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Some Theorems on Subtraction Groups

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ABSTRACT:- *In this paper, some additional conditions relating to subtraction algebra, the so called subtraction semi group and subtraction group are introduced, and some theorems are investigated.*

KEY WORDS:- *subtraction algebra and subtraction group.*

Preliminaries

Definition;1.1:- A non-empty set ' X ' together with a binary operation " – '' is said to be a subtraction algebra if for all $a, b, c \in X$ the following conditions hold.

(SA.1) $a - (b - a) = a$

(SA.2) $a - (a - b) = b - (b - a)$

(SA.3) $(a - b) - c = (a - c) - b$

Example;1.1:- Let $X = \{0, \text{x}, \text{y}, 1\}$in which "–" is defined by

-	0	x	y	1
0	0	0	0	0
x	x	0	x	0
y	y	y	0	0
1	1	y	x	0

TABLE. 1

Then $(X, -)$ is a subtraction algebra.

In a subtraction algebra the following holds

(i) $a - 0 = a$ *and* $0 - a = 0$
(ii) $(a - b) - a = 0$
(iii) $(a - b) - b = a - b$
(iv) $(a - b) - (b - a) = a - b,$

In [3] it has been proved that in every subtraction algebra X there exists an element 0 such that $0 = a - a$ for all $a \in X$.

Definition;1.2:- A non-empty set ' X' together with a binary operation "–" and "**.**" is said to be subtraction semigroup if for all $a, b, c \in X$ the following conditions hold.

(SS.1) $(X, -)$ is a subtraction algebra.

(SS.2) $(X, -)$ is a semigroup.

(SS.3) $a(b - c) = ab - ac$ *and* $(a - b)c = ac - bc$.

Example;1.2:- Let$X = \{0, \text{x}, \text{y}, 1\}$ in which "–" and "**.**" are defined by

-	0	x	y	1
0	0	0	0	0
x	x	0	x	0
y	y	y	0	0
1	1	y	x	0

.	0	x	y	1
0	0	0	0	0
x	0	x	0	x
y	0	0	y	y
1	0	x	y	1

TABLE.1.2

Then $(X, -\ )$ is a subtraction semigroup.

Definition;1.3:- A non-empty set ' X ' together with a binary operation "–" and "**.**" is said to be subtraction group if the following conditions hold.

(SG.1) $(X, -, .)$ is a subtraction semigroup and

(SG.2) $X - \{0\}$ is a group with the multiplication inherited from X.

Theorem;1.1:- Let X_0 is a subtraction group and $X = X_0 - \{0\}$. Define $\eta: X_0 \to X_0$ by$\eta(x) = e - x$, where 'e' is the identity in$X$. Then

(T.1) $a\eta(a^{-1}b\eta(b^{-1}a) = a,$

(T.2) $a\eta(ab^{-1}\eta(ba^{-1}) = a,$

(T.3) $a\eta^2(a^{-1}b) = b\eta^2(b^{-1}a)$ and

(T.4) $a\eta(a^{-1}c)\eta[(\eta(a^{-1}c))^{-1}a^{-1}b] = a\eta(a^{-1}b)\eta[(\eta(a^{-1}b))^{-1}a^{-1}c] \quad \forall\, a, b, c \in X.$

Proof:- (T.1) $a\eta(a^{-1}b\eta(b^{-1}a) = a\eta(a^{-1}b(e - b^{-1}a)$

$$= a\eta(a^{-1}b - e)$$

$$= a[e - (a^{-1}b - e)]$$

$$= ae[\because \textit{by eqn.} (\text{SA}.1)]$$

$= a.$

(T.2) $a\eta(ab^{-1}\eta(ba^{-1}) = a\eta(ab^{-1}(e - ba^{-1})$

$$= a\eta(ab^{-1} - e)$$

$$= a[e - (ab^{-1} - e)]$$

$$= ae[\because \textit{by eqn} (\text{SA}.1)]$$

$$= a.$$

(T.3) $\text{L.H.S} = a\eta^2(a^{-1}b)$

$$= a\eta(e - a^{-1}b)$$
$$= a[e - (e - a^{-1}b)]$$
$$= ae - a(e - a^{-1}b)$$
$$= a - (a - b)]$$
$$R.H.S = b\eta^2(b^{-1}a)$$
$$= b\eta(e - b^{-1}a)$$
$$= b[e - (e - b^{-1}a)]$$
$$= b - b(e - b^{-1}a)$$
$$= b - (b - a)$$

By (SA.2) of definition 1.1, we get,

$$a\eta^2(a^{-1}b) = b\eta^2(b^{-1}a).$$

(T.4) $L.H.S = a\eta(a^{-1}c)\eta[(\eta(a^{-1}c))^{-1}a^{-1}b]$

$$= a\eta(a^{-1}c)\eta[((e - a^{-1}c))^{-1}a^{-1}b]$$
$$= a\eta(a^{-1}c)\eta[((e - a^{-1}c))^{-1}a^{-1}b]$$
$$= a\eta(a^{-1}c)\eta[(a(e - a^{-1}c))^{-1}b]$$
$$= a\eta(a^{-1}c)\eta[(a - c)^{-1}b]$$
$$= a\eta(a^{-1}c)[e - (a - c)^{-1}b]$$
$$= a(e - a^{-1}c)[e - (a - c)^{-1}b]$$
$$= (a - c)[e - (a - c)^{-1}b]$$

$= (a - c) - b.$

$$R.H.S = a\eta(a^{-1}b)\eta[(\eta(a^{-1}b))^{-1}a^{-1}c]$$
$$= a\eta(a^{-1}b)\eta[((e - a^{-1}b))^{-1}a^{-1}c]$$
$$= a\eta(a^{-1}b)\eta[(a(e - a^{-1}b))^{-1}c]$$
$$= a\eta(a^{-1}b)\eta[(a - b)^{-1}c]$$
$$= a\eta(a^{-1}b)[(e - (a - b)^{-1}c]$$
$$= a(e - a^{-1}b)[(e - (a - b)^{-1}c]$$
$$= (a - b)[(e - (a - b)^{-1}c]$$

$= (a - b) - c.$

By (SA.2) of definition 1.1,we get

$$a\eta(a^{-1}c)\eta[(\eta(a^{-1}c))^{-1}a^{-1}b]$$
$$= a\eta(a^{-1}b)\eta[(\eta(a^{-1}b))^{-1}a^{-1}c].$$

Theorem;1.2:- Let X is a group and X_0 is the corresponding group with 0. Suppose $\eta: X \to X$ has the properties (T.1), (T.3) and (T.4) described in Theorem 1.1.Then X_0 is a subtraction group if we define $a - b = a\eta(a^{-1}b), 0 - y = 0$ *and* $y - 0 = y$ *for all* $a, b \in X$.

Proof:-First we see (SA.1), (SA.2) and (SA.3) for all $a, b, c \in X_0$.

Let us first prove for $a, b, c \in X$.

(SA.1) $a - (b - a) = a - (b\eta(b^{-1}a))$

$$= a\eta(a^{-1}b\eta(b^{-1}a)$$
$$= a. [\because by\ eqn\ \text{T}.1]$$

(SA.2) $a - (a - b) = a - (a\eta(a^{-1}b))$

$$= a\eta[a^{-1}a\eta(a^{-1}b)]$$
$$= a\eta^2(a^{-1}b).$$

Similarly $b - (b - a) = b\eta^2(b^{-1}a)$.

Therefore $a - (a - b) = b - (b - a). [\because by\ eqn\ \text{T}.3]$

(SA.3) $(a - b) - c = a\eta(a^{-1}b) - c$

$$= a\eta(a^{-1}b)\eta[\left(a\eta(a^{-1}b)\right)^{-1}c]$$
$$= a\eta(a^{-1}b)\eta[(\eta(a^{-1}b))^{-1}a^{-1}c]$$

Similarly
$(a - c) - b = a\eta(a^{-1}c)\eta[(\eta(a^{-1}c))^{-1}a^{-1}b]$

Therefore $(a - b) - c = (a - c) - c[\because by\ eqn\ \text{T}.3]$

Now if one or more elements of a, b, c are equal to zero, an easy check reveals that the condition

(SA.1), (SA.2) and (SA.3) listed at the beginning of this proof are valid in this case too.

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