce the concept π-Pure Rickart modules as a generalization of Pure Rickart modules as well as that of GPF-rings . An R-module M is called *Pure π-Rickart* if for every $f \in \mathrm{End}_R(M)$, there exists a positive integer n such that $\mathrm{Ker}\, f^{\,n}$ is a pure (in sense of Anderson and Fuller) submodule of M.

The work consists of five sections. In Section two, we supply some examples and properties of Pure Rickart modules (Remarks and Examples 2.2).It is shown that Pure Rickart rings are exactly PF-rings (Definition 2.1). We show that the Pure Rickart property does not always transfer from a module to each of its submodules or conversely (Examples 2.4). But (Proposition 2.6) tell us that that the Pure Rickart property is inherited from a module to each of its direct summands. At this place, we have already observed that the direct sum of Pure Rickart modules need not be Pure Rickart (Remark 2.7). This observation lead us to introduce the concept of relatively Pure Rickart Modules to study under what conditions the direct sum of Pure Rickart Modules is again Pure Rickart. Let M and N be R-modules. M is called relatively Pure Rickart to N if for every $f \in \mathrm{Hom}_R(M, N)$, ker f is a pure (in sense of Anderson and

Fuller) submodule of M. Thus, as special case M is a Pure Rickart module if and only if M is relatively Pure Rickart to M. The concept of relatively Pure Rickart is introduced and investigated in section three. We give many results which are useful in this study on direct sums. In section four, we introduce and study the concept of Pure π-Rickart modules. We Prove that the Pure π-Rickart rings are precisely GPF-rings (Definition 4.1). Further, it is shown that some results of Pure Rickart modules can be extended to Pure π-Rickart modules for this general settings.

2. Pure Rickart Modules

In this section we study the concept of Pure Rickart modules. Basic facts of this type of modules are investigated. We begin with the following definition.

Definition 2.1. *An R-module M is called Pure Rickart if for every* $f \in \mathrm{End}_R(M)$, $\mathrm{Ker}\, f$ *is a pure (in sense of Anderson and Fuller) submodule of M. If M = R, then R is called Pure Rickart ring if R is Pure Rickart as R-module. In other words, R is Pure Rickart ring if* $ann_R(a)$ *of R is pure ideal of R for each* $a \in R$.

Since for every $a \in R$ and $f \in \mathrm{End}_R(R) \cong R$. We can define $f : R \longrightarrow R$ by $f(r) = ra$ for each $r \in R$. It follows that $\mathrm{Ker}\, f = \{r \in R \mid f(r) = 0\} = \{r \in R \mid ra = 0\} = ann_R(a)$. Therefore when $M = R$, the concept of Pure Rickart modules coincides with that of PF-rings. A ring R is called *a PF-ring if every principal ideal is a flat ideal in R*. Equivalently, R is *a PF-ring if and only if for every* $a \in R$, $ann_R(a)$ *is a pure ideal of* R [7]. Hence every Rickart ring is a Pure Rickart ring. A ring R is called *a PP-ring*, in other words, *Rickart ring if for each* $a \in R$, $ann_R(a)$ *is a direct summand of* R [8].

Remarks and Examples 2.2.

(1) Clearly that every regular module is Pure Rickart. But the converse is not true in general. For example, the module $\mathbb{Q}$ as $\mathbb{Z}$-module is Pure Rickart since every endomorphism of $\mathbb{Q}$ is either zero or an isomorphism. But $\mathbb{Q}$ is not regular.

(2) It is evident that every Rickart module is Pure Rickart but not conversely. For example, consider the ring $R = (\prod_{i=1}^{\infty} \mathbb{Z}_2) / (\oplus_{i=1}^{\infty} \mathbb{Z}_2)$. By [13, Examples 2.5], every principal ideal of the power series ring $R_1 = R[[x]]$ over R is flat. it

follows that $R_1$is a PF-ring. That is, R_1 is Pure Rickart as R_1-module. But R_1 is not Rickart by [13, Examples 2.5].

(3) If M is Pure simple R-module. Then M need not be Pure Rickart module, where an R-module M is called *Pure simple* if $M \neq \{0\}$ and it has no pure submodules except $\{0\}$ and M [6]. For example, the $\mathbb{Z}$-module $\mathbb{Z}_4$ is Pure simple. But it is not Pure Rickart since there exists an endomorphism $f : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_4$ defined by $f(m) = m2$ for each $m \in \mathbb{Z}_4$, then $\ker f = \{\overline{0}, \overline{2}\}$ is not a pure submodule in $\mathbb{Z}_4$.

(4) If M is Quasi-Dedekind R-module. Then M is Pure Rickart, where an R-module M is called *Quasi-Dedekind* if for every $f \in \mathrm{End}_R(M)$, $\ker f = 0$ [10]. The converse is not true. For example, the $\mathbb{Z}$-module $\mathbb{Z}_6$ is regular. Then $\mathbb{Z}_6$ is Pure Rickart module but it is not Quasi-Dedekind.

(5) If M is a Pure Rickart and Pure simple R-module. Then M is Quasi-Dedekind.
Proof. Let M be a Pure Rickart R-module and $0 \neq f \in \mathrm{End}_R(M)$. Then $\ker f$ is a pure submodule of M. Since M is pure simple and $f \neq 0$, then $\ker f = 0$. Hence M is Quasi-Dedekind.

(6) Homomorphic image of a Pure Rickart module may not be Pure Rickart. For example. Consider the natural homomorphism $f : \mathbb{Z} \longrightarrow \mathbb{Z}_4$. It is clear that the $\mathbb{Z}$-module $\mathbb{Z}$ is Quasi-Dedekind, implies that $\mathbb{Z}$ is Pure Rickart. But the $\mathbb{Z}$-module $\mathbb{Z}_4$ is not Pure Rickart.

(7) Every integral domain is Pure Rickart.
Proof. Since for each $a \in R$, $ann_R(a) = 0$ which is a pure ideal of R.

(8) The converse of Remark (7) is not true in general. For example the ring $\mathbb{Z}_6$ is regular and hence it is Pure Rickart. But $\mathbb{Z}_6$ is not integral domain.

Recall that an R-module M is called *indecomposable* if $M \neq \{0\}$ and it cannot be written as a direct sum of non-zero submodules. That is M and $\{0\}$ are the only direct summands of M [4].

We have the following

Proposition 2.3. *Let R be a ring. the following are equivalent*

(1) *R is integral domain.*

(2) *R is Pure Rickart and Pure simple.*

(3) *R is PP-ring and indecomposable.*

Proof. (1) $\Rightarrow$ (2) Let R be an integral domain. Suppose R is not Pure simple, then there exists an ideal I of R such that $0 \neq I \neq R$ and I is a pure ideal in R. Thus for every ideal J of R, $J \cap I = JI$. Let $0 \neq a \in I$, then $< a > \cap I = < a > I$. So $a = ab$ for some $b \in I$. Then $a(1 - b) = 0$, but $a \neq 0$. Since R is integral domain then $1 - b = 0$, so $1 = b \in I$. Therefore $I = R$ which is a contradiction. Hence R is pure simple. The rest is clear by Remark and Example 2.2(7).

(2) $\Rightarrow$ (3) Let R be a Pure Rickart, then for each $a \in R$, $ann_R(a)$ is a pure ideal of R. Since R is a Pure simple, then $ann_R(a) = 0$ or $ann_R(a) = R$. That is $ann_R(a)$ is a direct summand of R, so R is a PP-ring. Further, if R is not indecomposable, then there exists a direct summand I of R such that $0 \neq I \neq R$. It follows that I is a pure ideal in R which is a contradiction.

(3) $\Rightarrow$ (1) Let R be a PP-ring, then for each $0 \neq a \in R$, $ann_R(a)$ is a direct summand of R. Since R is indecomposable, then $ann_R(a) = 0$ or $ann_R(a) = R$. If $ann_R(a) = R$, then $a.1 = 0$. It follows that $a = 0$ which is a contradiction. Hence must be $ann_R(a) = 0$, therefore R is integral domain.

The next two examples show that the Pure Rickart property does not always transfer from a module to each of its submodules or conversely.

Examples 2.4.

(1) If M is a Pure Rickart R-module and N is any submodule of M. Then N need not be Pure Rickart. For example, let M denote $\mathbb{Z}$-module $\mathbb{Q}\oplus\mathbb{Z}_2$. By [9, Example 2.5], M is Rickart module, so it is Pure Rickart. Now consider the submodule $N = \mathbb{Z}\oplus\mathbb{Z}_2$ of M. Then N is not a Pure Rickart since there exists an endomorphism $f : N \longrightarrow N$ defined by $f(m, \bar{n}) = (0, \bar{m})$ where $m, n \in \mathbb{Z}$. Thus $ker\ f = \{(m,\bar{n}) \in \mathbb{Z}\oplus\mathbb{Z}_2 \mid f(m,\bar{n}) = (0,\bar{0})\} = \{(m,\bar{n}) \mid \bar{m} = \bar{0}\} = 2\mathbb{Z}\oplus\mathbb{Z}_2$ is not a pure submodule in $\mathbb{Z}\oplus\mathbb{Z}_2$, since $(2,\bar{0}) = (1,\bar{0})2 \in \ker f \cap (\mathbb{Z}\oplus\mathbb{Z}_2)2$. On the other hand $(2,\bar{0}) \notin (\ker f)2 = (2\mathbb{Z}\oplus\mathbb{Z}_2)2 = 4\mathbb{Z}\oplus\bar{0}$.

(2) If each proper submodule of an R-module M is a Pure Rickart. Then M may not be Pure Rickart. For example, the $\mathbb{Z}$-module $\mathbb{Z}_4$ in which every proper submodule is regular module, and hence it is Pure Rickart module. But $\mathbb{Z}_4$ is not Pure Rickart.

Now we recall known the following lemma from [12].

Lemma 2.5. *Let M be an R-module and A, N be submodules. Then we have*

(1) *If A is a pure submodule in N, and N is a pure submodule in M. Then A is a pure submodule in M.*
(2) *If A is a pure submodule in M and $A \subseteq N$. Then A is a pure submodule in N.*

The following proposition shows that the Pure Rickart property is inherited from a module to each of its direct summands.

Proposition 2.6. *Every direct summand of a Pure Rickart module is Pure Rickart.*

Proof. Let M be a Pure Rickart R-module and A be a direct summand of M, then $M = A \oplus B$ for some submodule B of M. Let $f \in \mathrm{End}_R(A)$, then we have the following $M = A \oplus B \xrightarrow{\rho} A \xrightarrow{f} A \xrightarrow{i} M$, where ρ is the natural projection map of M onto A and i is the inclusion map. Say $= i f \rho$, then $g \in End_R(M)$. Therefore ker g is a pure submodule in M. We claim that ker g = ker $f \oplus B$. To show this, let $m \in$ ker $f + B$, $m = x + y$ where $x \in$ ker f and $y \in B$. Then $(m) = g(x + y) = (i f \rho)(x + y) = i(f(x)) = f(x) = 0$. Therefore $m \in$ ker g implies that ker $f + B \subseteq$ ker g. For the reverse inclusion, let $m \in$ ker $g \subseteq M = A \oplus B$. let $m = x + y$ where $x \in A$ and $y \in B$, then $g(m) = g(x + y) = (i f \rho)(x + y) = 0$. Thus $i(f(x)) = 0$ and hence $f(x) = 0$. That is $x \in$ ker f and since $y \in B$. Thus $m = x + y \in$ ker $f + B$ and hence ker $g \subseteq$ ker $f + B$. That is, ker g = ker $f + B$. Clearly ker $f \cap B = 0$. Therefore ker g = ker $f \oplus$ B and hence ker f is a pure submodule in ker g. But ker g is pure in M, then by lemma 2.5(1), ker f is pure in M. But A is containing ker f. Again by lemma 2.5(2), ker f is pure in A. Therefore A is a Pure Rickart R-module.

We end this section by the following obeservation

Remark 2.7. The direct sum of Pure Rickart modules may be not be Pure Rickart. For example, by Example 2.4(1) the module $\mathbb{Z} \oplus \mathbb{Z}_2$ as $\mathbb{Z}$-module is not Pure Rickart, while each of $\mathbb{Z}$ and $\mathbb{Z}_2$ is a Pure Rickart module.

3. Relatively Pure Rickart Modules

Remark 2.7 shows that a direct sum of Pure Rickart modules need not be Pure Rickart. In this section we define relatively Pure Rickart property in order to investigate when are direct sums of Pure Rickart modules also Pure Rickart.

Recall an R-module M is called *relatively Rickart to an R-module N* if for every homomorphism $f : M \longrightarrow N$, ker f is a direct summand of M [9]. In view of the above definition, an R-module M is Rickart if and only if M is relatively Rickart to M.

Definition 3.1. *Let M and N be R-modules. M is called relatively Pure Rickart to N if for every $f \in Hom_R(M, N)$, ker f is a pure (in sense of Anderson and Fuller) submodule of M.*

Thus, as special case, M is Pure Rickart if and only if M is relatively Pure Rickart to M.

Remarks and Examples 3.2.

(1) It is clear every relatively Rickart module is relatively Pure Rickart, but the converse is not true in general. For example, the module R_1 as R_1-module in Remark and Example 2.2(2), R_1 is Pure Rickart, then it is relatively Pure Rickart. But R_1 is not Rickart and hence R_1 is not relatively Rickart.

(2) Obviously every regular R-module M is relatively Pure Rickart to any R-module N.

(3) Let M and N be R-modules. If M is relatively Pure Rickart to N, then N need not be relatively Pure Rickart to M. For example, let $\mathbb{Z}_n$ and $\mathbb{Z}$ as $\mathbb{Z}$-modules. Then $\mathbb{Z}_n$ is relatively Pure Rickart to $\mathbb{Z}$ for each positive integer n greater than one, in fact $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}) = 0$. On the other hand, $\mathbb{Z}$ is not relatively Pure

Rickart to $\mathbb{Z}_n$, since there exists the natural homomorphism $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_n)$ defined by $f(m) = \bar{m}$ for each $m \in \mathbb{Z}$. Thus $\ker f = n\mathbb{Z}$ is not pure in $\mathbb{Z}$.

(4) If M is a Pure Rickart R-module, then M need not be relatively Pure Rickart to an R-module N. For example, the $\mathbb{Z}$-module $\mathbb{Z}$ is Pure Rickart, But $\mathbb{Z}$ is not relatively Pure Rickart to $\mathbb{Z}_n$ as $\mathbb{Z}$-module for each positive integer $n > 1$.

(5) If M is relatively Pure Rickart to an R-module N, then M may not be Pure Rickart. For example, consider the $\mathbb{Z}$-module $\mathbb{Z}_4$ is relatively Pure Rickart to the $\mathbb{Z}$-module $\mathbb{Z}_3$, because $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_4, \mathbb{Z}_3) = 0$. But $\mathbb{Z}_4$ is not Pure Rickart.

(6) If M is a Pure simple or Quasi-Dedekind R-module, then M need not be relatively Pure Rickart to an R-module N. For example, the $\mathbb{Z}$-module $\mathbb{Z}$ is Pure simple and Quasi-Dedekind but not relatively Pure Rickart to the $\mathbb{Z}$-module $\mathbb{Z}_n$, for each positive integer $n > 1$.

Recall that an R-module M is said to have the *Pure Intersection Property* (briefly PIP) if the intersection of any two pure submodules is again pure [1].

Lemma 3.3. *Let M be an R-module. Then M has the PIP if and only if every pure submodule of M has the PIP* [1].

Lemma 3.4. *Let M be an R-module with the PIP, then for every decomposition* $M = A \oplus B$ *and for every* $f \in \mathrm{Hom}_R(A, B)$, $\ker f$ *is a pure submodule in M* [1].

Theorem 3.5. *Let M be an R-module with the PIP and* $A \oplus B$ *is a pure submodule of M. Then A is relatively Pure Rickart module to B.*

Proof. Assume that M has the PIP. Then by lemma 3.3, every pure submodule of M has the PIP. So $A \oplus B$ has the PIP. By lemma 3.4, for every $f \in \mathrm{Hom}_R(A, B)$, $\ker f$ is a pure submodule in $A \oplus B$. But $\ker f \subseteq A$ and A is direct summand in $A \oplus B$, so A is pure in $A \oplus B$. Therefore by lemma 2.5(1), $\ker f$ is a pure submodule in A. Hence A is relatively Pure Rickart to B.

As an immediate consequences we have

Corollary 3.6. *Let M and N be R-modules. If M⊕N has the PIP, then M is relatively Pure Rickart to N.*

Corollary 3.7. *Let M be R-module. If M ⊕ M has the PIP, then M is Pure Rickart module.*

Remark 3.8. The converse of Corollary 3.6 is not true in general. For example, the module $\mathbb{Z}_2$ as $\mathbb{Z}$-module is regular. Then it is relatively Pure Rickart for any R-module N. Let $N = \mathbb{Z}_4$ as $\mathbb{Z}$-module and M be denote $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ as $\mathbb{Z}$-module. We show that M does not have the PIP. Let $A = \bar{0} \oplus \mathbb{Z}_4$, and $B = (\bar{1},\bar{1})\mathbb{Z}$ be the submodule generated by $(\bar{1},\bar{1})$. It is easy to see that B is a direct summand of M. Then A and B are both pure submodules in M. But $A \cap B = \{(\bar{0},\bar{0}),(\bar{0},\bar{2})\}$ is not a pure submodule in M, since $(\bar{0},\bar{1})2 = (\bar{0},\bar{2}) \in (A \cap B) \cap M2$. On the other hand, $(A \cap B)2 = \{\bar{0},\bar{0}\}$. That is, $(A \cap B) \cap M2 \neq (A \cap B)2$.

Our next results on relatively Pure Rickart modules will be useful in this study on direct sums.

Theorem 3.9. *Let M and N be R-modules. The following statements are equivalent*

(1) *M is relatively Pure Rickart to N.*
(2) *For every direct summand A of M and any submodule B of N, A is relatively Pure Rickart to B.*

Proof. (1) ⇒(2) Assume M is relatively Pure Rickart to N. Let A be a direct summand of M and B is any submodule in N. Let $f \in \mathrm{Hom}_R(A, B)$. Consider the following $M = A \oplus B \xrightarrow{\rho} A \xrightarrow{f} B \xrightarrow{i} N$ for a submodule H of M where ρ is the natural projection map of M onto A and i is the inclusion map. Say $g = i f \rho \in \mathrm{Hom}_R(M, N)$. This implies that $\ker g$ is a pure submodule in M. Next, we show that $\ker g = \ker f \oplus H$. Let $m \in \ker f + H$, $m = x + y$ where $x \in \ker f$ and $y \in H$. Then $g(m) = g(x + y) = (i f \rho)(x + y) = (i f)(x) = i(f(x)) = f(x) = 0$. Thus $m \in \ker g$, implies that $\ker f + H \subseteq \ker g$. For the reverse inclusion, let $m \in \ker g \subseteq M = A \oplus H$, $m = x + y$ where $x \in A$ and $y \in H$. Then $0 = (m) = g(x + y) = (i f \rho)(x + y) = i(f(x)) = f(x)$.

Therefore $x\in$ ker f and since $y \in H$, then $m = x + y \in$ ker $f + H$. Hence ker g= ker $f + H$. Clearly, ker $f \cap H = 0$, it follows that ker g = ker $f \oplus H$. Thus ker f is a direct summand in ker g. Then ker f is pure submodule in ker g. But ker g is a pure submodule in M, then by lemma 2.5(1), ker f is a pure submodule in M. But A is containing ker f, so by lemma 2.5(2), ker f is a pure submodule in A. Therefore A is relatively Pure Rickart to B.

(2) $\Rightarrow$(1) It is clear by taking $A = M$ and $B = N$.

Proposition 3.10. *Let* $\{M_i\}_{i\in\Lambda}$ *be a family of R-modules where* $\Lambda = \{1,2,\ldots,n\}$ *and N be an R-module. The following statements are equivalent*

(1) *If N has the PIP, then N is relatively Pure Rickart to* $\oplus_{i=1}^{n} M_i$.

(2) *N is relatively Pure Rickart to* M_i *for all* $i = 1,2,\ldots,n$.

Proof. (1) $\Rightarrow$(2) It is clear from Theorem 3.9.

(2) $\Rightarrow$ (1) Assume that N is relatively Pure Rickart to M_i for all $i = 1,2,\ldots,n$ and N has the PIP. To show N is relatively Pure Rickart to $\oplus_{i=1}^{n} M_i$, let $f \in \mathrm{Hom}_R(N, \oplus_{i=1}^{n} M_i)$ and $\rho_i : \oplus_{i=1}^{n} M_i \longrightarrow M_i$ be the natural projection map of $\oplus_{i=1}^{n} M_i$ onto M_i for all $i = 1,2,\ldots,n$. Let us consider the following $N \xrightarrow{f} \oplus_{i=1}^{n} M_i \xrightarrow{\rho} M_i$. It is evident that $\mathrm{Im} f = \sum_{i=1}^{n} \mathrm{Im}\, \rho_i f$. Hence $f = (\rho_1 f, \rho_2 f, \ldots, \rho_n f) = (\rho_i f)_{i\in\Lambda}$. Then ker f = $\mathrm{Ker}(\sum_{i=1}^{n} \mathrm{Im}\, \rho_i f) = ann_N(\sum_{i=1}^{n} \rho_i f) = \bigcap_{i=1}^{n} ann_N(\rho_i f) = \bigcap_{i=1}^{n} Ker(\rho_i f)$. But $\rho_i f \in \mathrm{Hom}_R(N, M_i)$ and N is relatively Pure Rickart to M_i. Thus ker $\rho_i f$ is a pure submodule in N. Because N has the PIP, therefore ker $f = \bigcap_{i=1}^{n} Ker(\rho_i f)$ is a pure submodule in N, and hence N is relatively Pure Rickart to $\oplus_{i=1}^{n} M_i$

As an immediat result we have the following

Corollary 3.11. *Let* $\{M_i\}_{i\in\Lambda}$ *be a family of R-modules where* $\Lambda = \{1,2,\ldots,n\}$. *Then the following are equivalent*

(1) *If* M_j *has the PIP for all* $j = 1,2,\ldots,n$, *then* M_j *is relatively Pure Rickart to* $\oplus_{i=1}^{n} M_i$.

(2) M_j *is relatively Pure Rickart to* M_i *for all* $i = 1,2,\ldots,n$.

We end this section by the following two results

Proposition 3.12. *Let R be a ring. The following statements are equivalent*

(1) $\oplus_{\Lambda} R$ *is a Pure Rickart R-module for any index set* Λ.

(2) *All projective R-modules are Pure Rickart modules.*

(3) *All free R-modules are Pure Rickart modules.*

Proof. (1) $\Rightarrow$ (2) Let M be a projective R-module then there exists a free R-module F and an epimorphism $f : F \longrightarrow M$. Since $F \cong \oplus_{\Lambda} R$ for some index set Λ. We have the following short exact sequence $0 \longrightarrow \ker f \xrightarrow{i} \oplus_{\Lambda} R \xrightarrow{f} M$. But M is a projective then the sequence splits. Thus $\oplus_{\Lambda} R \cong \ker f \oplus M$. Because $\oplus_{\Lambda} R$ is a Pure Rickart module, therefore by Proposition 2.6, M is Pure Rickart module.

(2) $\Rightarrow$ (1) It is clear and (1) $\Leftrightarrow$ (3) Similar proof of (2) $\Leftrightarrow$ (1).

Proposition 3.13. *Let R be a ring. The following statements are equivalent*

(1) *R is regular ring.*

(2) *All R-modules are regular.*

(3) *All R-modules are relatively Pure Rickart to any R-module.*

(4) *All R-modules have the PIP.*

(5) *All injective R-modules have the PIP.*

(6) *All injective R-modules are regular.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) it follows by [1, Theorem 1.12].

(1) $\Rightarrow$ (3) It is Clear.

(3) $\Rightarrow$ (1) Let I be an ideal of R. Since all R-modules are relatively Pure Rickart to any R-module.Then the R-module R is relatively Pure Rickart to the module R / I as R-module. Because there exists the natural homomorphism $\pi : R \longrightarrow R / I$. Therefore $\ker \pi = I$ is a pure ideal of R, implies that R is regular.

4. Pure π-Rickart Modules

In this section we introduce the concept of Pure π-Rickart modules. Some basic properties of this type of modules are investigated. We show that Pure π-Rickart rings are precisely GPF-rings. First, we give the following definition.

Definition 4.1. *An R-module M is called Pure π-Rickart if for every $f \in End_R(M)$, there exists a positive integer n* such that $\ker f^n$ *is a pure (in sense of Anderson and Fuller) submodule of M. If M = R, then R is called Pure π-Rickart ring if R is Pure π-Rickart as R-module*. In other words, *R is Pure π-Rickart ring if for every $a \in R$, there exists a positive integer n* such that $ann_R(a^n)$ *is a pure ideal of R.*

Since for every $a \in R$ and $f \in End_R(R) \cong R$. We can define $f: R \longrightarrow R$ by $f(r) = ra$ for each $r \in R$.It follows that $\text{Ker} f = \{r \in R \mid f(r) = 0\} = \{r \in R \mid ra = 0\} = ann_R(a)$ and hence $\ker f^n = ann_R(a^n)$. Therefore when $M = R$, the concept of Pure π-Rickart modules coincides with that of GPF-rings. A ring R is called *GPF-ring* if for every $a \in R$, there exists a positive integer n such that $ann_R(a^n)$ is a pure ideal of R [2]. Hence every GPP-ring (and hence PP-ring) is Pure π-Rickart. A ring R is called *GPP-ring*, if for every $a \in R$, there exists a positive integer n such that $ann_R(a^n)$ is a direct summand of R [8]. Further, since every PF-ring is GPF-ring, then PF-rings are Pure π-Rickart.

Remarks and Examples 4.2.

(1) It is evedint that every Pure Rickart module (and hence every regular module) is Pure π-Rickart, but the converse is not true in general. For example, consider the $\mathbb{Z}$-module $\mathbb{Z}_4$. It is not hard to see that for every $f \in End_{\mathbb{Z}}(\mathbb{Z}_4)$, there exists a positive integer n such that $\ker f^n$ is a pure submodule in $\mathbb{Z}_4$. Then $\mathbb{Z}_4$ is a Pure π-Rickart, but it is not Pure Rickart by Remarks and Examples 2.2(3). Moreover, one can easily see that the $\mathbb{Z}$-module $\mathbb{Z}_n$ is Pure π-Rickart for each positive integer n.

(2) Obviously that every π-Rickart module is Pure π-Rickart but not conversely. Since as we mentioned that π-Rickart rings and Pure π-Rickart rings are

precisely GPP-rings and GPF-rings respectively. It is well- known that GPF-ring need not be GPP-ring. So Pure π-Rickart modules need not be π-Rickart.

We give the following result

Proposition 4.3. *Let R be a ring such that the set of all nilpotent elements* $L(R) = 0$. *the following are equivalent*

(1) *R is integral domain.*

(2) *R is Pure π-Rickart and Pure simple*.

(3) *R is GPP-ring and indecomposable*.

Proof. (1) $\Rightarrow$ (2) Let R be an integral domain. Then by Proposition 2.3, R is Pure simple and Pure Rickart, and hence it is Pure π-Rickart .

(2) $\Rightarrow$ (3) Let R be a Pure π-Rickart. Then for every $a \in R$, there exists a positive integer n such that $ann_R(a^n)$ is a pure ideal of R. Since R is Pure simple, then $ann_R(a^n) = 0$ or $ann_R(a^n) = R$. That is $ann_R(a^n)$ is a direct summand of R, so R is a GPP-ring. Further, if R is not indecomposable, then there exists a direct summand I of R such that $0 \neq I \neq R$. It follows that I is a pure ideal in R which is a contradiction.

(3) $\Rightarrow$ (1) Let R be a GPP-ring, then for every $0 \neq a \in R$, there exists a positive integer n such that $ann_R(a^n)$ is a direct summand of R. Since R is indecomposable, then $ann_R(a^n) = 0$ or $ann_R(a^n) = R$. If $ann_R(a^n) = R$, then $a^n.1 = 0$. It follows that $a^n = 0$, but $L(R) = 0$. So $a = 0$, which is a contradiction. Hence must be $ann_R(a^n) =$ 0, implies that $ann_R(a) = 0$. Therefore R is integral domain.

Proposition 4.4. *Every direct summand of Pure π-Rickart module is Pure -Rickart.*

Proof. By similar proof of Proposition 2.6.

Proposition 4.5. *Let R be a ring. The following statements are equivalent*

(1) $\oplus_{\Lambda} R$ *is a Pure π-Rickart R-module for any index set* Λ.

(2) *All projective R-modules are Pure π-Rickart modules*.

(3) *All free R-modules are Pure π-Rickart modules*.

Proof. By similar argument of Proposition 3.12.

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