Complementary Tree Domination in Splitting Graphs of Graphs

S. Muthammai[1], P. Vidhya[2]

[1]Government Arts College for Women (Autonomous)
Pudukkottai - 622 001, India
[2]S.D.N.B. Vaishnav College for Women (Autonomous)
Chennai - 600 044, India

Abstract- *Let G = (V, E) be a simple graph. A dominating set D is called a complementary tree dominating set if the induced subgraph <V–D> is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. For a graph G, let V′(G) = {v′ : v ∈ V(G)} be a copy of V(G). The splitting graph Sp(G) of G is the graph with the vertex set V(G) ∪ V′(G) and edge set {uv, u′v, uv′ : uv ∈ E(G)}. In this paper, complementary tree domination number of splitting graphs of graphs are determined.*

Keywords- *Dominating set, complementary tree dominating set.*

I. Introduction

E. Sampathkumar and H.B. Walikar [4] introduced the concept of splitting graphs. V. Swaminathan and A. Subramanian [6] have obtained the domination number of splitting graphs. T.N. Janakiraman, S. Muthammai and M. Bhanumathi [3] have characterized self-centered, bi-eccentric splitting graphs and obtained bounds for global domination number and neighbourhood number.

Given a graph G, let V′(G) = {v′ : v ∈ V(G)} be a copy of V(G). The splitting graph Sp(G) of G is the graph with the vertex set V(G) ∪ V′(G) and edge set {uv, u′v, uv′ : uv ∈ E(G)}. For each vertex v ∈ V(G), there is a corresponding vertex v′ ∈ V(Sp(G)) and each edge uv of a graph G produces three edges, uv, u′v and uv′ in Sp(G). Therefore G is an induced subgraph of Sp(G).

In this paper, bounds and exact values of complementary tree domination number of splitting graphs of standard graphs are determined. Also relationship between complementary tree domination number of a graph and its splitting graph is established.

Example 1.1. A graph G and its splitting graph are given in the following figure.

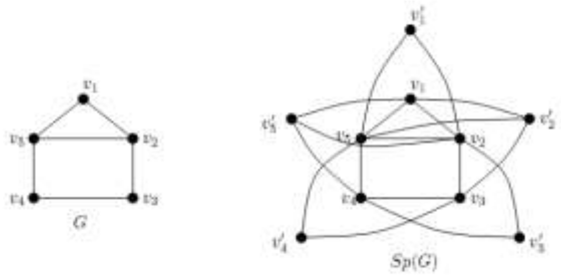

Figure 1.

In the following, a necessary and sufficient condition for a ctd-set of a graph G to be a ctd-set of its splitting graph Sp(G) is found.

Theorem 1.1. A ctd-set D of a connected graph G = (V, E) is also a ctd-set of Sp(G) if and only if

(i) <D> has no isolated vertices

(ii) For each v ∈ D, N(v) ∩ (V−D) ≠ ϕ and

(iii) <V−D> ≅ K_2 and if v_1, v_2 ∈ V−D, then $(N(v_1) \cap D) \cap (N(v_2) \cap D) \neq \phi$.

Proof. Let D be a ctd-set of both G and Sp(G).

(i) Let v ∈ D be an isolated vertex in <D>. Then, its duplicate vertex v′ in Sp(G) is not adjacent to any of the vertices in D. Hence, D is not a ctd-set of Sp(G). Therefore, <D> has no isolated vertices.

(ii) Let there exists a vertex v ∈ D such that N(v) ∩ (V−D) = ϕ. Then, its duplicate vertex v′ of v is isolated in <V−D>.

(iii) If P_3 (a path on 3 vertices) is an induced subgraph of <V−D>, then <V(Sp(G))−D> contains C_4. Therefore, <V−D> ≅ K_2. Let v_1, v_2 ∈ V−D and $(N(v_1) \cap D) \cap (N(v_2) \cap D) = \phi$. Then, there exists a vertex u ∈ D such that uv_1, uv_2 ∈ E(G) and <u′, v_1, v_2> ≅ C_3 in Sp(G). Hence, $(N(v_1) \cap D) \cap (N(v_2) \cap D) \neq \phi$.

Conversely, if (i) is true, then D is dominating set of Sp(G). If (ii) holds, then <V(Sp(G))−D> is connected and if (iii) holds, then <V(Sp(G))−D> is acyclic. Therefore, <V(Sp(G))−D> is a tree. Hence, D is also a ctd-set of Sp(G). □

Observation 1.1.

(i) For the cycle C_n, $\gamma_{ctd}(Sp(C_n))$ = n−2, where n ≥ 6. For, if $V(C_n) = \{v_1, v_2, \ldots, v_n\}$, then

$\{v_1, v_2, v'_4, ..., v'_{n-1}\}$ is a minimum ctd-set of $Sp(C_n)$.

(ii) $\gamma_{ctd}(Sp(C_3)) = 2 = \gamma_{ctd}(Sp(C_4))$, $\gamma_{ctd}(Sp(C_5)) = 4$.

(iii) For a wheel W_n, $\gamma_{ctd}(Sp(W_n)) = n-1$, $n \geq 7$. $W_n = C_{n-1} + K_1$. For $n \geq 7$, let $v_1, v_2, ..., v_{n-1}$ be the vertices of degree 3 and v be the vertex of degree n–1 in W_n. Then, $\{v_1, v_2, v'_4, ..., v'_{n-1}, v, v'\}$ is a minimum γ_{ctd}-set of $Sp(W_n)$. Hence, $\gamma_{ctd}(Sp(W_n)) = n-1$, $n \geq 7$.

(iv) $\gamma_{ctd}(Sp(K_n)) = n$, $n \geq 4$. If $v_1, v_2, ..., v_n$ are the vertices of K_n, then, $\{v'_1, v_2, ..., v_n\}$ is a γ_{ctd}-set of $p(K_n)$.

(v) $\gamma_{ctd}(Sp(K_{m,n})) = 2m$ where $m \leq n$ and $m, n \geq 2$. Let $A = \{u_1, u_2, ..., u_m\}$ and $B = \{v_1, v_2, ..., v_n\}$ be the bipartition of $V(K_{m,n})$. Then, $\{v_1, u_2, ..., u_m, u'_1, ..., u'_m\}$ is a γ_{ctd}-set of $Sp(K_{m,n})$.

(vi) $\gamma_{ctd}(Sp(K_1 + P_n)) = n$, $n \geq 3$. Let $V(K_1) = \{v\}$ and $V(P_n) = \{v_1, v_2, ..., v_n\}$. Then, $\{v'_2, ..., v'_{n-1}, v, v'\}$ is a γ_{ctd}-set of $Sp(K_1 + P_n)$.

II. Bounds for Complementary Tree Domination Number of Splitting Graphs of Graphs

Theorem 2.1. For any connected graph G with $p \geq 2$, $2 \leq \gamma_{ctd}(Sp(G)) \leq 2p-2$.

Proof. Sp(G) has 2p vertices and radius of Sp(G) is atleast 2. Hence, $\gamma_{ctd}(Sp(G)) \geq 2$. Also, there is no vertex of degree 2p–1 in Sp(G), $|D| \leq 2p-2$. Therefore, $2 \leq \gamma_{ctd}(Sp(G)) \leq 2p-2$. □

Theorem 2.2. $\gamma_{ctd}(Sp(G)) = 2$ if and only if $G \cong C_4$, C_3 (or) K_2.

Proof. Let D be a γ_{ctd}-set of Sp(G) such that |D| = 2.
Let D = {u, v}, where u, v ∈ V(Sp(G)).
Case 1. u and v are vertices of G.
Then, D is also a γ_{ctd}-set of G. By Theorem 1.1, it can be seen that $G \cong C_4$.
Case 2. Let u ∈ V(G) and v ∈ V′(G).
Subcase 2.1. v = u′.
That is, D = {u, u′} is a γ_{ctd}-set of Sp(G) and $G \cong C_3$.
Subcase 2.2. v ≠ u′.
Let v = w′, for some w ∈ V(G) and w′ ≠ u′. Then, D = {u, w′} is a γ_{ctd}-set of Sp(G). w′ ≠ u′ implies that u′ is not adjacent to both u and w′, which is not true.
Therefore, v = u′.
Case 3. u = w′ and v = x′ where w, x ∈ V(G).
If p ≥ 3, then any vertex of V′(G) – {w′, x′} is not adjacent to both w′, x′.
Therefore, p = 2 and hence $G \cong K_2$.
Conversely, if $G \cong C_4$, C_3 (or) K_2, then $\gamma_{ctd}(Sp(G)) = 2$. □

In the following, $\gamma_{ctd}(Sp(G)) = 2p-2$ is found for the graph G.

Theorem 2.3. $\gamma_{ctd}(Sp(G)) = 2p-2$ if and only if $G \cong K_2$.

Proof. Assume, $\gamma_{ctd}(Sp(G)) = 2p-2$. Let D be a γ_{ctd}-set of Sp(G) having 2p–2 vertices.
Let V(Sp(G))–D = {u, v}. Since <V(Sp(G))–D>$\cong K_2$ and v, u ∉ V′(G), either
(i) u, v ∈ V(G) or
(ii) u ∈ V(G) and v = w′, for some w ∈ V(G), w ≠ u.
Case 1. u, v ∈ V(G).
Let w ∈ D then w′ ∈ D. If both u and v are adjacent to w, then w′ is adjacent to both u and v and w is adjacent to both u′ and v′. Then, D – {u′} is a ctd-set of Sp(G), which is a contradiction.
Let exactly one of u and v (say u) is adjacent to w. Then, also D – {u′} is a ctd-set of Sp(G). Therefore, no vertex of V(G) is an element of D and hence, u′, v′ ∈ D. Hence, $G \cong K_2$.
Case 2. u ∈ V(G) and v = w′, for some w ∈ V(G), w ≠ u.
Since <V(Sp(G)) – D> $\cong K_2$, w′ is adjacent to u in <V(Sp(G)) – D>. That is, w is adjacent to u in G. Since D is a ctd-set, w′ is adjacent to some vertex, say x in D, x ∈ V(G). Then, x′ is adjacent to w in D. w′ is adjacent to x implies that w is adjacent to x in V(G) and hence in Sp(G).

(i) If u is also adjacent to x, then D – {w} is a ctd-set of Sp(G).
(ii) If u is adjacent to some vertex, say y in D, then D – {x} is a ctd-set of Sp(G).
(iii) If w′ is adjacent to some vertex, say z in D, then also D – {x} is a ctd-set of Sp(G).

Therefore, u is adjacent to w and w′ is adjacent to x only. In this case $G \cong$ <u, w, x> $\cong P_3$, a path on three vertices and {u′, v′, w′} is a γ_{ctd}-set of P_3 and hence, $\gamma_{ctd}(Sp(G)) = 3 \neq 2p-2$.
Hence, Case 2 is not possible.
From Case 1, it is concluded that $G \cong K_2$.
Conversely, if $G \cong K_2$, then $\gamma_{ctd}(Sp(G)) = 2p-2$. □

Remark 2.1. If p ≥ 3, then $\gamma_{ctd}(Sp(G)) \leq 2p-3$.
Equality holds, if $G \cong P_3$.

Observation 2.1. $\gamma_{ctd}(G) \leq \gamma_{ctd}(Sp(G))$.
Equality holds, if $G \cong K_4 - e$.

Theorem 2.4. If G is a connected graph such that $\delta(G) \geq 2$, then $\gamma_{ctd}(Sp(G)) \leq 2p-4$.

Proof. Let e = (u, v) ∈ E(G), where u, v ∈ V(G).
Let D = {u, v, u′, v′} ⊆ V(Sp(G)) and let D′ = V(Sp(G))–D.
Since $\delta(G) \geq 2$, each vertex in V(Sp(G))–D′ (= D) is adjacent to atleast one vertex in D′ and <V(Sp(G))–D′> $\cong P_4$ in Sp(G) and hence, D′ is a ctd-set of Sp(G).
Therefore, $\gamma_{ctd}(Sp(G)) \leq |D'| = |V-D| = 2p-4$.

Equality holds, if $G \cong K_4$, the complete graph on 4 vertices. □

Corollary 2.1. Let G be a connected graph such that $\delta(G) \geq 2$. If G contains a P_3 as an induced subgraph such that central vertex of P_3 is of degree atleast 3 and other two vertices in P_3 is degree atleast 2, then $\gamma_{ctd}(Sp(G)) \leq 2p-5$.

Proof. Let $V(P_3) = \{u, v, w\}$. Let $D = \{u, v, w, u', w'\}$ and then $D' = V(Sp(G)) - D$ is a ctd-set of Sp(G). Therefore, $\gamma_{ctd}(Sp(G)) \leq 2p-5$. □

Theorem 2.5. Let G be a connected, non complete graph with $\delta(G) \geq 3$, then $\gamma_{ctd}(Sp(G)) \leq 2p-5$.

Proof. Since G is not complete, G contains a P_3 as an induced subgraph. Let the vertices of P_3 be u, v and w where v is the central vertex of P_3. Let $D = \{u, v, w, u', w'\}$ and let $D' = V(Sp(G))-D$.
Since $\delta(G) \geq 3$, each vertex in $V(Sp(G))-D'$ (= D) is adjacent to some vertex in D′ and $<V(Sp(G))-D'> \cong K_{1,4}$ with v as the central vertex of $K_{1,4}$ in Sp(G). Therefore, D′ is a ctd-set of Sp(G) and hence, $\gamma_{ctd}(Sp(G)) \leq 2p-5$. Equality holds, if $G \cong K_5-e$. □

Theorem 2.6. Let G be a connected graph such that $\delta(G) \geq 2$, then $\gamma_{ctd}(Sp(G)) \leq 2p-\delta(G)-1$.

Proof. Let v be a vertex of maximum degree in G. Let $S = \{u' \in V(Sp(G)) : u \in N(v)\}$ and $D' = V(Sp(G))-S-\{v\}$. Then, $V(Sp(G))-D' = S \cup \{v\}$. Let $u \in N(v)$. Since $\delta(G) \geq 2$, $\deg(u) \geq 2$ and hence, u is adjacent to a vertex of G other than v. Let u be adjacent to w such that $w \neq v$. Then, $u' \in S$ is adjacent to w, where w is a vertex in D′. That is, $u' \in S$ is adjacent to a vertex in D′. Also, v is adjacent to atleast one vertex in G and hence in S. Therefore, D′ is a dominating set of V(Sp(G)). Moreover, $<V(Sp(G))-D'> \cong K_{1,\Delta(G)}$ and hence, D′ is a ctd-set of Sp(G). Therefore,

$$\gamma_{ctd}(Sp(G)) \leq |V(Sp(G)) - S - \{v\}|$$
$$= 2p - \Delta(G) - 1.$$

Equality holds, if $G \cong K_p$, $p \geq 4$. □

Theorem 2.7. For any connected graph G with p vertices, $\gamma_{ctd}(Sp(G)) \leq p + \gamma_{ctd}(G)$.

Proof. Let D be a minimum ctd-set of G and hence, $|D| = \gamma_{ctd}(G)$.
Therefore, $<V(G)-D>$ is a tree.
Now, the set $D' = D \cup V'(G)$ is a ctd-set of Sp(G).
Hence, $\gamma_{ctd}(Sp(G)) \leq |D'| = \gamma_{ctd}(G) + p$. □

Theorem 2.8. If $\gamma_{ctd}(G) = 1$ then $\gamma_{ctd}(Sp(G)) \leq t+1$, where t is the number of vertices of G of degree atleast 2.

Proof. Assume $\gamma_{ctd}(G) = 1$.
Then $G \cong T + K_1$, where T is a tree with atleast two vertices.
Let $V(K_1) = \{v\}$ and let $D' = \{v_1' : \deg(v_1) \geq 2\}$.
Then, $D' \subseteq V(Sp(G))$ and $|D'| = t-1$, where t is the number of vertices in G of degree atleast 2. Let $D = \{v, v'\} \cup D'$, then, $D \subseteq V(Sp(G))$ and all the vertices in $<V(Sp(G))-D>$ are adjacent to v and $<V(Sp(G))-D>$ is the tree obtained from the tree T by attaching m pendant edges at each of the supports u of T, where $\deg_T(u) = m$, $m \geq 1$.
Therefore, D is a ctd-set of Sp(G) and hence, $\gamma_{ctd}(G) \leq |D| = t+1$.
Equality holds, if $G \cong P_n+K_1$ and
$G \cong K_{1,n} + K_1$, $n \geq 2$. □

Theorem 2.9. Let D be a minimum ctd-set of a connected graph G with least number of edges. Let $\{S_1, S_2, ..., S_r\}$ ($r \geq 1$) be the star decomposition of $<V-D>_G$ such that $|V(S_i)| \geq 2$, i = 1, 2, ..., r. Then, $\gamma_{ctd}(Sp(G)) \leq 2\gamma_{ctd}(G) + r$.

Proof. Let $T' = \{u' : u \in D\}$ and $T'' = \{x_i' : x_i \text{ is the centre of } S_i\}$. For each star S_i with claws $y_1, y_2, ..., y_{ti-1}$, the corresponding vertices $y_1', y_2', ..., y_{ti-1}'$, are dominated by those vertices in D dominating $y_1, y_2, ..., y_{ti-1}$ in $<V-D>_G$.
Then, the set $D' = D \cup T' \cup T''$ is a ctd-set of Sp(G). Therefore

$$\gamma_{ctd}(Sp(G)) \leq |D| + |T'| + |T''|$$
$$= \gamma_{ctd}(G) + \gamma_{ctd}(G) + r$$
$$= 2\gamma_{ctd}(G) + r$$

That is, $\gamma_{ctd}(Sp(G)) \leq 2\gamma_{ctd}(G) + r$.
The above bound is attained, if $G \cong K_4-e$.
Let v_1, v_2, v_3, v_4 be the vertices of K_4-e, where v_1 and v_3 have degree 3 and v_2 and v_4 have degree 2.
The set $D = \{v_1\}$ is a minimum ctd-set of K_4-e, and hence, $\gamma_{ctd}(G) = 1$.
Also, $<V-D> \cong P_3$, with v_3 as the central vertex. Then, $D' = \{v_1, v_1', v_3'\}$ is a minimum ctd-set of Sp(G). Hence, $\gamma_{ctd}(Sp(G)) = 3 = 2\gamma_{ctd}(G) + 1$. □

Theorem 2.10. Let G be a unicyclic graph. Then, $\gamma_{ctd}(G) = \gamma_{ctd}(Sp(G))$ if and only if $G \cong C_n$, $n \neq 3, 5$ and G is the graph obtained by attaching pendant edges at exactly one vertex of C_4.

Proof. Assume $\gamma_{ctd}(G) = \gamma_{ctd}(Sp(G))$.
Case 1. The cycle in G is C_3.
If $\gamma_{ctd}(G) = 1$, then $\gamma_{ctd}(Sp(G)) = 2$. Hence, $G \neq C_3$. Let the vertices of C_3 be v_1, v_2, v_3. Let a pendant edge be attached at exactly one vertex of C_3, say at v_1. Let the pendant edge be (v_1, v_4). Then, $\{v_1, v_1', v_4, v_4'\}$ is a minimum ctd-set of Sp(G) and hence, $\gamma_{ctd}(Sp(G)) = 4$, whereas $\gamma_{ctd}(G) = 2$. Hence,

$\gamma_{ctd}(G) > \gamma_{ctd}(Sp(G))$. Similarly, if either two or more edges are attached at exactly one vertex of C_3 (or) pendant edges are attached at vertices of C_3, then also $\gamma_{ctd}(Sp(G)) > \gamma_{ctd}(G)$.

Case 2. The cycle in G is C_4.

If $G \cong C_4$, then $\gamma_{ctd}(G) = \gamma_{ctd}(Sp(G)) = 2$.

Let the vertices of C_4 be v_1, v_2, v_3, v_4 in order.

(i) Let m ($m \geq 1$) pendant edges be attached at exactly one vertex of C_4, say at v_1, then the set consisting of v_1, v_4 and the pendant vertices forms a minimum ctd-set of G, whereas the set consisting of v_1, v_4 together with the duplicate vertices corresponding to pendant vertices in G forms a minimum ctd-set of Sp(G). Hence, $\gamma_{ctd}(G) = m+2 = \gamma_{ctd}(Sp(G))$.

(ii) If exactly one pendant edge is attached at each of two or more vertices of C_4, then $\gamma_{ctd}(Sp(G)) > \gamma_{ctd}(G)$, since if G contains P_3 as an induced subgraph, Sp(G) contains C_4 as an induced subgraph. Therefore, for each P_3 in G, a vertex is to be added in the ctd-set D′ of Sp(G), for <V(Sp(G))–D′> to be a tree.

Case 3. The cycle in G is C_5.

If $G \cong C_5$, then $\gamma_{ctd}(G) = 3$ and $\gamma_{ctd}(Sp(G)) = 4$. Also, if one or more pendant edges are attached at the vertices of C_5, then $\gamma_{ctd}(Sp(G)) > \gamma_{ctd}(G)$.

Case 4. G contains C_n ($n \geq 6$) as the unique cycle.

If $G \cong C_n$ ($n \geq 6$) then $\gamma_{ctd}(G) = \gamma_{ctd}(Sp(G)) = n-2$.

As in Case 2, if one or more pendant edges are attached at atleast one of the vertices of C_5, then $\gamma_{ctd}(Sp(G)) > \gamma_{ctd}(G)$.

The same result holds, if paths of length atleast 2 are attached at the vertices of C_n, $n \geq 3$.

From the above cases, it is concluded that, $\gamma_{ctd}(G) = \gamma_{ctd}(Sp(G))$ if $G \cong C_n$ ($n \neq 3, 5$) and G is the graph obtained by attaching pendant edges at exactly one vertex of C_4.

Converse follows easily. □

Theorem 2.11. Let G be a connected graph with p vertices ($p \geq 3$) $V(Sp(G)) = V(G) \cup V'(G)$. Then, V′(G) is a ctd-set of G if and only if G is a tree.

Proof. Assume V′(G) is a ctd-set of Sp(G). Then, each vertex in V(Sp(G))–V(G) is adjacent to atleast one vertex in V′(G) and <V(Sp(G)–V′(G)> is a tree. That is, <V(G)> is a tree.

Conversely, assume G is a tree. Let D = V′(G). That is, D contains all the duplicate vertices of G. Since G is connected, each vertex v in V(Sp(G))–D = V(G) is adjacent to $deg_G(v)$ vertices in D and V(Sp(G)–D = V(G) is a tree. Hence, D is a ctd-set of G. □

Remark 2.2. For a tree T with p vertices, $\gamma_{ctd}(Sp(G)) \leq p$. This bound is attained, if $G \cong K_{1,n}$, $n \geq 1$.

References

[1] F. Harary, *Graph Theory*, Narosa Publishing House, Reprint, 1969.

[2] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker Inc., New York, 1998.

[3] T.N. Janakiraman, S. Muthammai and M. Bhanumathi, "On Splitting Graphs," *Ars Combinatoria*, vol. 82, pp. 211–221, 2007.

[4] E. Sampathkumar and H.B. Walikar, "On the splitting graph of a graph," *J. Karnataka Univ. Sci.*, vol. 25 and 26 (combined) pp. 13–16, 1980-1981.

[5] S. Muthammai, M. Bhanumathi and P. Vidhya, "Complementary tree domination number of a graph," *International Mathematical Forum*, vol. 6, no. 26, pp. 1273–1282, 2011.

[6] V. Swaminathan and A. Subramanian, "Domination number of splitting graph,| *J. Combin. Inform. System Sci.*, vol. 26, no. 1–4, pp. 17–22, 2001.