Prime Cyclic Rings

Meram Munirathnam[#1] *And* Dr. D. Bharathi [#2]

[#1]*Adhoc Lecturer, Dept. of mathematics, RGUKT, R.K.Valley, Idupulapaya(vi),Vempalli (md), Kadapa(Dt),Andhra Pradesh, India, Pin:517330.*

[#2]*Assoc. Professor, Dept. of mathematics, S.V.University, Tirupati, Chittoor(dt), Andhra Pradesh, India, Pin:517502.*

Abstract:

Kleinfeld [1] proves that every prime cyclic ring where 2x=0 implies x=0 is commutative and associative that. In this Paper we improve on this result by showing that every prime cyclic ring is associative and commutative without assuming 2x=0 implies x=0.

Key Words: *Cyclic Ring, Prime Ring, Semi-prime Ring, Associative Ring and Commutative Ring.*

Introduction:

We define a cyclic ring to be a not necessarily associative ring R that satisfies the cyclic identity

$$x(yz) = y(zx) \qquad (1)$$

For all x,y,z in R.

A ring is called prime ring if every pair of ideals I and J of R, $IJ=0$ implies that either $I=0$ or $J=0$. R is called semiprime if for every ideal I of R, $I^2=0$ implies $I=0$. It is clear that prime implies semiprime.

Main Results:

LEMMA 1: Let R be a cyclic ring. Then the following identities hold:

(i) $(xy)(st)=(tx)(sy)$,

(ii) $(xy)((st)(uv))= (tx)((sy)(uv))$,

(iii) $(xy,st,uv)=0$,

(iv) $(x,y,st)(uv)=0$.

PROOF: (i) By cyclic identity (1), we have

$$\begin{aligned}(xy)(st) &= s(t(xy))\\ &= s(y(tx))\\ &= (tx)(sy).\end{aligned}$$

Therefore $(xy)(st)= (tx)(sy)$.

(ii) Using cyclic identity (1) and part (i) we obtain

$$\begin{aligned}(xy)((st)(uv)) &= (uv)((xy)(st))\\ &= (uv)((tx)(sy))\\ &= (tx)((sy)(uv)).\end{aligned}$$

(iii) Using cyclic identity (1) use parts (i) and (ii) we obtain

$$\begin{aligned}(xy)((st)(uv)) &= (xy)((vs)(ut)) \text{ (from part (i))}\\ &= (sx)((vy)(ut)) \text{ (from part(ii))}\\ &= (sx)(tv)(uy)) \text{ (from part (i))}\\ &= (vs)((tx)(uy)) \text{ (from part (ii))}\\ &= (vs)((yt)(ux)) \text{ (from part (i))}\\ &= (vs)(u(x(yt))) \text{ (from 1)}\\ &= u((x(yt))(vs)) \text{ (from 1)}\\ &= u(v(s(x(yt)))) \text{ (from 1)}\\ &= (s(x(yt)))(uv) \text{ (from 1)}\end{aligned}$$

$= ((yt)(sx))(uv)$ (from 1))

$= ((xy)(st))(uv)$ (from part (i))

(iv) We use part (i) and (ii) and (iii) and cyclic identity (1) to obtain

$((xy)(st))(uv) = (xy)((st)(uv))$ (from part (iii))

$= (xy)((vs)(ut))$ (from part (i))

$= (ys)((vx)(ut))$ (from part (ii))

$= (ys)((xt)(uv))$ (from part (i))

$= ((ys)(xt))(uv)$ (from part (iii))

$=((st)(xy))(uv)$ (from part (i))

$= (x(y(st)))(uv)$ (from (1))

Therefore $(x,y,st)(uv)=0$.♦

COROLLARY 1: Let R be a cyclic ring. Then R^2 is associative.

PROOF: From part (iii) of lemma (1), every element of R^2 is a finite sum of products of elements in R.♦

LEMMA 2: Let R be a cyclic ring.

(i) If R satisfies the identity (x,y,st)=0,then R also satisfies the identities [xy,st]=0 and (x,y,z)(st)=0.

(ii) if R is associative, then R satisfies the identity [x,y](st)=0.

PROOF: (i) let us assume that R satisfies the identity

$$(x,y,st)=0. \qquad (2)$$

Then $(xy)(st) = x(y(st))$ (from (2))

$= (st)(xy)$ (from (1))

That is $[xy,st]=0$.

Let $x,y,z,s,t \in R$ and using (1)

Consider $((xy)z)(st)$

$((xy)z)(st) = (xy)(z(st))$(from (2))

By applying (1) to this continuously we have

$= (z(st))(xy)$

$= x(y(z(st)))$

$= x((st)(yz))$

$= x((yz)(st))$

$= (x(yz))(st)$.

Therefore $(x,y,z)(st)=0$.

(ii) Since R is associative then $(x,y,st)=0$.

By Part (i), if R satisfies the identity $(x,y,st)=0$ then also satisfies $[xy,sy]=0$.

This implies $(xy)(st)=(st)(xy)$

Now use part (i) continuously gives

$= y((st)x)$

$= y(s(tx))$

$= y(x(st))$

$= (yx)(st)$.

Thus $[x,y](st)=0$.♦

LEMMA 3: Let R be a cyclic ring and let $I=\{x\in/xR^2=0\}$. Then I is an ideal of R.

PROOF: Let $x\in I$, $y,s,t\in R$, then by identity (1),we have $(xy)(st) = s(t(xy))$

$= s(x(ty))$

$= 0$.

Therefore I is a right ideal of R.

$(yx)(st) = s(t(yx))$

$= s(x(ty))$

$= 0.$

Therefore I is a left ideal of R.

Hence I is a ideal of R.♦

THEOREM 1: Let R be a prime cyclic ring then R is associative and commutative.

Proof: Let N= $I \cap R^2$.Clearly N is an ideal of R (from lemma (3))

Now $N^2 \subseteq NR^2 = 0$.

Since R is prime, N=0.

By lemma (1) part (iv) we have

$(x,y,st) \in N$ for all $x,y,s,t \in R$.

By lemma (2(i)), if R satisfies $(x,y,st)=0$ then $(x,y,z)(st)=0$.

Which gives $A.I=0$ for all $s,t \in I$

Since R is prime either A=0 or I=0.

If I =(0) then N=0, implies R is associative.

If A = (0) then R is associative.♦

References:

1. Kleinfeld, M. "Rings with x(yz)=y(zx)", Comm.Algebra.13(1995), 5085 -5093.