Anti - Fuzzy Ideals in Ci–Algebras

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Y. B. Jun , K. J. Lee and S. Z. Song [1] introduced the concept of fuzzy ideal in BE – algebra in 2008 – 09. Here we have studied the concept of anti - fuzzy ideal in CI – algebra and obtained several results including characteristic Property.

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§.1. PRELIMINARIES

Definition (1.1):- A system $(X; *, 1)$ consisting of a non –empty set X , a binary operation $*$ and a fixed element 1 , is called a CI – algebra [2] if the following conditions are satisfied :

1. (CI 1) $x * x = 1$
2. (CI 2) $1 * x = x$
3. (CI 3) $x * (y * z) = y * (x * z)$

for all $x, y , z \in X$.

Definition(1.2):- A non – empty subset I of a CI – algebra X is called an ideal [1] of X if

(1) $x \in X$ and $a \in I \Rightarrow x * a \in I$;
(2) $x \in X$ and $a , b \in I \Rightarrow (a * (b * x)) * x \in I$.

Lemma (1.3) *:* - In a CI – algebra following results are true:

(1) $x * ((x * y) * y) = 1$
(2) $(x * y) * 1 = (x * 1) * (y * 1)$
(3) $1 \leq x$ imply $x = 1$

for all $x, y \in X$.

Lemma (1.4) : - (i) Every ideal I of X contains 1.

(ii) If I is an ideal of X then $(a * x) * x \in I$ for all $a \in I$ and $x \in X$

(iii) If I_1 and I_2 are ideals of X then so is $I_1 \cap I_2$.

Now we mention some results which appear in [4].

Theorem (1.5):- Let $(X; *, 1)$ be a system consisting of a non –empty set X , a binary operation $*$ and a fixed element 1. Let $Y = X \times X$. For $u = (x_1, x_2)$, $v = (y_1, y_2)$ a binary operation "$\odot$ "is defined in Y as

$$u \odot v = (x_1 * y_1, x_2 * y_2)$$

Then $(Y; \odot, (1, 1))$ is a CI – algebra iff $(X; *, 1)$ is a CI –algebra .

Theorem (1.6):- Let A and B be subsets of a CI – algebra X .

Then A x B is an ideal of $Y = X \otimes X$ iff A and B are ideals of X .

§ 2. FUZZY IDEALS

Here we discuss definitions and results of fuzzy ideals in a CI – algebra similar to the definition and results given by Jun , Lee and Song [1]

for a BE - algebra.

Definition (2.1):- Let $(X; *, 1)$ be a CI – algebra and let μ be a fuzzy set in X . Then μ is called a fuzzy ideal of X if it satisfies the following conditions:

$$(\forall\ x , y \in X)\ (\mu\ (x * y) \geq \mu(y)), \tag{2.1}$$

$$(\forall\ x , y, z \in X)\ (\mu\ ((x * (y * z)) * z) \geq \min\ \{\mu\ (x) , \mu(y)\}) \tag{2.2}$$

The following characteristic property of a fuzzy ideal can be proved as appears in [1].

Theorem (2.2):- Let μ be a fuzzy set in a CI – algebra $(X; *, 1)$ and let

$$U(\mu ; \alpha) = \{ x \in X: \mu(x) \geq \alpha\} \text{ where } \alpha \in [0, 1]. \tag{2.3}$$

Then μ is a fuzzy ideal of X iff

$$(\forall\ \alpha \in [0, 1])\ (U\ (\mu\ ; \alpha) \neq \phi \Rightarrow U\ (\mu ; \alpha) \text{ is an ideal of } X). \tag{2.4}$$

Some elementary properties of a fuzzy ideal are noted below:

Proposition (2.3):- Let μ be a fuzzy ideal of X.

Then (a) $\mu(1) \geq \mu(x)$ for all $x \in X$

(b) $\mu((x * y) * y) \geq \mu(x)$ for all $x, y \in X$

(c) $x, y \in X$ and $x \leq y \Rightarrow \mu(x) \leq \mu(y)$.

Proposition (2.4):- Let μ_1 and μ_2 be fuzzy ideals of X and let $\mu = \mu_1 \cap \mu_2$. Then μ is a fuzzy ideal of X

Proof : For $\alpha \in [0 , 1]$, we have

$$\begin{aligned} U\ (\mu ; \alpha) &= \{ x \in X : \mu(x) \geq \alpha\} \\ &= \{ x \in X : \mu_1 (x) \geq \alpha\} \cap \{ x \in X ; \mu_2 (x) \geq \alpha\} \\ &= U\ (\mu_1 ; \alpha) \cap U\ (\mu_2 ; \alpha). \end{aligned}$$

Since $U\ (\mu_1 : \alpha)$ and $U\ (\mu_2 : \alpha)$ are ideals in X, $U\ (\mu ; \alpha)$ is an ideal in X. Using theorem (2.2) we see that μ is a fuzzy ideal of X.

§ 3. ANTI - FUZZY IDEALS IN CI - ALGEBRAS

On the basis of definition given in § 2 the concept of anti – fuzzy ideal can be developed.

Definition (3.1):- A fuzzy set μ on a CI – algebra (X; $*$, 1) is called an anti - fuzzy ideal if it satisfies:

$$(\forall\ x , y \in X)\ (\mu\ (x * y) \le \mu(y)) \tag{3.1}$$

$$(\forall\ x , y, z \in X)\ (\mu\ ((x * (y * z)) * z) \le \max\ \{\mu\ (x) , \mu(y)\}) \tag{3.2}$$

Now we obtain a necessary and sufficient condition for a fuzzy set to be an anti- fuzzy ideal.

Theorem (3.2):- Let μ be a fuzzy set on X and for every $\alpha \in [\,0, 1]$, let

$$V\ (\mu\ ; \alpha) = \{\ x \in X: \mu(x) \le\ \alpha\}. \tag{3.3}$$

Then μ is an anti -fuzzy ideal of X iff

$$(\forall\ \alpha \in [0, 1])\ (V\ (\mu\ ; \alpha) \ne \phi \Rightarrow V\ (\mu\ ; \alpha) \text{ is an ideal of } X). \tag{3.4}$$

Proof : Suppose μ is an anti- fuzzy ideal of X. Let $\alpha \in [0, 1]$ be such that $V\ (\mu\ ; \alpha) \ne \phi$. Let x, y $\in$ X be such that y $\in V\ (\mu\ ; \alpha)$. Then $\mu\ (y) \le \alpha$. So $\mu(x * y) \le \mu\ (y) \le \alpha \Rightarrow x * y \in V\ (\mu\ ; \alpha)$. Again let x $\in$ X and a, b $\in V\ (\mu\ ; \alpha)$. Then $\mu\ (a) \le \alpha$ and $\mu\ (b) \le \alpha$. So (3 . 2) implies that

$$\mu\ ((a * (b * x)) * x) \le\ \max\ \{\ \mu\ (a), \mu\ (b)\} \le \alpha.$$

This implies that $(a * (b * x)) * x \in V\ (\mu\ ; \alpha)$.

Hence $V\ (\mu\ ; \alpha)$ is an ideal of X.

Conversely, suppose μ satisfies condition (3.4). Let $\mu\ (a * b) > \mu\ (b)$ for some a, b $\in$ X. Then $\mu\ (a * b) > \alpha_o > \mu\ (b)$ where $\alpha_o = ½\ [\mu\ (a * b) + \mu\ (b)]$. This means that $a * b \notin V\ (\mu\ ; \alpha_o)$ and $b \in V\ (\mu\ ; \alpha_o)$.. This contradicts the fact that $V\ (\mu\ ; \alpha)$ is an ideal of X.

Let a, b , c $\in$ X be such that

$$\mu\ ((a * (b * c)\ ) * c) > \max\ \{\mu\ (a)\ ,\ \mu\ (b)\ \}.$$

We put $\beta_o = ½\ [\mu\ ((a * (b * c)) *,c)\ + \max\ \{\ \mu\ (a)\ , \mu\ (b)\}]$.

Then $\mu\ ((a * (b * c))\ * c) >\ \beta_o > \max\ \{\ \mu\ (a)\ , \mu\ (b)\}$.

This gives $(a * (b * c)) * c \notin V\ (\mu\ ; \beta_o)$ for a, b $\in V\ (\mu\ ; \beta_o)$.

This is a contradiction. This proves that μ is an anti fuzzy ideal of X.

Theorem (3.3):- Let μ_1 and μ_2 be anti– fuzzy ideals of X and let $\mu = \mu_1 \cap\ \mu_2$. Then μ is an anti - fuzzy ideal.

Proof : We have , for any $\alpha \in [\ 0\ , 1]$,

$$\begin{aligned} V\ (\mu\ ; \alpha) &= \{\ x \in X : \mu(x) \le \alpha\} \\ &= \{\ x \in X : \mu_1\ (x) \le \alpha\} \cap \{\ x \in X\ ; \mu_2\ (x) \le \alpha\} \\ &= V\ (\mu_1 : \alpha) \cap V\ (\mu_2 : \alpha) \end{aligned}$$

Since $V(\mu_1 : \alpha)$ and $V\ (\mu_2 : \alpha)$ are ideals in X , $V\ (\mu\ ;\ \alpha)$ is an ideal of X.

Hence μ is an anti - fuzzy ideal of X.

Theorem (3.4):- Let μ be a fuzzy set defined on a CI – algebra X. Then μ is a fuzzy ideal of X iff $\nu = (1 - \mu)$ is an anti - fuzzy ideal of X.

Proof : Under the notations (3.3) and (3.4), for any $\alpha \in [0, 1]$,

we see that

$$\begin{aligned} V(\nu ; \alpha) &= \{ x \in X : \nu(x) \leq \alpha \} \\ &= \{ x \in X : 1 - \mu(x) \leq \alpha \} \\ &= \{ x \in X : 1 - \alpha \leq \mu(x) \} \\ &= \{ x \in X : \mu(x) \geq 1 - \alpha \} \\ &= U(\mu ; 1 - \alpha) \end{aligned}$$

The above identity implies that μ is a fuzzy ideal of X iff ν is an anti - fuzzy ideal of X.

Lemma (3.5):- If μ is an anti- fuzzy ideal of X then

$$\mu(1) \leq \mu(x) \text{ for all } x \in X.$$

Proof : For $x \in X$, we have

$$\mu(x * x) \leq \mu(x)$$

i.e., $\mu(1) \leq \mu(x)$.

Proposition (3.6) :- If μ is an anti - fuzzy ideal of X then

$$(\forall\, x, y \in X)\ (\mu((x * y) * y) \leq \mu(x))$$

Proof : We take y =1 and z = y in (3.2) , we get

$$\mu((x * y) * y) = \mu((x * (1 * y)) * y) \leq \max\{\mu(x), \mu(1)\}.$$

Then using lemma (3.5) we have

$$\mu((x * y) * y) \leq \mu(x).$$

Corollary (3. 7) :- If μ is an anti fuzzy ideal of X then

$x \leq y \Rightarrow \mu(y) \leq \mu(x)$.

Proof : We $x \leq y \Rightarrow x * y = 1$

Now $\mu(y) = \mu(1 * y) = \mu((x * y) * y) \leq \mu(x)$.

§ 4. ANTI - FUZZY IDEALS IN CARTESIAN PRODUCT ALGEBRA

Now we establish some results for anti - fuzzy ideals on Cartesion product of CI – algebras.

Theorem (4.1):- Let μ be a fuzzy set on a CI – algebra X and let Y = X x X. Let μ_1, μ_2, μ_3 be fuzzy sets on Y defined as

$$\mu_1(x, y) = \mu(x)$$

$$\mu_2(x, y) = \mu(y)$$

$$\mu_3 (x , y) = \max \{\mu(x) , \mu(y)\}$$

Then (a) μ_1 is an anti - fuzzy ideal of Y iff μ is an anti fuzzy ideal of X.

(b) μ_2 is an anti - fuzzy ideal of Y iff μ is an anti - fuzzy ideal of X.

(c) μ_3 is an anti - fuzzy ideal of Y iff μ is an anti -fuzzy ideal of X.

Proof :- For any real $\alpha \in [0, 1]$, let

$$V (\mu ; \alpha) = \{x \in X: \mu(x) \leq \alpha\};$$

$$V_1 (\mu_1 ; \alpha) = \{(x, y) \in Y : \mu_1 (x , y) = \mu(x) \leq \alpha\};$$

$$V_2 (\mu_2 ; \alpha) = \{(x, y) \in Y : \mu_2 (x , y) = \mu(y) \leq \alpha\};$$

and $V_3 (\mu_3 ; \alpha) = \{(x, y) \in Y : \mu_3 (x , y)) \leq \alpha\};$

Then we see that $V_1 (\mu_1 ; \alpha) = V (\mu; \alpha) \text{ x } X$

$$V_2 (\mu_2 ; \alpha) = X \text{ x } V (\mu; \alpha)$$

$$V_3 (\mu_3 ; \alpha) = V (\mu, \alpha) \text{ x } V (\mu,\alpha)$$

Now using theorem (1.6) we see that

(i) $V_1 (\mu_1 ; \alpha)$ is an ideal in Y iff $V (\mu; \alpha)$ is an ideal in X ;
(ii) $V_2 (\mu_2 ; \alpha)$ is an ideal in Y iff $V (\mu; \alpha)$ is an ideal in X ;
(iii) $V_3 (\mu_3 ; \alpha)$ is an ideal in Y iff $V (\mu; \alpha)$ is an ideal in X.

For all real $\alpha \in [0, 1]$

Now using theorem (3.2) we get the result.

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