More Results on Co – Isolated Locating Domination Number of Graphs

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Abstract— Let G (V, E) be a simple, finite and undirected connected graph. A nonempty set $S \subseteq V$ of a graph G is a dominating set, if every vertex in V – S is adjacent to atleast one vertex in S. A dominating set $S \subseteq V$ is called a locating dominating set, if for any two vertices v, w $\in$ V – S, $N(v) \cap S \neq N(w) \cap S$. A locating dominating set $S \subseteq V$ is called a co – isolated locating dominating set (cild – set), if there exists atleast one isolated vertex in <V – S >. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set. The locating domination number $\gamma_{ld}(G)$ and co – isolated locating domination number $\gamma_{cild}(G)$ are defined in the same way. A partition of V(G), all of whose classes are cild – sets in G is called a co – isolated locating domatic partition of G. The maximum number of classes of a co – isolated locating domatic partition of G is called the co – isolated locating domatic number of G and denoted by $d_{cild}(G)$. In this paper, connected graphs satisfying the relation $\gamma_{cild}(G) \leq \gamma_{ld}(G) \leq \gamma(G)$ are constructed. Also the bounds for $d_{cild}(G)$ are obtained.

Keywords— *Dominating set, locating dominating set, co – isolated locating dominating set, co – isolated locating domination number, locating domatic number, co – isolated locating domatic number.*

I. Introduction

Let G = (V, E) be a simple graph of order p and size q. For v $\in$V(G), the neighborhood $N_G(v)$ (or simply N(v)) of v is the set of all vertices adjacent to v in G. If a graph and its complement are connected, then the graph is said to be a doubly connected graph. The concept of domination in graphs was introduced by Ore [11]. A non – empty set $S \subseteq V(G)$ of a graph G is a dominating set, if every vertex in V(G) – S is adjacent to some vertex in S. A special case of dominating set S is called a locating dominating set. It was defined by D. F. Rall and P. J. Slater in [12]. A dominating set $S \subseteq V$ is called a locating dominating set, if for any two vertices v, w $\in$ V – S, $N(v) \cap S \neq N(w) \cap S$. A locating dominating set $S \subseteq V$ is called a co – isolated locating dominating set (cild – set), if there exists atleast one isolated vertex in <V – S >. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set. The locating domination number $\gamma_{ld}(G)$ and co – isolated locating domination number $\gamma_{cild}(G)$ are defined in the same way. We call a set of vertices a γ-set if it is a dominating set with cardinality $\gamma(G)$. Similarly, γ_{ld} and γ_{cild} – sets are defined. The domatic number of a graph was defined by E.J. Cockayne and S.T. Hedetniemi[3]. The location - domatic number of a graph was introduced by B.Zelinka[13]. A partition of V(G), all of whose classes are cild – sets in G is called a co – isolated locating domatic partition of G. The maximum number of classes of a co – isolated locating domatic partition of G is called the co – isolated locating domatic number of G and denoted by $d_{cild}(G)$). In this paper, the connected graphs satisfying the relation $\gamma_{cild}(G) \leq \gamma_{ld}(G) \leq \gamma(G)$ are constructed. Also the bounds for $d_{cild}(G)$ are obtained.

II. Prior Results

The following results are obtained in [7], [8], [9] & [10]

Theorem 2.1[7]:

For every nontrivial simple connected graph G with p vertices, $1 \leq \gamma_{cild}(G) \leq p - 1$.

Theorem 2.2[7]:

For any connected graph G, $\gamma_{cild}(G) = 1$ if and only if $G \cong K_2$.

Theorem 2.3[7]:

If $G \cong K_p$, then $\gamma_{cild}(G) = p - 1$.

Theorem 2.4[7]:

For any connected graph G, $\gamma_{cild}(G) = 2$ if and only if G is one of the following graphs.

(i) P_p (p = 3, 4, 5)

(ii) C_p (p = 3, 4, 5)

(iii) G is a graph obtained by attaching a pendant edge at a vertex of degree 2 in $K_4 - e$.

(iv) G is a graph C_5 with a chord.

(v) G is a graph obtained by attaching either a path of length 2 at a vertex of C_3 (or) exactly one pendant edge at two vertices of C_3.

Theorem 2.5[8]:

Let G be a connected graph with p ($p \geq 4$) vertices. Then $\gamma_{cild}(G) = p - 1$ if and only if V(G) can be partitioned into two sets X and Y, such that Y is independent and each vertex in X is adjacent to each in Y and the subgraph <X> induced by X is one of the following.

(i) <X> is a complete subgraph of G

(ii) <X> is totally disconnected

(iii) Any two non – adjacent vertices in V(<X>) have common neighbors in <X>.

Theorem 2.6[8]:

For a path P_p on p vertices, $\gamma_{cild}(P_p) = \left\lceil \frac{2p+4}{5} \right\rceil$, $p \geq 3$.

Theorem 2.7 [8]:

If C_p ($p \geq 3$) is a cycle on p vertices, then $\gamma_{cild}(C_p) = \left\lceil \frac{2p}{5} \right\rceil$.

Lemma 2.8[9]:

If G is a connected graph, then $\delta(G) \leq \gamma_{cild}(G)$, where $\delta(G)$ is the minimum degree of G.

Theorem 2.9[9]:

Let G be a doubly connected graph of order $p \geq 5$ such that $diam(G) = diam(\bar{G}) = 2$. Then G contains a co – isolated locating dominating set of cardinality $p - 3$.

Theorem 2.10[10]:

Let G = (V, E) be a connected cubic graph with p vertices ($p \geq 4$).

Then $\left\lceil \frac{p+1}{3} \right\rceil \leq \gamma_{cild}(G) \leq \frac{p}{2}$.

III. MAIN RESULTS

In the following, the maximum number of vertices in the complement of a γ_{cild} – set is found and the corresponding graph is constructed.

Theorem 3.1:

Let S be a γ_{cild} – set of a connected graph G. If S has k vertices, then the number of vertices in V – S is atmost $2^k - 1$.

Proof:

Since S is a γ_{cild} – set of G, for any two vertices u, $v \in V(G) - S$, $N(u) \cap S$ and $N(v) \cap S$ are distinct. Therefore if each vertex in V – S is adjacent to exactly one vertex in S then the maximum number of vertices in V – S in this way is kC_1. If each vertex in V – S is adjacent to exactly two vertices in S then the maximum number of vertices in V – S in this way is kC_2. Proceeding in a similar way if each vertex in V – S is adjacent to exactly k vertices in S then the maximum number of vertices in V – S in this way is kC_k. Hence $|V - S| \leq kC_1 + kC_2 + \ldots + kC_k = 2^k - 1$.

Remark 3.1:

If a γ_{cild} – set S of a connected graph G has k vertices, then G has atmost $k + 2^k - 1$ vertices.

Theorem 3.2:

A connected graph G can be constructed with a co – isolated locating dominating set S of G having k vertices and V – S with $2^k - 1$ vertices.

Proof:

Let S be a co – isolated locating dominating set of a connected graph G with p vertices and |S| = k. Let $S = \{v_1, v_2, \ldots, v_k\}$. Then V – S has p – k vertices. The construction of a graph G with V – S having $p - k = 2^k - 1$ vertices is as follows.

Step 1:

Choose a vertex u_1 in V – S and make it adjacent to exactly one vertex in S say v_1. Then choose u_2 in $V - S - \{u_1\}$ and make it adjacent to a vertex say v_2 in S such that $v_1 \neq v_2$. Repeating this procedure, there exist k (= kC_1) vertices $u_1, u_2, \ldots, u_k$ in V – S such that u_i is adjacent to v_i in S. Let $D_1 = \{u_1, u_2, \ldots, u_k\} \subseteq V - S$.

Step 2:

Choose a vertex $u_{k+1} \in V - S - D_1$ and make it adjacent to exactly two vertices in S, say v_1 and v_2. Then choose $u_{k+2} \in V - S - (D_1 \cup \{u_{k+1}\})$ and make it adjacent to vertices say v_2 and v_3 in S. Repeating this procedure, there exist kC_2 vertices $u_{k+1}, u_{k+2}, \ldots, u_{(k+C2)}$ (where $kC_1 + kC_2 < p - k$) in V – S, each adjacent to exactly two distinct vertices in S.

Let $D_2 = \{u_{k+1}, u_{k+2}, \ldots, u_{(k+kC2)}\} \subseteq V - S$.

Let $t_{(k-2)} = k + kC_2 + kC_3 + \ldots + kC_{(k-3)} + kC_{(k-2)}$.

Proceeding in this way, at Step (k – 2), there exist $kC_{(k-2)}$ vertices $u_{t(k-3)+1}, u_{t(k-3)+2}, \ldots, u_{t(k-2)}$, where $t(k-2) = t(k-3) + kC_{(k-2)}$, each is adjacent to (k – 2) distinct vertices in S, where $kC_1 + kC_2 + \ldots + kC_{(k-2)} < p - k$.

Let $D_{(k-2)} = \{u_{t(k-3)+1}, u_{t(k-3)+2}, \ldots, u_{t(k-2)}\}$

Step k – 1:

Choose a vertex $u_{t(k-2)+1}$ in $V - S - (D_1 \cup D_2 \cup \ldots \cup D_{(k-2)})$ and make it adjacent to exactly (k – 1) vertices in S.

Then choose $u_{t(k-2)+2}$ in $V - S - ((D_1 \cup D_2 \cup \ldots \cup D_{(k-2)}) \cup \{u_{t(k-2)+1}\})$ and make it adjacent to another set of (k – 1) vertices in S. Repeating the procedure there exist kC_{k-1} vertices $u_{t(k-2)+1}, u_{t(k-2)+2}, \ldots, u_{t(k-1)}$, where $t(k-1) = t(k-2) + kC_{(k-1)}$ in V – S, each is adjacent to (k – 1) distinct vertices in S, where $kC_1 + kC_2 + \ldots + kC_{(k-1)} < p - k$.

Let $D_{(k-1)} = \{u_{t(k-2)+1}, u_{t(k-2)+2}, \ldots, u_{t(k-1)}\}$

Step k:

Choose a vertex $u_{t(k-1)+1}$ in $V - S - (D_1 \cup D_2 \cup \ldots \cup D_{(k-1)})$ and make it adjacent to all the k vertices in S. Since S is a locating dominating set, there is no other vertex in V – S adjacent to a vertex in S. Therefore V – S has $k + kC_2 + \ldots + kC_{k-1} + 1 = kC_1 + kC_2 + \ldots + kC_{k-1} + kC_k = 2^k - 1$ vertices.

Example 3.1:

The graph G in Fig 3.1 is a bipartite graph with cild – set S having 3 vertices and $|V - S| = 7 = 2^3 - 1$.

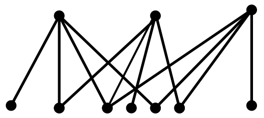

Fig. 3.1

Remark 3.2:

Properties of the graph G obtained in Theorem 3.2.

i) G is a bipartite graph.
ii) The girth of G is 4.
iii) The circumference G is 2k.
iv) G contains cycles C_{2n}, n = 2, 3, …, k as its induced subgraphs.

Proposition 3.1:

Every superset of a cild – set of a connected graph G need not be a cild – set of G.

Proof:

Let S be a cild – set of G and let A be the set of isolated vertices in V – S. If $S_1 = S \cup A$, then S_1 will not be a cild – set since $\langle V - S_1\rangle$ does not have any isolated vertices.

Example 3.2:

For the cycle C_5 with vertex set $V(C_5) = \{v_1, v_2, v_3, v_4, v_5\}$, the set $S = \{v_2, v_5\}$ is a cild – set. The set $S_1 = S \cup \{v_3\}$ is also a cild – set whereas the set $S_2 = S \cup \{v_1\}$ is not a cild – set, since $V - S_2$ does not contain any isolated vertices.

Remark 3.3:

Since every cild-set is a locating dominating set as well as a dominating set,
$\gamma_{cild}(G) \le \gamma_{ld}(G) \le \gamma(G)$.

In the following, the graphs satisfying $\gamma_{cild}(G) \le \gamma_{ld}(G) \le \gamma(G)$ are obtained.

Theorem 3.3:

There exists a connected bipartite graph G with $\gamma_{cild}(G) = \gamma_{ld}(G) = \gamma(G)$.

Proof:

The construction of a bipartite graph G with $\gamma_{cild}(G) = \gamma_{ld}(G) = \gamma(G)$ is as follows

Step 1:

Consider a cycle C_{2p} with p (≥ 3) vertices and $V(C_{2p}) = \{ u_1, v_1, u_2, v_2, \ldots, u_p, v_p \}$.

Step 2:

Attach exactly one pendant vertex at the vertices $v_1, v_2, \ldots, v_p$. Let the newly introduced pendant vertices be $v_1', v_2', \ldots, v_p'$.

Let G' be the graph obtained from the above construction. This graph G' is a unicyclic graph and $|V(G)| = 3p$ and $\gamma_{cild}(G) = p$, since the set $\{v_1, v_2, \ldots, v_p\}$ forms a γ_{cild} – set of G. This set is also a γ_{ld} – set and γ – set of G.

Example 3.3:

In Fig. 3.2, G is a bipartite graph with $\gamma_{cild}(G) = \gamma_{ld}(G) = \gamma(G) = 5$.

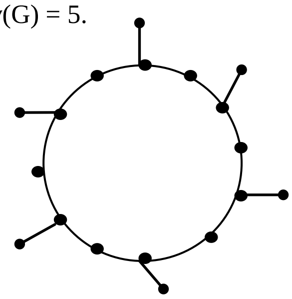

Fig. 3.2.

Theorem 3.4:

There exists a connected graph G with $\gamma(G) < \gamma_{ld}(G) = \gamma_{cild}(G)$.

Proof:

Consider the cycle C_p, p ≥ 3, p ≠ 5, 7, 10. Then $\gamma(C_p) = \left\lceil \frac{p}{3} \right\rceil$ while, $\gamma_{ld}(C_p) = \gamma_{cild}(C_p) = \left\lceil \frac{2p}{5} \right\rceil$.

Therefore, $\gamma(G) < \gamma_{ld}(G) = \gamma_{cild}(G)$.

Theorem 3.5:

There exists a connected graph G such that $\gamma(G) < \gamma_{ld}(G) < \gamma_{cild}(G)$.

Proof:

A graph G with $\gamma(G) < \gamma_{ld}(G) < \gamma_{cild}(G)$ is constructed as follows.

Step 1:

Consider a path P_p, where p = 5k with $V(P_p) = \{v_1, v_2, \ldots, v_p\}$.

Step 2:

Let p be even and let p = 2k. Let G^* be the graph obtained from the path P_p by adding edges $\bigcup_{i=1}^{k-1} \{(v_{2i-1}, v_{2i+1})\}$ and the edges $\bigcup_{i=1}^{k-1}\{(v_{2i}, v_{2i+2})\}$.

Let p be odd and let p = 2k + 1, the edges $\bigcup_{i=1}^{k}\{(v_{2i-1}, v_{2i+1})\}$ and $\bigcup_{i=1}^{k-1}\{(v_{2i}, v_{2i+2})\}$ are to be added to P_p to construct G*. If p ≡ 0(mod 5), then the set $S' = \bigcup_{i=0}^{k-1}\{v_{5i+3}\}$ is a dominating set, if p ≡ 1, 2(mod 5), then the set $S'' = S' \cup \{v_p\}$ is a dominating set and if p ≡ 3, 4 (mod 5), then the set $S''' = \bigcup_{i=0}^{k}\{v_{5i+3}\}$ is a dominating set of G^*. Therefore $\gamma(G^*) = \left\lceil \frac{p}{5} \right\rceil$. Further a γ_{cild} – set of P_p is also a γ_{ld} – set of G^*.

Therefore by Theorem 2.3., $\gamma_{ld}(G^*) = \gamma_{cild}(P_p) = \left\lfloor \frac{2p+4}{5} \right\rfloor$. If S_1 and S are γ_{cild} – sets of P_p and G^*

respectively, then $S = S_1 \cup \{v_3\}$, since $V – S$ contains atleast one isolated vertex.

Therefore $\gamma_{cild}(G^*) = \left\lfloor \frac{2p+4}{5} \right\rfloor + 1$.

Hence $(\frac{p}{5}) < \left\lfloor \frac{2p+4}{5} \right\rfloor < \left\lfloor \frac{2p+4}{5} \right\rfloor + 1$.

Example 3.4:

In Fig. 3.3, G is a graph with $\gamma(G) < \gamma_{ld}(G) < \gamma_{cild}(G)$.

The set $S = \{v_3, v_8, v_{11}\}$ is a γ – set and $\gamma(G) = 3$, the set $S_1 = \{v_2, v_4, v_7, v_9, v_{11}\}$ is a γ_{ld} – set of G and $\gamma_{ld}(G) = 5$ and the set $S_2 = S_1 \cup \{v_3\}$ is a γ_{cild} – set of G and $\gamma_{cild}(G) = 6$.

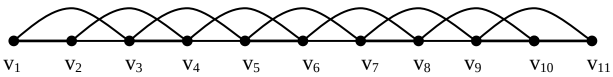

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Fig. 3.3

Remark 3.4:

(i) There exist integers a = 1, b = 2, c ≥ 3 and a graph G with $\gamma(G) < \gamma_{ld}(G) < \gamma_{cild}(G)$ where $\gamma(G) = a$, $\gamma_{ld}(G) = b$ and $\gamma_{cild}(G) = c$. If $G \cong K_p – e$, $p \geq 4$. $\gamma(K_p – e) = 1$, $\gamma_{ld}(K_p – e) = 2$ and $\gamma_{cild}(K_p – e) = p – 1$.

(ii) There exist integers a = 2, b > a and c = b + 1 with $\gamma(G) = a$, $\gamma_{ld}(G) = b$ and $\gamma_{cild}(G) = c$ and hence $\gamma(G) < \gamma_{ld}(G) < \gamma_{cild}(G)$. For a complete bipartite graph $K_{m,n}$; m, n ≥ 2. $\gamma(K_{m,n}) = a = 2$, $\gamma_{ld}(K_{m,n}) = b = m + n – 2$, $\gamma_{cild}(K_{m,n}) = c = m + n – 1$.

Therefore 2 < m + n – 2 < m + n – 1.

Observation 3.1:

Let S be a minimum cild-set of a connected graph G.

(i) If a connected spanning subgraph H of G is obtained by removing the edges having both its ends in <V – S> then $\gamma_{cild}(G) = \gamma_{cild}(H)$.

Example 3.5:

Let G be a graph obtained by taking the two cycles C_p and C_p' with $V(C_p) = \{u_1, u_2, \ldots, u_p\}$ and $V(C_p') = \{v_1, v_2, \ldots, v_p\}$ and then adding the edges of the form $\bigcup_{i=1}^{p} u_i v_i$.

If H is a connected spanning subgraph of G obtained from G by removing a path P_3 with vertices $u_i, u_{i+1}, v_{i+1}, v_{i-1}$, then $\gamma_{cild}(G^*) = \gamma_{cild}(H)$.

Example 3.6:

In Fig. 3.4, H is a connected spanning subgraph of G and $\gamma_{cild}(G) = 5 = \gamma_{cild}(H)$

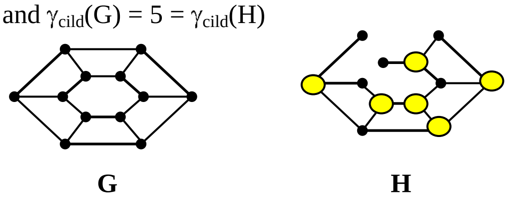


Fig. 3.4

(ii) If a graph H is obtained by removing the edges having one end in S and the other end in V – S such that $N(u) \cap S = N(v) \cap S$, for u, v ∈ V – S, then $\gamma_{cild}(G) < \gamma_{cild}(H)$.

Let G be the graph obtained from a cycle C_p,(p ≥ 3) by subdividing each edge of C_p twice and then attaching a cycle C_3 at each vertex of C_p. Let H be a spanning subgraph obtained from G by removing an edge in each cycle C_3. Then $\gamma_{cild}(G) = 2p$, since a set S containing vertices of the cycle C_p and one vertex from each cycle C_3 is a γ_{cild} – set of G. Also $\gamma_{cild}(H) = 3p$, since S ∪ {one vertex from each cycle C_3 in G but not in S} is a the γ_{cild} – set of H. Hence $\gamma_{cild}(G) < \gamma_{cild}(H)$.

Example 3.7:

In Fig. 3.5, H is a connected spanning subgraph of G and $\gamma_{cild}(G) = 12$ and $\gamma_{cild}(H) = 18$ and hence $\gamma_{cild}(G) < \gamma_{cild}(H)$.

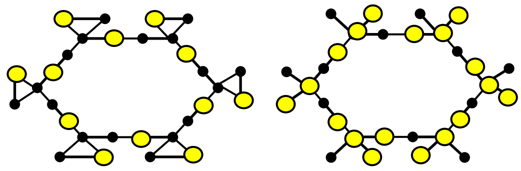


Fig. 3.5

(iii) If the graph H is obtained by removing the edges having one end in S and the other end in V – S such that $N(u) \cap S \neq N(v) \cap S$, where u ∈ S and v ∈ V – S, then $\gamma_{cild}(G) > \gamma_{cild}(H)$.

Example 3.8: $\gamma_{cild}(K_p) = p – 1 (p \geq 5)$, whereas $\gamma_{cild}(C_p) = \left\lceil \frac{2p}{5} \right\rceil$. Hence $\gamma_{cild}(G) > \gamma_{cild}(H)$.

In the following, an algorithm to find a cild – set of a connected graph is given.

Algorithm:

Given G, a connected graph with vertex set V(G) and edge set E(G).

Step 1: Choose any arbitrary vertex v ∈ V(G) and set $v = v_0$. Set S = A = ∅ (the empty set).

Step 2: Let $S = N(v_0)$ and A = {collection of all subsets of S} (except the empty set).

Step 3: If there exists a vertex u ∈ V – S, such that $d(u) = d(v_0)$ then goto step 4; otherwise go to step 5.

Step 4: Set S = S ∪ {u} and A = {collection of all subsets of S}. Goto step 3.

Step 5: If there exists a vertex u ∈ V – S, such that N(u) ∉ A then goto step 6; otherwise goto step 7.

Step 6: Set S = S ∪ {u} and A = {collection of all subsets of S}. Goto step 5.

Step 7: If there exist vertices u, v ∈ V – S, such that N(u) ∩ A = N(v) ∩ A then goto step 8; otherwise goto step 9.

Step 8: Set S = S ∪ {u} or S = S ∪ {v}. Goto step 7.

Step 9: The set S which is obtained from the above steps is a cild – set of G.

Theorem 3.6:

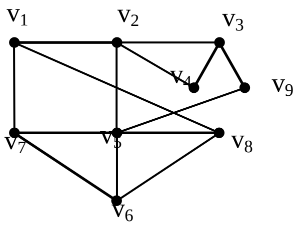
The set S constructed by the above algorithm is a cild – set of G.

Proof:

By Step 5, all the vertices of V – S are dominated by S, since there exists no vertex $u \in V - S$ such that $N(u) \notin A$ = {collection of all subsets of S}. Therefore S is dominating set of G. By Step 7, for any two vertices $u, v \in V - S$, $N(u) \cap A$ and $N(v) \cap A$ are distinct. Therefore, S is a locating – dominating set. Also by Step 2, $N(v_0) \subseteq S$ and therefore there exists atleast one isolated vertex in V – S. Hence S is a cild – set of G.

Example 3.9:

For the graph given in Fig. 3.6, a cild – set is found using the algorithm



G

Fig. 3.6

Set $v_0 = v_9$ (Step 1) and set $S = N(v_9) = \{ v_3, v_5 \}$; A = {collection of all subsets of S} (Step 2) and since there exists no vertex $u \in V - S$ such that $d(u) = d(v_0)$ (Step 3), and there exists a vertex $v_1 \in V - S$ such that $N(v_1) = \{ v_8, v_7, v_2 \} \notin A$ (Step 5), set $S = S \cup \{v_1\} = \{ v_1, v_3, v_5 \}$ and A = {collection of all subsets of S} (Step 6). Since there exists no vertex $u \in V - S$ such that $N(u) \notin A$ (Step 5) and since there exist vertices $v_7, v_8 \in V - S$ such that $N(v_7) \cap A = N(v_8) \cap A = \{v_1, v_5\}$(Step 7), set $S = S \cup \{v_7\} = \{ v_1, v_3, v_5, v_7 \}$ (Step 8). Since there exists no vertices $u, v \in V - S$, such that $N(u) \cap A = N(v) \cap A$(Step 7), the set $S = \{ v_1, v_3, v_5, v_7 \}$ is a cild – set of G by Step 9.

In the following co-isolated locating domatic number is defined.

Definition 3.1:

A partition of V(G), all of whose classes are cild – sets in G is called a co – isolated locating domatic partition of G. The maximum number of classes of a co – isolated locating domatic partition of G is called the co – isolated locating domatic number of G and denoted by $d_{cild}(G)$.

Example 3.10:

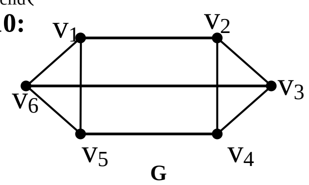


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Fig. 3.7

For the graph G given in Fig. 3.7, the sets $S_1 = \{v_1, v_5, v_3\}$ and $S_2 = \{v_2, v_4, v_6\}$ are co – isolated locating dominating sets. Therefore, $d_{cild}(G) = 2$.

Observation 3.2:

1. $d_{cild}(K_{m,n}) = 1$ for $m, n \geq 2$ and $d_{cild}(K_{1,1}) = 2$.
2. $d_{cild}(K_n) = 1$ for $n \geq 3$.
3. $d_{cild}(C_p) = 2$ for $p \geq 5$.
4. $d_{cild}(P_p) = 2$, for $p \geq 4$.
5. If T is a tree obtained from P_p^+ by subdividing each edge joining supports exactly once, then $d_{cild}(T) = 2$.

Theorem 3.7:

For any connected graph G, $d_{cild}(G) = 1$ or 2.

Proof:

It is sufficient to prove that $d_{cild}(G) \ngeq 3$. Suppose $d_{cild}(G) = 3$. Then there exist 3 pairwise disjoint cild – sets S_1, S_2, S_3 in G. Therefore there exists atleast one isolated vertex in each of the sets $V - S_1$, $V - S_2$ and $V - S_3$. Let x_i be an isolated vertex in $V - S_i$, i = 1, 2, 3. That is,

$x_1 \in V - S_1 = S_2 \cup S_3$ and $N(x_1) \subseteq S_1$. Similarly $x_2 \in V - S_2 = S_1 \cup S_3$ and $N(x_2) \subseteq S_2$ and $x_3 \in V - S_3 = S_2 \cup S_1$ and $N(x_3) \subseteq S_3$.

$x_1 \in S_2 \cup S_3$ implies either $x_1 \in S_2$ or $x_1 \in S_3$ since S_2 and S_3 are disjoint. Assume $x_1 \in S_2$. Also $N(x_1) \subseteq S_1$ implies $x_1 \notin S_3$ and $N(x_1) \notin S_3$ which shows that S_3 is not a cild – set of G, a contradiction. Therefore $d_{cild}(G) \neq 3$. Similarly is the case when $d_{cild}(G) \geq 4$. Therefore, $d_{cild}(G) = 1$ or 2.

Theorem 3.8:

For a connected graph G, $d_{cild}(G) = 1$ if one of the following conditions holds.

(i) There exists atleast one vertex of degree p – 1 in G.

(ii) If G has a support, then this support has atleast two leaves.

(iii)There exist three distinct vertices v_1, v_2, v_3 in G such that $N_G(v_1) = N_G(v_2) = N_G(v_3)$.

Proof:

Let G be a connected graph satisfying one of the conditions given in the Theorem.

Case 1: G has atleast one vertex of degree p – 1

Let $v \in V(G)$ such that $d(v) = p - 1$. Assume $d_{cild}(G) \geq 2$. Let D_1 and D_2 be two disjoint co – isolated locating dominating sets in G. Assume $v \in D_1$. Then $v \notin D_2$, since $D_1 \cap D_2 = \emptyset$. Therefore $v \in V - D_2$. This implies that $V - D_2$ does not have an isolated vertex. Hence D_2 is not a cild – set, a contradiction. Hence $d_{cild}(G) = 1$.

Case 2: If G has a support, then this support has atleast two leaves

Let u be a support of G and let u be adjacent to k leaves $u_1, u_2, \ldots, u_k$, where $k \geq 2$. Then any cild – set of G either contains all the k leaves or (k – 1) leaves and u. Assume $d_{cild}(G) \geq 2$. Let D_1 and D_2 be two disjoint co – isolated locating dominating sets of G.

Subcase 2.a: D_1 contains all the k leaves $u_1, u_2, \ldots, u_k$.

Then D_2 contains a support and $u_1, u_2, \ldots, u_k \in V - D_2$ for which $N(u_i) \cap D_2 = \{u\}$ for all i = 1, 2, …, k. Hence D_2 is not a cild – set, a contradiction.

Subcase 2.b: D_1 contains (k – 1) leaves $u_1, u_2, \ldots, u_{k-1}$ and u.

Then V – D_2 contains $u_1, u_2, \ldots, u_{k-1}$, u and $u_1, u_2, \ldots, u_{k-1}$ are not adjacent to any of the vertices in D_2. Therefore, D2 is not a dominating set of G, a contradiction. Hence $d_{cild}(G) = 1$.

Case 3: There exist three distinct vertices v_1, v_2, v_3 in G such that $N_G(v_1) = N_G(v_2) = N_G(v_3)$.

Suppose $d_{cild}(G) \geq 2$. Let D_1 and D_2 be two disjoint co – isolated locating dominating sets in G. Assume $v_1, v_2 \in D_1$. Then $v_1, v_2 \in V - D_2$ for which $N(v_1) \cap D_2 = N(v_2) \cap D_2$. Therefore D_2 is not a cild – set of G. Hence $d_{cild}(G) = 1$.

Remark 3.5:

For any connected graph G,

$d_{cild}(G) \leq \delta(G) + 1$.

Definition 3.2:

A graph G is called cild – domatically full, if $d_{cild}(G) = \delta(G) + 1$.

Remark 3.6:

For any connected graph G, $d_{cild}(G) = \delta(G) + 1$ if and only if $\delta(G) = 1$ and $d_{cild}(G) = 2$ since $d_{cild}(G) = 1$ or 2. If $G \cong C_5$ + e where e is a pendant edge attached at a vertex of C_5, then $d_{cild}(G) = 2$.

Theorem 3.9:

For any integer k, there exists a regular bipartite graph G with 2k vertices for which $\gamma_{cild}(G) = k$ and $d_{cild}(G) = 2$, where $k \geq 3$.

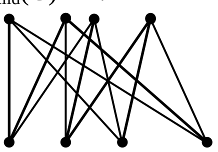
Proof:

Let S be a γ_{cild} – set of G. Assume $S = \bigcup_{j=0}^{k-1}\{v_i\}$ and $V - S = \bigcup_{j=0}^{k-1}\{u_i\}$.

For i = 0, 1, …, k – 1, let $N(u_i) = \bigcup_{j=0}^{k-2}\{v_{i+j}\}$; where the subscripts are taken modulo k. That is, each vertex in V – S is adjacent to k – 1 distinct vertices in S and therefore each vertex in S has degree k – 1. The graph G thus constructed is a (k – 1) - regular bipartite graph. The sets S and V – S form a co-isolated locating domatic partition of G and hence $d_{cild}(G) = 2$.

Example 3.12

The graph G given in Fig. 3.8 contains 8 vertices and $d_{cild}(G) = 2$.



G

Fig. 3.8

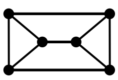
In the following Nordhaus – Gaddum type results are obtained.

Remark 3.7:

For a doubly connected graph G with atleast four vertices,

(i) $2 \leq d_{cild}(G) + d_{cild}(\bar{G}) \leq 4$

(ii) $1 \leq d_{cild}(G) \cdot d_{cild}(\bar{G}) \leq 4$. Also these bounds are sharp.

The upper bound holds, if G ≅ 

For this graph G, $d_{cild}(G) = 2$ and $d_{cild}(\bar{G}) = 2$.

The lower bound holds for all trees having a support with atleast two leaves.

For example, if G is a tree obtained by attaching two pendant edges at the central vertex of P_7, then $d_{cild}(G) = d_{cild}(\bar{G}) = 1$.

The inequality is strict, if $G \cong P_5$, since $d_{cild}(G) = 2$ and $d_{cild}(\bar{G}) = 1$.

Theorem 3.10:

For a connected graph G with p(≥ 2) vertices,

(i) $3 \leq d_{cild}(G) + \gamma_{cild}(G) \leq p$

(ii) $2 \leq d_{cild}(G) \cdot \gamma_{cild}(G) < 2(p-1)$. Also these bounds are sharp.

Proof:

(i) $\gamma_{cild}(G) = 1$ if and only if $G \cong K_2$ for which $d_{cild}(G) = 2$. Therefore $d_{cild}(G) + \gamma_{cild}(G) \geq 3$. Also $\gamma_{cild}(G) \leq p - 1$ and $d_{cild}(G) = 1$ or 2. Therefore $d_{cild}(G) + \gamma_{cild}(G) \leq p + 1$.

$d_{cild}(G) + \gamma_{cild}(G) = p + 1$, if and only if $d_{cild}(G) = 2$ and $\gamma_{cild}(G) = p - 1$. (If $d_{cild}(G) = 1$, then $\gamma_{cild}(G) = p$, but $\gamma_{cild}(G) \leq p - 1$). But the graphs G for which $\gamma_{cild}(G) = p - 1$ are characterized in Theorem 2.5 and for these graphs $d_{cild}(G) = 1$.

Therefore $d_{cild}(G) + \gamma_{cild}(G) \leq p$ and hence $3 \leq d_{cild}(G) + \gamma_{cild}(G) \leq p$

The lower bound is attained, if $G \cong C_3$.

The upper bound is strict, if $G \cong C_4$ + e for which $\gamma_{cild}(G) = 3$ and $d_{cild}(G) = 1$.

Hence $d_{cild}(G) + \gamma_{cild}(G) = 4 < 5$.

(ii) By a similar argument, $d_{cild}(G) \cdot \gamma_{cild}(G) \neq 1$. Also $d_{cild}(G) \cdot \gamma_{cild}(G) \nleq 2(p-1)$.

Therefore $2 \leq d_{cild}(G) \cdot \gamma_{cild}(G) < 2(p-1)$.

The lower bound is attained if $G \cong C_3$ + e.

The upper bound is strict, if $G \cong P_4$ for which $\gamma_{cild}(G) = 2 = p - 2$ and $d_{cild}(G) = 2$.

Hence $d_{cild}(G) \cdot \gamma_{cild}(G) = 4 < 6$.

IV. Conclusion

An algorithm for finding a cild – set of a graph and a necessary condition for any connected graph G with $d_{cild}(G) = 1$ are found.

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