Stability of Impulsive Functional Differential Equations via Lyapunov Functionals

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Abstract

In this paper, we consider the impulsive stabilization problems for a class of impulsive functional differential equation of the form

$$\begin{cases} x'(t) = f(t, x_t), & t \geq t_0 \\ \Delta x = I_k(t, (x_t{}^-)), & t = t_k, k \in Z^+ \end{cases}$$

Our method is based on the application of the Liapunov second method together with Liapunov functionals.

Key Words: stability, impulsive differential equation, Liapunov functional.

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1. Introduction

System of differential equations with impulse effect are an adequate apparatus for mathematical simulation of a number of processes and phenomena in science and technology. Recently the number of publications dedicated to their investigation for stability grows constantly and has taken shape of a developed theory presented in monographs [1-3]. Systems of functional differential equations have been much less studied.

In the present paper Liapunov's direct method with Liapunov functional is proposed for the discussion of problems on stability of impulsive differential equations for system of functional equations with impulse effect.

2. Preliminaries

Consider the impulsive functional differential equation

$$\begin{cases} x'(t) = f(t, x_t), & t \neq t_k\ t \geq t_0 \\ \Delta x = I_k(t, (x_t{}^-)), & t = t_k, k \in Z^+ \end{cases} \qquad (1)$$

Where $f: J \times PC \to R^n, \Delta x = x(t) - x(t^-), t_0 < t_1 < \cdots t_k < t_{k+1} < \cdots$, With $t_k \to \infty$ as $k \to \infty$ and $I_k: J \times S(\rho) \to R^n$, where $J = [t_0, \infty), S(\rho) = \{x \in R: |x| < \rho\}. PC = PC([-\tau, 0], R^n)$ denotes the space of piecewise right continuous functions $\varphi: [-\tau, 0] \to R^n$ with sup-norm $\|\varphi\|_\infty = sup_{-\tau \leq s \leq 0}|\varphi(s)|$ and the norm $\|\varphi\|_2 = (\int_{-\tau}^{0}|\varphi(s)|^2 ds)^{1/2,}$ where τ is a positive constant, $\|.\|$ is a norm in $R^n. x_t \in PC$ is defined by $x_t(s) = x(t + s)$ for $-\tau \leq s \leq 0. x'(t)$ denotes the right-hand derivative of $x(t). Z^+$ is the set of all positive integers,

Let $f(t, 0) = 0$ and $J(0) = 0$, then $x(t) = 0$ is the zero solution of (1). Set $PC(\rho) = \{\varphi \in PC: \|\varphi\|_\infty < \rho\}, \forall \rho > 0$.

DEFINITION 1.1 Let σ be the initial time, $\forall\ \sigma \in R$, the zero solution of (1) is said to be

(a) stable if , for each $\sigma \geq t_0$ and $\varepsilon > 0$, there is a $\delta = \delta(\sigma, \varepsilon) > 0$ such that , for $\varphi \in PC(\delta)$, a solution $x(t, \sigma, \varphi)$ satisfies $|x(t, \sigma, \varphi)| < \varepsilon$ for $t \geq t_0$.

(b) uniformly stable if it is stable and δ in the definition of stability is independent of σ

(c) asymptotically stable if it is stable and, for each $t_0 \in R_+$, there is an $\eta = \eta(t_0) > 0$ such that, for $\varphi \in PC(\eta), x(t, \sigma, \varphi) \to 0 \ as \ t \to \infty$

(d) uniformly asymptotically stable if it is uniformly stable and there is an $\eta > 0$ and , for each $\varepsilon > 0$, a $T = T(\varepsilon) > 0$ such that , for $\varphi \in PC(\eta), |x(t, \sigma, \varphi)| < \varepsilon$ for $t \geq t_0 + T$

DEFINITION 1.2 A functional $V(t, \varphi): J \times PC(\rho) \to R_+$ belong to class $v_o(.)$ (a set of Liapunov like functional if) (a) V is continuous on $[t_{k-1}, t_k) \times PC(\rho)$ for each $k \in Z_+$, and for all $\varphi \in PC(\rho)$ and $k \in Z_+$, the limit $\lim_{(t,\varphi) \to (t_k^-, \varphi)} V(t, \varphi) = V(t_k^-, \varphi)$ exists.

(b) V is locally Lipchitzian in φ in each set in $PC(\rho)$ and $V(t, 0) = 0$

The set $\mathfrak{R}$ is defined by

$\mathfrak{R} = \{W \in C(R_+, R_+)$: strictly increasing and $W(0) = 0$

3. Main Results

Theorem 1. Assume that there exist $V_1, V_2 \in v_0(.)$ and $W_1, W_2, W_3, W_4 \in \mathfrak{R}$ such that

(i) $W_1|\varphi(0)| \leq V(t, \varphi) \leq W_2|\varphi(0)|$ where $V(t, \varphi) = V_1(t, \varphi) + V_2(t, \varphi)$

(ii) $for \ each \ k \in Z^+ \ and \ x \in S(\rho_1)$

$$\left|V(t_k, x + I_k(t_k, x)) - V(t_k^-, x)\right| \leq \beta_k V(t_k^-, x)$$

$$where \ \beta_k \geq 0 \ \ with \ \ \sum_{k=1}^{\infty} \beta_k < \infty$$

(iii) $aV'_1(t, x_t) + bV'_2(t, x_t) \leq -\lambda(t) W_3(\inf\{|x(s)|; t - h \leq s \leq t\})$

$$and \qquad pV'_1(t, x_t) + qV'_2(t, x_t) \leq 0$$

$$where \ a^2 + b^2 \neq 0, p^2 + q^2 \neq 0 \ and \quad \int_0^{\infty} \lambda(s) ds = \infty$$

then the zero solution is uniformly stable and asymptotically stable.

Proof Let $\beta = \prod_{k=1}^{\infty}(1 + \beta_k)$ then $\beta \in [1, \infty)$. For any $\sigma \geq t_0$ and $\varepsilon > 0 (\varepsilon < \rho_1)$. We may choose a $\delta = \delta(\varepsilon) > 0$ such that $\beta W_2(\delta) < W_1(\varepsilon)$.

Let $x(t) = x(t, t_0, \varphi)$ be solution of (1.1) where $\varphi \in PC_\delta$.

From (ii)

$$|V(t_k) - V(t_k^-)| \leq \beta_k V(t_k^-) \tag{1}$$

From (iii)

$$aV'_1(t, x_t) + bV'_2(t, x_t) \leq -\lambda(t) W_3(\inf\{|x(s)|; t - h \leq s \leq t\})$$

Integrating both sides from $\sigma \ to \ t \ (t > t_0)$, we have

$$aV_1(t) + bV_2(t) \leq aV_1(\sigma) + bV_2(\sigma) - \int_{\sigma}^{t} \lambda(s) W_3(\inf\{|x(u)|; s - h \leq u \leq s\}) \, ds + \sum_{\sigma \leq t_k \leq t} [V(t_k) - Vt_k^-)]$$

Now, $\sigma \geq t_0, \varepsilon > 0\ (\varepsilon < \rho_1), we\ define\ \varepsilon_1 = W_2^{-1}\left(\frac{W_1(\varepsilon)}{2}\right)$

$$aV_1(t) + bV_2(t) \leq aV_1(\sigma) + bV_2(\sigma) - W_3(\varepsilon_1)\int_{\sigma}^{t} \lambda(s)\,ds + \sum_{\sigma \leq t_k \leq t} [V(t_k) - Vt_k^{-})]$$

And so by using (1)

$$aV_1(t) + bV_2(t) \leq aV_1(\sigma) + bV_2(\sigma) + \sum_{\sigma \leq t_k \leq t} \beta_k V(t_k^{-})$$

By theorem 1.5.1 in [1], we see that for all $t > \sigma$

$$aV_1(t) + bV_2(t) \leq aV_1(\sigma) + bV_2(\sigma) \prod_{\sigma \leq t_k \leq t} (1 + \beta_k)$$

$$\leq [aV_1(\sigma) + bV_2(\sigma)]\,\beta$$

i.e. $aV_1(t) + bV_2(t) \leq \beta[aV_1(\sigma) + bV_2(\sigma)],\ t \geq \sigma$

Then $W_1|x(t)| \leq V(t) \leq \beta[aV_1(\sigma) + bV_2(\sigma)] \leq \beta W_2(\delta) < W_1(\varepsilon), t \geq \sigma$

and so the zero solution of (1) is uniform stable.

To prove asymptotic stability, for a given $t_0 \in R_+, and$ a fixed $0 < H_2 < H_1, take$

$\eta = \eta(t_0) = \delta(t_0, H_2) > 0$, where δ is that in the definition of stability and for a given $\varphi \in PC_\eta$, let $x(t) = x(t, t_0, \varphi)$be solution of (1). Suppose for contradiction that $x(t) \nrightarrow 0$ as $t \to \infty$. Then there is a sequence $\{T_i\}$ and an $\varepsilon_0 > 0$ with $T_i \to \infty$ and $|x(T_i)| > \varepsilon_0$. Define $\varepsilon_2 = W_2^{-1}\left(\frac{W_1(\varepsilon_0)}{2}\right), then\ there\ is\ a\ sequence\ \{s_i\}\ with\ s_i \to \infty$ and $|x(s_i)| < \varepsilon_2$. Otherwise there is an $S \geq t_0$ such that $|x(t)| \geq \varepsilon_2$ for $t \geq S\ and$

$$aV_1(t) + bV_2(t) \leq aV_1(S+t) + bV_2(S+t) - \int_{S+h}^{t} \lambda(s) W_3(\inf\{|x(s)|; t-h \leq s \leq t\})ds + \sum_{S+h \leq t_k \leq t} V(t_k) - V(t_k^{-})$$

$$\leq aV_1(S+t) + bV_2(S+t) - W_3(\varepsilon_2)\int_S^t \lambda(s)ds$$

$$\to -\infty \quad as \quad t \to \infty$$

Which is a contradiction and hence zero solution is asymptotic stable.

Theorem 2. Assume that there exist $V_1, V_2 \in v_0(.)$ and $W_1, W_2, W_3, W_4 \in \Re$ such that

(i) $W_1|\varphi(0)| \leq V(t,\varphi) \leq W_2|\varphi(0)|$ where $V(t,\varphi) = V_1(t,\varphi) + V_2(t,\varphi)$

(ii) $V\big(t_k, x + I_k(t_k, x)\big) - V(t_k^{-}, x) \leq \beta_k V(t_k^{-}, x), k \in Z^+, \beta_k \geq 0$

(iii) $aV'_1(t, x_t) + bV'_2(t, x_t) \leq -\lambda(t) W_3(\inf\{|x(s)|; t-h \leq s \leq t\})$
$and \quad pV'_1(t, x_t) + qV'_2(t, x_t) \leq 0$
Where $a^2 + b^2 \neq 0, p^2 + q^2 \neq 0\ and\ \lim_{S\to\infty}\int_t^{t+S} \lambda(s)ds = \infty$ uniformly in $t \in R_+$

Then the zero solution of (1) is uniformly stable and asymptotic stable.

Proof: Uniform Stability can be prove as Stability in Theorem 1.

For asymptotic stability,

Set $\eta = \delta(H_2)$ for a fixed $0 < H_2 < H_1$ and δ in the definition of uniform stability. For given $t_0 \in R_+, \varphi \in C_\eta$, let $x(t) = x(t, \sigma, \varphi)$ be a solution of (1). Let $\varepsilon > 0$ be given and take $\delta = \delta(\varepsilon) > 0$ of uniform stability. Define $\delta_1 = W_2{}^{-1}(\frac{W_1(\delta)}{2})$. Choose an $S = S(\varepsilon) > 0$ with

$$\int_t^{t+S} \lambda(s)ds > 2(|a|W_2(H_2) + |b|W_3(H_2))/W_4(\delta_1)$$

For $t \in R_+$ and an integer $N = N(\varepsilon) \geq 1$ with $N\mu(\delta_1)W_1(\delta)/2 > 2(|p|W_2(H_2) + |q|W_3(H_2))$

Define $T = T(\varepsilon) = N(S + 2h)$. Suppose, for contradiction, that $||x_t|| \geq \delta$ for $t_0 \leq t \leq t_0 + T$.

From the supposition , for $1 \leq i \leq N$, there is a

$$t_0 + (i-1)(S+2h) + h + S \leq T_i \leq t_0 + i(S+2h)$$

Such that $|x(T_i)| \geq \delta$. Thus, there is an $s_i < t_i < T_i$ with $|x(t_i)| = \delta_1$ and $|x(t)| > \delta_1$ for $t_i < t \leq T_i$. We obtain

$$pv_1\big(t_0 + i(S+2h)\big) + qv_2\big(t_0 + i(S+2h)\big) - (pv_1\big(t_0 + (i-1)(S+2h)\big) + qv_2(t_0 + (i-1)(S+2h)))$$

$$\leq pv_1(T_i) + qv_2(T_i) - (pv_1(t_i) + qv_2(t_i))$$

$$\leq -\mu(\delta_1)\big(v_1(T_i) - v_1(t_i)\big) \ \leq\ -\mu(\delta_1)W_1(\delta)/2$$

And

$$-2\big(|p|W_2(H_2) + |q|W_3(H_2)\big)$$

$$\leq\ pv_1\big(t_0 + N(S+2h)\big) + qv_2\big(t_0 + N(S+2h)\big) - (pv_1(t_0) + q(v_2(t_0))$$

$$= \sum_{i=1}^{N}(pv_1\big(t_0 + i(S+2h)\big) + qv_2(t_0 + i(S+2h))) - (pv_1\big(t_0 + (i-1)(S+2h)\big) + qv_2(t_0 + (i-1)(S+2h)))$$

$$\leq -N\mu(\delta_1)W_1(\delta)/2 \ <\ -2(|p|W_2(H_2) + |q|W_3(H_2)),$$

This inequality also holds true as per condition (ii)

Which is a contradiction.

Consequently $||x_{t'}|| < \delta$ for some $t_0 \leq t' \leq t_0 + T$ and $|x(t)| < \varepsilon$ for $t \geq t_0 + T$. This completes the proof.

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