Fuzzy Ideals in some specific BE – algebras

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Abstract

The concept of fuzzy ideals in BE – algebra have been introduced by Y. B. Jun, K. J. Lee and S. Z. Song in 2008-09. They investigated characteristic property of a fuzzy ideal and developed several properties. Here we study the concept of fuzzy ideals in Cartesian product of BE – algebra and the BE – algebra of all functions defined on a BE – algebra.

Key words: BE – algebra, Ideals, Fuzzy ideal, Cartesian product.

Mathematics Subject Classification: 06F35, 03G25, 08A30, 03B52

§.1. PRELIMINARIES :

Definition (1.1): A system $(X; *, 1)$ consisting of a non –empty set X , a binary operation $*$ and a fixed element 1 is called a BE – algebra ([2]) if the following conditions are satisfied :

1. (BE 1) $x * x = 1$
2. (BE 2) $x * 1 = 1$
3. (BE 3) $1 * x = x$
4. (BE 4) $x * (y * z) = y * (x * z)$

for all $x, y, z \in X$.

Definition(1.2): A non – empty subset I of a BE – algebra X is called an ideal ([4,5]) of X if

(1) $x \in X$ and $a \in I \Rightarrow x * a \in I$;

(2) $x \in X$ and $a, b \in I$

$\Rightarrow (a * (b * x)) * x \in I$.

Lemma (1.3) : In a BE – algebra following identities hold ([2]).

(1) $x * (y * x) = 1$
(2) $x * ((x * y) * y) = 1$

for all $x, y \in X$.

Lemma (1.4) : (i) Every ideal I of X contains 1.

(ii) If I is an ideal of X then $(a * x) * x \in I$ for all $a \in I$ and $x \in X$.

(iii) If I_1 and I_2 are ideals of X then so is $I_1 \cap I_2$.

Now we mention some results which appear in ([3]).

Theorem (1.5): Let $(X; *, 1)$ be a system consisting of a non –empty set X , a binary operation $*$ and a fixed element 1. Let $Y = X \times X$. For $u = (x_1, x_2)$, $v = (y_1, y_2)$ in Y, a binary operation $\odot$ is defined as

$$u \odot v = (x_1 * y_1,\ x_2 * y_2)$$

Then $(Y; \odot, (1, 1))$ is a BE – algebra iff $(X; *, 1)$ is a BE –algebra .

Theorem (1.6): Let A and B be subsets of a BE – algebra X . Then $A \times B$ is an ideal of $Y = X \times X$ iff A and B are ideals of X . In particular, $\{1\} \times A$ and $A \times \{1\}$ are ideals in Y for every ideal A of X.

§ 2. FUNCTION ALGEBRA OF BE – ALGEBRA:

We have the following result:

Theorem (2.1): Let $(X; *, 1)$ be a BE – algebra and let F(X) be the class of all functions $f : X \to X$. Let a binary operation o be defined in F(X) as follows:

For $f, g \in F(X)$ and $x \in X$,

$$(f \circ g)(x) = f(x) *, g(x).$$

Then $(F(X) ; o , \tilde{1})$ is a BE – algebra, where $\tilde{1}$ is defined as $\tilde{1}(x) = 1$ for all $x \in X$.

Proof : Let $f, g, h \in F(X)$. Then for $x \in X$, we have

(i) $(f \circ f)(x)$
$= f(x) * f(x)$
$= 1$
$= \tilde{1}(x)$

$\Rightarrow f \circ f = 1^{\sim}$;

(ii) $(f \circ 1^{\sim})(x)$
$= f(x) * 1^{\sim}(x)$
$= f(x) * 1$
$= 1$
$= 1^{\sim}(x)$
$\Rightarrow f \circ 1^{\sim} = 1^{\sim}$;

(iii) $(1^{\sim} \circ f)(x)$
$= 1^{\sim}(x) * f(x)$
$= f(x)$
$\Rightarrow 1^{\sim} \circ f = f$;

and (iv) $(f \circ (g \circ h))(x)$

$= f(x) * (g \circ h)(x)$

$= f(x) * (g(x) * h(x))$

$= g(x) * (f(x) * h(x))$

$= g(x) * (f \circ h)(x)$

$= (g \circ (f \circ h))(x)$

$\Rightarrow f \circ (g \circ h) = g \circ (f \circ h)$.

This prove that $(F(X); \circ, 1^{\sim})$ is a BE – algebra.

Corollary (2.2): If $(X; *, 1)$ is transitive or self – distributive then so is $(F(X); \circ, 1^{\sim})$.

Theorem (2.3): Let I be an ideal of X and let F(I) be the collection of all functions $f \in F(X)$ such that $f(x) \in I$ for all $x \in X$. Then F(I) is an ideal of F(X).

Proof : For $f \in F(X)$ and $g \in F(I)$ we have, $(f \circ g)(x) = f(x) * g(x) \in I$.

So $f \circ g \in F(I)$.

Again for $g, h \in F(I)$ and $f \in F(X)$, we have

$((g \circ (h \circ f)) \circ f)(x)$

$= (g \circ (h \circ f))(x) * f(x)$

$= (g(x) * ((h(x) * f(x))) * f(x) \in I$.
So $(g \circ (h \circ f)) \circ f \in F(I)$. Hence F(I) is an ideal of F(X).

Notation (2.4): For any set $A \subseteq X$, let F(A) denote the set of all functions $f \in F(X)$ such that $f(x) \in A$ for all $x \in X$.

§ 3. FUZZY IDEALS:

Here we discuss definitions and results of fuzzy ideals given by Jun , Lee and Song ([4]).

Definition (3.1): Let $(X; *, 1)$ be a BE – algebra and let μ be a fuzzy set in X . Then μ is called a fuzzy ideal of X if it satisfies the following conditions ([5]:

$(\forall x, y \in X)(\mu(x * y) \geq \mu(y))$, (3.1)
$(\forall x, y, z \in X)(\mu((x * (y * z)) * z) \geq \min\{\mu(x), \mu(y)\})$ (3.2)

The following characteristic property of a fuzzy ideal appears in ([5]).

Theorem (3.2): Let μ be a fuzzy set in a BE – algebra $(X; *, 1)$ and let

$U(\mu; \alpha) = \{x \in X: \mu(x) \geq \alpha\}$, where $\alpha \in [0, 1]$. (3.3)

Then μ is a fuzzy ideal of X iff
$(\forall \alpha \in [0, 1])(U(\mu; \alpha) \neq \phi \Rightarrow U(\mu; \alpha)$ is an ideal of X). (3.4)

Some elementary properties of a fuzzy ideal are noted below:

Proposition (3.3): Let μ be a fuzzy ideal of X. Then (a) $\mu(1) \geq \mu(x)$ for all $x \in X$
(b) $\mu((x * y) * y) \geq \mu(x)$ for all $x, y \in X$
(c) $x, y \in X$ and $x \leq y \Rightarrow \mu(x) \leq \mu(y)$.

Proposition (3.4): Let μ_1 and μ_2 be fuzzy ideals of X and let $\mu = \mu_1 \cap \mu_2$. Then μ is a fuzzy ideal of X

Proof : For $\alpha \in [0, 1]$, we have
$U(\mu; \alpha) = \{x \in X : \mu(x) \geq \alpha\}$
$= \{x \in X : \mu_1(x) \geq \alpha\} \cap \{x \in X ; \mu_2(x) \geq \alpha\}$
$= U(\mu_1; \alpha) \cap U(\mu_2; \alpha)$

Since $U(\mu_1; \alpha)$ and $U(\mu_2; \alpha)$ are ideals in X, $U(\mu; \alpha)$ is an ideal in X. Using theorem (3.2) we see that μ is a fuzzy ideal of X.

§ 4. FUZZY IDEALS IN CARTESIAN PRODUCT ALGEBRA:

Now we establish some results for fuzzy ideals on Cartesian product of BE – algebras.

Theorem (4.1): Let μ be a fuzzy set on a BE – algebra X and let $Y = X \times X$. Let μ_1, μ_2, μ_3 be fuzzy sets on Y defined as

$\mu_1(x, y) = \mu(x)$

$\mu_2(x, y) = \mu(y)$

μ_3 (x , y) = min {μ(x) , μ(y)}

Then (a) μ_1 is a fuzzy ideal of Y iff μ is a fuzzy ideal of X.

(b) μ_2 is a fuzzy ideal of Y iff μ is a fuzzy ideal of X.

(c) μ_3 is a fuzzy ideal of Y iff μ is a fuzzy ideal of X.

Proof :- For any real $\alpha \in [0, 1]$, let

$U(\mu ; \alpha) = \{x \in X: \mu(x) \geq \alpha\}$;

$U_1(\mu_1 ; \alpha) = \{(x, y) \in Y : \mu_1(x, y) = \mu(x) \geq \alpha\}$;

$U_2(\mu_2 ; \alpha) = \{(x, y) \in Y : \mu_2(x, y) = \mu(y) \geq \alpha\}$;

and $U_3(\mu_3 ; \alpha) = \{(x, y) \in Y : \mu_3(x, y) \geq \alpha\}$.

Then we see that

$U_1(\mu_1 ; \alpha) = U(\mu ; \alpha) \times Y$

$U_2(\mu_2 ; \alpha) = X \times U(\mu ; \alpha)$

$U_3(\mu_3 ; \alpha) = U(\mu ; \alpha) \times U(\mu ; \alpha)$

Now using theorem (1.6) we see that

(i) $U_1(\mu_1 ; \alpha)$ is an ideal in Y iff $U(\mu ; \alpha)$ is an ideal in X

(ii) $U_2(\mu_2 ; \alpha)$ is an ideal in Y iff $U(\mu ; \alpha)$ is an ideal in X

(iii) $U_3(\mu_3 ; \alpha)$ is an ideal in Y iff $U(\mu ; \alpha)$ and $U(\mu ; \alpha)$ are ideals in X.

for all real $\alpha \in [0, 1]$

Now using theorem (3.2) we get the result.

§ 5. FUZZY IDEALS IN FUNCTION ALGEBRA:

Definition (5.1): Let μ be a fuzzy set defined on a finite BE – algebra $(X ; *, 1)$. Let $(F(X) ; o , \tilde{1})$ be the BE – algebra discussed in theorem (2.1). We extend $\overline{\mu}$ on F (X) as

$$\overline{\mu}(f) = \min \{\mu(f(x)) : x \in X\}.$$

We prove the following result.

Lemma(5.2) : $F(U(\mu ; \alpha)) = U(\overline{\mu} ; \alpha)$ for every $\alpha \in [0, 1]$.

Proof : First of all we observe that for any $\alpha \in [0, 1]$,

$$U(\overline{\mu} ; \alpha) \neq \phi \Leftrightarrow U(\mu; \alpha) \neq \phi \qquad (5.1)$$

Let $U(\overline{\mu} ; \alpha) \neq \phi$ and $f \in U(\overline{\mu} ; \alpha)$

Then $\overline{\mu}(f) \geq \alpha$. So min $\{\mu(f(x)) : x \in X\} \geq \alpha$.

This implies that $f(x) \geq \alpha$

for some $x \in X$,

i.e. $f(x) \in U(\mu ; \alpha)$ and so

$U(\mu ; \alpha) \neq \phi$.

Again let $U(\mu ; \alpha) \neq \phi$ and $a \in U(\mu ; \alpha)$.

Then $\mu(a) \geq \alpha$. If we choose $f_a \in F(X)$ such that $f_a(x) = a$ for all $x \in X$, then $\overline{\mu}(f_a) = \min \{\mu(f_a(x)) : x \in X\} = \mu(a) \geq \alpha$, i. e. $f_a \in U(\overline{\mu} ; \alpha)$ and so $U(\overline{\mu} ; \alpha) \neq \phi$.

Now we see that

$f \in F(U(\mu ; \alpha))$

$\Rightarrow f(x) \in U(\mu ; \alpha)$ for all $x \in X$

$\Rightarrow \mu(f(x)) \geq \alpha$ for all $x \in X$

$\Rightarrow \min \{\mu(f(x)) : x \in X\} \geq \alpha$

$\Rightarrow \overline{\mu}(f) \geq \alpha$

$\Rightarrow f \in U(\overline{\mu} ; \alpha)$

So $F(U(\mu ; \alpha)) \subseteq U(\overline{\mu} ; \alpha)$ (5. 2)

Again

$f \in U(\overline{\mu} ; \alpha)$

$\Rightarrow \overline{\mu}(f) \geq \alpha$

$\Rightarrow \min \{\mu(f(x)) : x \in X\} \geq \alpha$

$\Rightarrow \mu(f(x)) \geq \alpha$ for all $x \in X$

$\Rightarrow f(x) \in U(\mu ; \alpha)$ for all $x \in X$

$\Rightarrow f \in F(U(\mu; \alpha))$

This gives $U(\overline{\mu};\alpha) \subseteq F(U(\overline{\mu};\alpha))$. (5. 3)

From (5.2) and (5.3) we get the result.

Corollary (5.3) : If $U(\mu;\alpha)$ is an ideal in X then $U(\overline{\mu};\alpha)$ is an ideal in F(X).

Proof : This follows from above lemma and theorem (2.3).

Theorem (5.4): If μ is a fuzzy ideal of X then so is $\overline{\mu}$ on F(X).

Proof : Let μ be a fuzzy ideal of X . Then for every $\alpha \in [0,1]$

$U(\mu;\alpha) \neq \phi \Rightarrow U(\mu;\alpha)$ is an ideal.

So $F(U(\mu;\alpha))$ is an ideal in F (X) by theorem (2.3).

From corollary (5.3), $U(\mu;\alpha)$ is an ideal in F(X) if $\alpha \in [0, 1]$ and $U(\overline{\mu};\alpha) \neq \phi$.

This proves that $\overline{\mu}$ is a fuzzy ideal of F(X).

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