

Some Results on Divisibility of Hyper Groups

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Abstract: In this paper we establish some results on divisible hyper groups as a generalization of the usual hyper groups. We investigate some of their properties besides constructing some divisible hyper groups.

Key words: Hyper groups defined from groups, Divisibility canonical hyper group, Divisibility sub canonical hyper group, Divisibility sub hyper group.

2000 Mathematical Subject Classification: 20N20.

1 INTRODUCTION: The concept of hyper structure theory (Hyper compositional algebra) was brought out in 1934, by the French Mathematician Marty.F[4] at the eighth congress of Scandinavian Mathematicians, when he defined hyper groups and began to analyse their properties and applied them to groups.

Algebraic hyper structures represent a natural extension of classical algebraic structures. In classical algebraic structures, the composition of 2 elements is an element, while in an algebraic hyper structures the composition of 2 elements is a set. Because, of its extensive applications in many branches of mathematics and applied sciences it has now a days become a well established branch in algebraic theory. Further, people have developed the semi hyper groups which are the simplest algebraic hyper structures having closure and associative properties. The canonical hyper groups are the special type of hyper groups, which are found by Mitta [5] who is the first one studied them extensively. Divisible semi groups have long been studied by Tamura[7], Brown[1], Hildebrandt[2], Wasanawichit[9] and Triphop[8] for examples. Divisible abelian groups have been characterized in terms of Z-injectively[3]. A.Wasanawichit is the first author have studied the divisibility of some periodic semi groups(that is, semi groups whose elements have finite order) in [8].

In this paper, the notion of divisibility is defined extensively. Divisible semi hyper groups are defined and the divisibility of some hyper groups defined from groups will be investigated. First, we present the basic definitions.

2 PRELIMINARIES In this section, some of the basic definitions are summarized which are needed in sequel.

Definition2.1[6] Let H be a non-empty set and $P^*(H)$ is the set of all non-empty subsets of H.

Then a hyper operation \times on H is a mapping from $H \times H$ into $P^*(H)$. That is $\times: H \times H \rightarrow P^*(H)$.

Definition2.2[6] Let H be a non-empty set and \times be hyper operation on H. That is $\times: H \times H \rightarrow P^*(H)$, where $P^*(H) = P(H) - \emptyset$. Then the couple of (H, \times) is called hyper groupoid.

Definition2.3[6] A hyper groupoid (H, \times) is called a semi hyper group if for all a, b, c of H, $(a \times b) \times c = a \times (b \times c)$ which means that

$$\bigcup_{u \in a \times b} u \times c = \bigcup_{v \in b \times c} a \times v.$$

Definition2.4[6] A hyper groupoid (H, \times) is called a quasi hyper groupoid if for all a of H, $a \times H = H \times a = H$. This condition is called the reproduction axiom.

Definition2.5[6] A hyper groupoid (H, \times) which is both a semi hyper group and a quasi hyper group is called a hyper group.

Example2.6 Consider the Klein four-group $K_4 = \{e, a, b, c\}$. It is abelian and isomorphic to the dihedral group of order 4. It is also isomorphic to the direct sum $Z_2 \oplus Z_2$. Multiplication table for Klein four group is given by $a \times b = aPb$.

\cdot	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Now consider the subgroup $P = \{e, a, b\}$. Then the canonical hyper group (G, \times_p) constructed with P is represented in the following table.

\odot	e	a	b	c
e	e	a	b	c
a	a	$\{e, a, b\}$	$\{a, b, c\}$	$\{e, b, c\}$
b	b	$\{a, b, c\}$	$\{e, a, b\}$	$\{e, a, c\}$
c	c	$\{e, b, c\}$	$\{e, a, c\}$	$\{e, a, b\}$

From the above table (K_4, \times_p) is a semi hyper group. Now, $a \times_p K_4 = a \cdot \{e, a, b\} \cdot \{e, a, b, c\}$
 $= \{a.e.e\} \cup \{a.e.a\} \cup \{a.e.b\} \cup \{a.e.c\} \cup \{a.a.e\}$
 $\cup \{a.a.a\} \cup \{a.a.b\} \cup \{a.a.c\} \cup \{a.b.e\} \cup \{a.b.a\}$
 $\cup \{a.b.b\} \cup \{a.b.c\}$
 $= \{e, a, b, c\}$
 $= K_4$

$\Rightarrow K_4$ is a quasi hyper group and hence K_4 is a hyper group.

Definition2.7[6] A hyper groupoid (H, \times) is called a canonical hyper group. If the following properties are satisfied

- (i) for all x, y, z \in H, $x \times (y \times z) = (x \times y) \times z$
- (ii) for all x, y, z \in H, $x \times y = x \times y$
- (iii) there exists $e \in H$ such that $e \times x = x \forall x \in H$

(iv) for all $x \in H$ there exists a unique element $x' \in H$ such that $e \in x \times x'$

(v) for all $x, y, z \in H$, if $z \in x \times y$, then $y \in xz'$.

Example2.8[6] Consider the group $K_4 = \{e, a, b, c\}$. It is an abelian group $P = \{e, a\}$. Then the canonical hyper group (G, \times_p) constructed with P is represented in the following table

\times_p	e	a	b	c
E	e	a	b	c
A	a	$\{e, a\}$	$\{b, c\}$	$\{b, c\}$
B	b	$\{b, c\}$	$\{e, a\}$	$\{e, a\}$
C	c	$\{b, c\}$	$\{e, a\}$	$\{e, a\}$

Definition2.9[6] A subset K of a canonical hyper group H is called a sub canonical hyper group of H , if K forms a canonical hyper group with respect to the hyper operation on H .

Example2.10 Consider the example 2.8 satisfies the sub canonical hyper group conditions.

Definition2.11[6] Let (H, \times) be a hyper groupoid and K be a non-empty subset of H . Then (K, \times) is called a sub hyper group of H , if K itself forms a hyper operation in H .

Example2.12[6] Let $G = \{1, -1, i, -i\}$ be an abelian group under the multiplication. Let $P = \{1, i\} \subseteq G$. Then the hyper group (G, \times_p) is represented by the table $a \times_p b = aPb$

\times_p	1	-1	i	-i
1	1	-1	i	-i
-1	-1	$\{1, i\}$	$\{1, -i\}$	$\{-1, i\}$
i	i	$\{1, -i\}$	$\{-1, -i\}$	$\{1, i\}$
-i	-i	$\{-1, i\}$	$\{1, i\}$	$\{-1, i\}$

Definition2.13[6] Let (H, \circ) and $(K, *)$ be hyper groupoids and $f: H \rightarrow K$. We say that f is a good homomorphism if for all $(a, b) \in H^2$, $f(a \circ b) = f(a) * f(b)$.

Example2.14 Suppose that the function $f: H \rightarrow K_4$ where $H = \{a, b, c\}$ and $K_4 = \{e, a, b, c\}$ defined by $f(x) = a \cdot x$.

Then $f(a) = a \cdot a = \{a, b, c\}$

$f(b) = a \cdot b = \{e, a, b\}$

$f(c) = a \cdot c = \{e, a, c\}$

$f(a \cdot a) = f(\{a, b, c\}) = a \cdot \{a, b, c\}$

$= \{a \cdot a\} \cup \{a \cdot b\} \cup \{a \cdot c\}$

$= \{a, b, c\} \cup \{e, a, b\} \cup \{e, a, c\} = \{e, a, b, c\}$

$f(a) \cdot f(a) = \{e, a, b, c\}$

The other cases are similar.

Hence f is a good homomorphism.

Definition2.15[6] If H is a non-empty set and for all $x, y \in H$, we define $x \circ y = H$, then (H, \circ) is a hyper group, called the hyper group.

3 Some Results on Divisibility of Hyper groups

Definition3.1[6] A semi hyper group (H, \cdot) is called divisible if for any $x \in H$ and $n \in \mathbb{N}$, there is an element $y \in H$ such that $x \in (y, \circ)^n$ where $(y, \circ)^n$ denotes the subset $y \circ y \circ \dots \circ y$ (n copies) of H .

Example3.2 Consider $G = \{1, -1, i, -i\}$ an abelian group under the multiplication. Let $P = \{1, i\} \subseteq G$, then the hyper operation \times_p The hyper group defined in the example2.12 is divisible.

For, $1 \in (i, \cdot)^n$

$n = 2, 1 \in i \cdot i = \{-1, -i\}$

$n = 3, 1 \in i \cdot i \cdot i = \{-1, -i\} \cdot i = \{1, -1, i, -i\}$

The other cases are similar.

Definition3.3 The hyper group (H, \circ) with $x \circ y = H$ for all $x, y \in H$ is divisible.

Example3.4 Consider the hyper group $H = \{a, b, c\}$. Now, consider the subset $P = \{a, b, c\}$ then the table for multiplication and hyper multiplication are given below

\cdot	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

\odot	a	b	c
a	H	H	H
b	H	H	H
c	H	H	H

Theorem3.5 A canonical hyper group is divisible.

Proof. Let (H, \times) be a canonical hyper group, then H is a semi hyper group.

Let $a \in H$ and $n \in \mathbb{N}$

Then $a \in (b, \cdot)^n$ for some $b \in H$

Hence H is divisible.

Example3.6 Consider the example 2.6,

To prove K_4 is divisible.

Now, $e \in (c, \cdot)^n$

$n = 2, e \in c \cdot c = \{a, b\}$

$n = 3, e \in c \cdot c \cdot c = \{e, c\}$

The other cases are similar for K_4 is divisible.

Theorem3.7 Let H be a canonical hyper group. Then any sub canonical hyper group K of H is divisible.

Proof. Let K be a sub canonical hyper group of H .

Now, to prove $a \in K$, $n \in \mathbb{N}$ and $b \in K$ then $a \in (b, \cdot)^n$ is divisible.

Let $a \in K$, $n \in \mathbb{N}$. Then $a \in H$, $n \in \mathbb{N}$

Since, H is divisible, then $a \in (b, \cdot)^n \in H$ for some

$b \in H$. Then $a \in (b, \cdot)^n \in K \subseteq H$ is divisible.

Example3.8 From the example 2.6, $P = \{a, b\}$ is a sub canonical hyper group.

To prove P is divisible.

Now, $a \in (b, \cdot)^n$

$n = 2, a \in b \cdot b = \{a, b\}$

Hence we can get P is divisible.

Theorem3.9 A sub hyper group is divisible.

Proof. Let (K, \times) be a sub hyper group of (H, \times) .

That is K itself forms a hyper operation in H .

To prove K is divisible.

Since, $K \subseteq H$, then K is a semi hyper group.

Now, $a \in K$, $n \in \mathbb{N}$, there is an element $b \in K$ such that $a \in (b, \cdot)^n \in K \subseteq H$.

Since, H is divisible, then K is divisible.

Example3.10 Consider the sub hyper group

$P = \{1, i\}$ of a hyper group $G = \{1, -1, i, -i\}$

To prove P is divisible.

Here, $1 \in (i, \cdot)^n$

$n = 2, 1 \in i \cdot i = \{-1, -i\}$

$n = 3, 1 \in i \cdot i \cdot i = \{1, i, -i\}$

$n = 4, 1 \in i \cdot i \cdot i \cdot i = \{1, -1, i, -i\}$

Hence we can get P is divisible.

Theorem 3.11 A hyper group is divisible.

Proof. Let H be a hyper group.

That is H satisfies both conditions semi hyper group and quasi hyper group.

To prove H is divisible.

That is to prove $a \in H, n \in \mathbb{N}$, then $a \in (b, \cdot)^n$ is divisible for some $b \in H$.

Let $a \in H$ and $n \in \mathbb{N}$.

Since, H is a semi hyper group, $b \in H$, then

$a \in (b, \cdot)^n$ for some n . Hence H is divisible.

Example 3.12 From the examples 2.6 and 3.2 are divisible.

Theorem 3.13 If K_1 and K_2 are sub canonical hyper group of a canonical hyper group H is divisible. Then $K_1 \cap K_2$ is divisible.

Proof. Let us assume that K_1 and K_2 are sub canonical hyper group of a canonical hyper group H is divisible.

Claim $K_1 \cap K_2$ is divisible.

Since, K_1 and K_2 are sub canonical hyper group and they satisfies the semi hyper group condition for all $x, y, z \in K_1, K_2$. Now, let us consider $a \in K_1 \cap K_2$ and $n \in \mathbb{N}$. This implies that $a \in K_1, a \in K_2$ and $n \in \mathbb{N}$ such that $a \in (b, \cdot)^n \subseteq K_1$ and $a \in (b, \cdot)^n \subseteq K_2$ is divisible for some $b \in K_1$ and $b \in K_2$. Hence we can get $a \in (b, \cdot)^n$ is divisible for all $b \in K_1 \cap K_2$.

Hence $K_1 \cap K_2$ is divisible.

Example 3.14 Let us consider the hyper group $K_4 = \{e, a, b, c\}$ and the sub canonical hyper groups $K_1 = \{a, b\}$ and $K_2 = \{a, b, c\}$.

Then $K_1 \cap K_2 = \{a, b\}$.

Since, $a \in K_1 \cap K_2$ and $n \in \mathbb{N}$, then $a \in (b, \cdot)^n$ is divisible $\forall b \in K_1 \cap K_2$. Hence $K_1 \cap K_2$ is divisible.

Example 3.15 Let us consider the hyper group $G = \{1, -1, i, -i\}$ and the sub canonical hyper groups $K_1 = \{1, i\}$ and $K_2 = \{1, i, -i\}$.

Then $K_1 \cap K_2 = \{1, i\}$.

Since, $1 \in K_1 \cap K_2$ and $n \in \mathbb{N}$, then $1 \in (i, \cdot)^n$ is divisible $\forall i \in K_1 \cap K_2$. Hence $K_1 \cap K_2$ is divisible.

Acknowledgement: The authors are highly grateful to the referees for their valuable comments and suggestions for improving this paper.

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