Oscillation of second-order nonlinear delay dynamic equations on time scales

H.A.Agwa[1], Ahmed M.M. Khodier[2], M.Hamam[3]

Department of Mathematics, Faculty of Education,

Ain Shams University, Roxy, Cairo, Egypt.

Abstract

In this paper, we establish some new oscillation criteria for second order nonlinear neutral delay dynamic equation of the form

$$(r(t)((m(t)y(t)+p(t)y(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+f(t,y(\delta(t)))=0$$

on a time scale $\mathbb{T}$*. The present results not only generalize and extend some existing results but also can be applied to the oscillation problems that are not covered before. Finally, we give some examples to illustrate our main results.*

Keywords: *Oscillation, Neutral equation, Delay equation, Dynamic equation, Time scale*.

Mathematics Subject Classification(2010): 34K11, 39A10, 34N05, 39A99.

1 Introduction

The theory of time scales was introduced by Hilger [5] in order to unify, extend and generalize ideas from discrete calculus, quantum calculus and continuous calculus to arbitrary time scale calculus. A time scale is an arbitrary closed subset of the reals. The cases when time scale equals to the reals or to the integers, the obtained results represent the classical theories of differential and difference equations. Many other interesting time scales exist (e.g., $\mathbb{T}=q^{\mathbb{N}_0}:=\{q^t: t\in\mathbb{N}_0 \text{ for } q>1\}$ which has important applications in quantum theory, $\mathbb{T}=h\mathbb{N}$ with $h>0$, $\mathbb{T}=\mathbb{N}^2$ and $\mathbb{T}=\mathbb{T}^n$ the space of harmonic numbers). For an introduction to time scale calculus and dynamic equations, we refer to the seminal books by Bohner and Peterson [3, 4]. In recent years, there has been much research activities concerning the oscillation of solutions of second-order nonlinear neutral delay dynamic equations on time scales, see [5-10, 12] and references cited therein.

In 2004, Agarwal et al. [1] considered the second order nonlinear neutral delay dynamic equation

$$(r(t)((y(t)+p(t)y(t-\tau))^{\Delta})^{\gamma})^{\Delta}+f(t,y(t-\delta))=0, t\in\mathbb{T}, \tag{1.1}$$

where $\gamma>0$ is a quotient of odd positive integers. In 2006, Saker [9] further studied Eq. (1.1) for an odd positive integer $\gamma\geq 1$.

Also, in 2006, Wu et al. [11] studied the second order nonlinear neutral delay dynamic equation with variable delays

$$(r(t)((y(t)+p(t)y(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+f(t,y(\delta(t)))=0, t\in\mathbb{T}, \tag{1.2}$$

where $\gamma\geq 1$. In 2007, Saker et al. [10] discussed Eq. (1.2) for an odd positive integer $\gamma\geq 1$. In 2010, Shao-Yan et al. [12] studied Eq. (1.2) for an odd positive integer $\gamma>0$. In 2013, Saker et al. [7] considered the second order nonlinear neutral delay dynamic equation with variable delays

$$(r(t)(m(t)y(t)+p(t)y(\tau(t)))^{\Delta})^{\Delta}+f(t,y(\delta(t)))=0, t\in\mathbb{T}, \tag{1.3}$$

where,

$f:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is a continuous function such that $uf(t,u)>0$ for all $u\neq 0$ and there exists $K>0$ such that $|f(u)|\geq K|u|$.

In this paper, we study the oscillation of the second order nonlinear neutral delay dynamic equation with variable delays of the form

$$(r(t)((m(t)y(t)+p(t)y(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+f(t,y(\delta(t)))=0, t\in\mathbb{T}, \tag{1.4}$$

where $\mathbb{T}$ is a time scale. Through out this paper, we consider the following hypotheses:

(H_1) γ is a quotient of odd positive integers.

(H_2) $\tau(t),\delta(t)\in C_{rd}(\mathbb{T},\mathbb{T})$, $\tau(t)\leq t$ and $\lim_{t\to\infty}(\tau(t))=\infty$, $\delta(t)\leq t$ and $\lim_{t\to\infty}(\delta(t))=\infty$.

(H_3) $m(t)$, $p(t)$ and $r(t)$ are real valued rd-continuous positive functions defined on $\mathbb{T}$ and

$$m(\tau(t))>p(t).$$

(H_4) $f:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is a continuous function such that $uf(t,u)>0$, for all $u\neq 0$, and there exists $q(t)\in C_{rd}(\mathbb{T},\mathbb{R}^+)$ such that $|uf(t,u)|\geq q(t)|u|^{\beta+1}$, where β is quotient of odd positive integers.

Also, the following two conditions are taken into consideration:

$$\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\Delta(t) = \infty \tag{1.5}$$

and

$$\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\Delta(t) < \infty \tag{1.6}$$

Now, defining

$$x(t):=m(t)y(t)+p(t)y(\tau(t)). \tag{1.7}$$

Eq. (1.4) reduces to

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}+f(t,y(\delta(t)))=0. \tag{1.8}$$

2 Preliminaries and Lemmas

Lemma 2.1. Assume that the conditions H_1-H_4 and (1.5) hold. If $y(t)$ is an eventually positive solution of Eq. (1.4), then there exists $t_1\in\mathbb{T}$ sufficiently large such that $x(t)>0$, $x^{\Delta}(t)>0$ and $(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}<0$ for $t\in[t_1,\infty)_{\mathbb{T}}$.

Proof. Since, $y(t)$ is an eventually positive solution of Eq. (1.4), then, by H_2 there exists $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $y(t)>0$, $y(\tau(t))>0$ and $y(\delta(t))>0$ for $t\geq t_1$.

From Eq. (1.7) and H_3, we get

$$x(t)\geq m(t)y(t) \text{ for } t\geq t_1.$$

Then,

$$x(t) > 0 \text{ for } t \geq t_1.$$

Also, by Eq. (1.8) and H_4, we have

$$(r(t)(x^\Delta(t))^\gamma)^\Delta \leq -q(t)y^\beta(\delta(t)) \text{ for } t \geq t_1.$$

Then,

$$(r(t)(x^\Delta(t))^\gamma)^\Delta < 0 \text{ for } t \geq t_1,$$

which implies that $(r(t)(x^\Delta(t))^\gamma)$ is decreasing on $[t_1,\infty)_{\mathbb{T}}$.

Now, we claim that $(r(t)(x^\Delta(t))^\gamma) > 0$ on $[t_1,\infty)_{\mathbb{T}}$. Therefore, we assume that this is not true. Hence, there exists $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $(r(t_2)(x^\Delta(t_2))^\gamma) < 0$. Since $(r(t)(x^\Delta(t))^\gamma)$ is decreasing on $[t_1,\infty)_{\mathbb{T}}$, then

$$r(t)(x^\Delta(t))^\gamma \leq r(t_2)(x^\Delta(t_2))^\gamma \text{ for } t \geq t_2.$$

Hence, we have

$$(x^\Delta(t)) \leq (r(t_2))^{\frac{1}{\gamma}} x^\Delta(t_2)(\tfrac{1}{r(t)})^{\frac{1}{\gamma}}. \tag{2.1}$$

Integrating inequality (2.1) from t_2 to $t \geq t_2$ and using H_3, we get

$$x(t) \leq x(t_2) + (r(t_2)^{\frac{1}{\gamma}})x^\Delta(t_2)\int_{t_2}^{t}(\frac{1}{r(s)})^{\frac{1}{\gamma}}\Delta s \to -\infty \text{as } (t\to\infty),$$

then $x(t) < 0$ which is a contradiction. Therefore, $(r(t)(x^\Delta(t))^\gamma) > 0$.

Hence, $x^\Delta(t) > 0$ for $t \geq t_1$. This completes the proof.

Remark 2.1. By Lemma (2.1), Equations (1.4), (1.7) and Hypothesis H_2 , H_4, we get

$$\begin{aligned}
0 &\geq (r(t)(x^\Delta(t))^\gamma)^\Delta + q(t)y^\beta(\delta(t)) \\
&\geq (r(t)(x^\Delta(t))^\gamma)^\Delta + \frac{q(t)}{m^\beta(\delta(t))}[x(\delta(t)) - p(\delta(t))y(\tau(\delta(t)))]^\beta \\
&\geq (r(t)(x^\Delta(t))^\gamma)^\Delta + \frac{q(t)}{m^\beta(\delta(t))}[x(\delta(t)) - \frac{p(\delta(t))}{m(\tau(\delta(t)))}x(\tau(\delta(t)))]^\beta \\
&\geq (r(t)(x^\Delta(t))^\gamma)^\Delta + \frac{q(t)}{m^\beta(\delta(t))}[x(\delta(t)) - \frac{p(\delta(t))}{m(\tau(\delta(t)))}x(\delta(t))]^\beta \\
&\geq (r(t)(x^\Delta(t))^\gamma)^\Delta + \frac{q(t)}{m^\beta(\delta(t))}[1 - \frac{p(\delta(t))}{m(\tau(\delta(t)))}]^\beta(x(\delta(t)))^\beta.
\end{aligned}$$

Define,

$$A(t,a) = \int_a^t (\tfrac{1}{r(s)})^{\frac{1}{\gamma}}\Delta s \text{ and } \psi(t,a) = \tfrac{A(\delta(t),a)}{A(\sigma(t),a)}.$$

Lemma 2.2. Assume that $H_1 - H_4$, (1.5)hold and $y(t)$ is an eventually positive solution of Eq. (1.4). Then there exists $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that

$$x(\delta(t)) \geq \psi(t,t_2)x(\sigma(t)) \text{for} t \geq \delta(t) \geq t_2.$$

Proof. Since $H_1 - H_4$ and (1.5) hold, then by Lemma (2.1) we have $(r(t)(x^\Delta(t))^\gamma)^\Delta < 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$, then $r(t)(x^\Delta(t))^\gamma$ is decreasing function in the interval $[t_1,\infty)_{\mathbb{T}}$.

For $t > t_2 \geq t_1$, we have

$$x(\sigma(t))-x(\delta(t))=\int_{\delta(t)}^{\sigma(t)}\frac{(r(s)(x^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{(r(s))^{\frac{1}{\gamma}}}\Delta s\leq r^{\frac{1}{\gamma}}(\delta(t))x^{\Delta}(\delta(t))\int_{\delta(t)}^{\sigma(t)}(\frac{1}{r(s)})^{\frac{1}{\gamma}}\Delta s.$$

Therefore,

$$\frac{x(\sigma(t))}{x(\delta(t))}\leq 1+\frac{r^{\frac{1}{\gamma}}(\delta(t))x^{\Delta}(\delta(t))}{x(\delta(t))}A(\sigma(t),\delta(t)). \tag{2.2}$$

Also,

$$x(\delta(t))\geq x(\delta(t))-x(t_2)=\int_{t_2}^{\delta(t)}\frac{(r(s)(x^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{(r(s))^{\frac{1}{\gamma}}}\Delta s\geq r^{\frac{1}{\gamma}}(\delta(t))x^{\Delta}(\delta(t))\int_{t_2}^{\delta(t)}(\frac{1}{r(s)})^{\frac{1}{\gamma}}\Delta s.$$

Hence,

$$\frac{r^{\frac{1}{\gamma}}(\delta(t))x^{\Delta}(\delta(t))}{x(\delta(t))}\leq\frac{1}{A(\delta(t),t_2)}. \tag{2.3}$$

From (2.2) and (2.3), we get

$$\frac{x(\sigma(t))}{x(\delta(t))}\leq 1+\frac{A(\sigma(t),\delta(t))}{A(\delta(t),t_2)}=\frac{A(\sigma(t),\delta(t))+A(\delta(t),t_2)}{A(\delta(t),t_2)}=\frac{A(\sigma(t),t_2)}{A(\delta(t),t_2)}.$$

Hence,

$$x(\delta(t))\geq\psi(t,t_2)x(\sigma(t)). \tag{2.4}$$

This completes the proof.

Lemma 2.3.[12]If $g(u)=Bu-Au^{\frac{\gamma+1}{\gamma}}$, where$A>0$, B are constants and γ is a quotient of odd positive integers, then g attains it's maximum value on $\mathbb{R}$ at $u^*=(\frac{B\gamma}{A(\gamma+1)})^{\gamma}$ and$max_{u\in\mathbb{R}}g=g(u^*)=\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^{\gamma}}$.

Lemma 2.4.[2] If x and z are Δ-differentiable on $\mathbb{T}$, then for $x\neq 0$ and any $t\in\mathbb{T}$, we have$x^{\Delta}(\frac{z^2}{x})^{\Delta}=(z^{\Delta})^2-xx^{\sigma}[(\frac{z}{x})^{\Delta}]^2$.

3 Main results

Case(1): when condition (1.5) holds.

Theorem 3.1. Assume thatthe conditionsH_1-H_4 hold and $\gamma\geq\beta>0$. Also, assume that there exists a positive rd-continuous Δ-differentiable function $z(t)$ such that for a constant $b>0$,

$$\lim_{t\to\infty} sup\int_{t_0}^{t}[Q(s)q(s)-\frac{1}{(\gamma+1)^{(\gamma+1)}}\frac{r(s)(z^{\Delta}(s))^{\gamma+1}}{z^{\gamma}(s)}]\Delta s=\infty, \tag{3.1}$$

where

$$Q(s)=\frac{z(s)}{m^{\beta}(\delta(s))}[1-\frac{p(\delta(s))}{m(\tau(\delta(s)))}]^{\beta}\frac{(\psi(s,t_2))^{\beta}}{b^{\gamma-\beta}(A(s,t_1))^{\gamma-\beta}}.$$

Then, every solution of Eq. (1.4) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose that $y(t)$ is a nonoscillatory solution of Eq. (1.4), then by H_1-H_4 there exists t_1 sufficiently large such that $y(t)>0$, $y(\tau(t))>0$ and$y(\delta(t))>0$. Define the function $w(t)$ by the Riccati substitution

$$w(t):=\frac{z(t)r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)} \text{ for } t\geq t_1, \tag{3.2}$$

where $x(t)$ is defined by Eq. (1.7). Then $w(t)>0$ and

$$\begin{aligned} w^{\Delta}(t) &= (r(x^{\Delta})^{\gamma})^{\Delta}(\frac{z}{x^{\gamma}})+(r(x^{\Delta})^{\gamma})^{\sigma}(\frac{z}{x^{\gamma}})^{\Delta} \\ &= (r(x^{\Delta})^{\gamma})^{\Delta}(\frac{z}{x^{\gamma}})+(r(x^{\Delta})^{\gamma})^{\sigma}(\frac{z^{\Delta}x^{\gamma}-z(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}) \\ &= (r(x^{\Delta})^{\gamma})^{\Delta}(\frac{z}{x^{\gamma}})+(r(x^{\Delta})^{\gamma})^{\sigma}\frac{z^{\Delta}}{(x^{\sigma}(t))^{\gamma}}-(r(x^{\Delta})^{\gamma})^{\sigma}\frac{z(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}} \\ &= (r(x^{\Delta})^{\gamma})^{\Delta}(\frac{z}{x^{\gamma}})+w^{\sigma}\frac{z^{\Delta}}{z^{\sigma}}-(r(x^{\Delta})^{\gamma})^{\sigma}\frac{z(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}. \end{aligned}$$

Using Remark (2.1), we have

$$w^{\Delta}(t)\leq z(\frac{-q}{m^{\beta}(\delta(t))})[1-\frac{p(\delta(t))}{m(\tau(\delta(t)))}]^{\beta}\frac{(x(\delta(t)))^{\beta}}{x^{\gamma}}+w^{\sigma}\frac{z^{\Delta}}{z^{\sigma}}-(r(x^{\Delta})^{\gamma})^{\sigma}\frac{z(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}.$$

By Lemma (2.2), we have

$$w^{\Delta}(t)\leq z(\frac{-q}{m^{\beta}\left(\delta(t)\right)})[1-\frac{p\left(\delta(t)\right)}{m\left(\tau\left(\delta(t)\right)\right)}]^{\beta}\frac{(\psi(t,t_2))^{\beta}(x^{\sigma})^{\beta}}{x^{\gamma}}+w^{\sigma}\frac{z^{\Delta}}{z^{\sigma}}-(r(x^{\Delta})^{\gamma})^{\sigma}\frac{z(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}},$$

for $t\geq t_2$. Since $x^{\Delta}>0$, we obtain

$$\begin{aligned} w^{\Delta}(t) &\leq z(\frac{-q}{m^{\beta}(\delta(t))})[1-\frac{p(\delta(t))}{m(\tau(\delta(t)))}]^{\beta}(\psi(t,t_2))^{\beta}\frac{x^{\beta}}{x^{\gamma}}+w^{\sigma}\frac{z^{\Delta}}{z^{\sigma}}-(r(x^{\Delta})^{\gamma})^{\sigma}\frac{z(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}} \\ &\leq z(\frac{-q}{m^{\beta}(\delta(t))})[1-\frac{p(\delta(t))}{m(\tau(\delta(t)))}]^{\beta}(\psi(t,t_2))^{\beta}\frac{1}{x^{\gamma-\beta}}+w^{\sigma}\frac{z^{\Delta}}{z^{\sigma}}-(r(x^{\Delta})^{\gamma})^{\sigma}\frac{z(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}. \end{aligned} \tag{3.3}$$

From Lemma (2.1), we have

$$x(t)-x(t_1)=\int_{t_1}^{t}\frac{(r(s)(x^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{(r(s))^{\frac{1}{\gamma}}}\Delta(s)\leq r^{\frac{1}{\gamma}}(t_1)x^{\Delta}(t_1)A(t,t_1).$$

Thus, there exists $T\in[t_1,\infty)_{\mathbb{T}}$ and a suitable constant $b>0$ such that

$$x(t)\leq bA(t,t_1) \text{ for } t\in[T,\infty)_{\mathbb{T}}. \tag{3.4}$$

Since $\gamma\geq\beta$, then

$$\frac{-1}{x^{\gamma-\beta}(t)}\leq\frac{-1}{b^{\gamma-\beta}(A(t,t_1))^{\gamma-\beta}} \text{ for } t\in[T,\infty)_{\mathbb{T}}. \tag{3.5}$$

Take $t_3=max\{t_2,T\}$, then from inequalities (3.3) and (3.5) we have

$$\begin{aligned} w^{\Delta}(t) &\leq z(\frac{-q}{m^{\beta}(\delta(t))})[1-\frac{p(\delta(t))}{m(\tau(\delta(t)))}]^{\beta}\frac{(\psi(t,t_2))^{\beta}}{b^{\gamma-\beta}(A(t,t_1))^{\gamma-\beta}}+w^{\sigma}\frac{z^{\Delta}}{z^{\sigma}}-(r(x^{\Delta})^{\gamma})^{\sigma}\frac{z(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}} \\ &\leq -q(t)Q(t)+w^{\sigma}\frac{z^{\Delta}}{z^{\sigma}}-(r(x^{\Delta})^{\gamma})^{\sigma}\frac{z(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}, \end{aligned} \tag{3.6}$$

for $t\in[t_3,\infty)_{\mathbb{T}}$,

where,

$$Q(s)=\frac{z(s)}{m^{\beta}\left(\delta(s)\right)}[1-\frac{p\left(\delta(s)\right)}{m\left(\tau\left(\delta(s)\right)\right)}]^{\beta}\frac{(\psi(s,t_2))^{\beta}}{b^{\gamma-\beta}(A(s,t_1))^{\gamma-\beta}}.$$

Using $\gamma>0$, Lemma (2.1) and Keller's chain rule, we get

$$
\begin{aligned}
(x^\gamma(t))^\Delta &= \gamma \int_0^1 [x(t) + h\mu(t)x^\Delta(t)]^{\gamma-1} dh\, x^\Delta(t) \\
&= \gamma \int_0^1 [(1-h)x + hx^\sigma]^{\gamma-1} dh\, x^\Delta(t).
\end{aligned}
$$

Hence,

$$
(x^\gamma(t))^\Delta \geq \begin{cases} \gamma x^{\gamma-1}(t)x^\Delta(t) & \text{if } \gamma \geq 1, \\ \gamma (x^\sigma)^{\gamma-1}(t) x^\Delta(t) & \text{if } \gamma < 1. \end{cases} \tag{3.7}
$$

From Lemma (2.1), we have $[r(t)(x^\Delta(t))^\gamma]^\Delta < 0$ for $t \in [t_3, \infty)_{\mathbb{T}}$. Therefore,

$$
r(\sigma(t))(x^\Delta(\sigma(t)))^\gamma \leq r(t)(x^\Delta(t))^\gamma. \tag{3.8}
$$

Hence,

$$
(x^\gamma(t))^\Delta \geq \begin{cases} \dfrac{\gamma x^{\gamma-1}(t) r^{\frac{1}{\gamma}}(\sigma(t)) x^\Delta(\sigma(t))}{r^{\frac{1}{\gamma}}(t)} & \text{if } \gamma \geq 1, \\ \dfrac{\gamma (x^\sigma)^{\gamma-1} r^{\frac{1}{\gamma}}(\sigma(t)) x^\Delta(\sigma(t))}{r^{\frac{1}{\gamma}}(t)} & \text{if } \gamma < 1, \end{cases} \tag{3.9}
$$

and inequality (3.6) becomes

$$
w^\Delta(t) \leq \begin{cases} -q(t)Q(t) + w^\sigma \dfrac{z^\Delta}{z^\sigma} - [r^\sigma(x^\Delta(\sigma(t)))^\gamma] \dfrac{z\gamma x^{\gamma-1}(t) r^{\frac{1}{\gamma}}(\sigma(t)) x^\Delta(\sigma(t))}{r^{\frac{1}{\gamma}}(t) x^\gamma(t) (x^\sigma)^\gamma} & \text{if } \gamma \geq 1, \\ -q(t)Q(t) + w^\sigma \dfrac{z^\Delta}{z^\sigma} - [r^\sigma(x^\Delta(\sigma(t)))^\gamma] \dfrac{z\gamma (x^\sigma)^{\gamma-1} r^{\frac{1}{\gamma}}(\sigma(t)) x^\Delta(\sigma(t))}{r^{\frac{1}{\gamma}}(t) x^\gamma(t) (x^\sigma)^\gamma} & \text{if } \gamma < 1. \end{cases}
$$

Since $x^\Delta(t) > 0$, we have

$$
w^\Delta(t) \leq -q(t)Q(t) + w^\sigma \frac{z^\Delta}{z^\sigma} - \frac{\gamma z}{r^{\frac{1}{\gamma}}} \frac{(r^\sigma)^{\frac{\gamma+1}{\gamma}} [(x^\Delta)^\sigma]^{\gamma+1}}{(x^\sigma)^{\gamma+1}} = -q(t)Q(t) + w^\sigma \frac{z^\Delta}{z^\sigma} - \frac{\gamma z}{r^{\frac{1}{\gamma}} (z^\sigma)^{\frac{\gamma+1}{\gamma}}} (w^\sigma)^{\frac{\gamma+1}{\gamma}}. \tag{3.10}
$$

Taking

$$
B = \frac{z^\Delta}{z^\sigma}, A = \frac{\gamma z}{r^{\frac{1}{\gamma}} (z^\sigma)^{\frac{\gamma+1}{\gamma}}} > 0 \text{ and } u = w^\sigma,
$$

then, Eq. (3.10) becomes

$$
w^\Delta(t) \leq -qQ + Bu - Au^{\frac{\gamma+1}{\gamma}}.
$$

Using Lemma (2.3), we get

$$
w^\Delta(t) \leq -qQ + \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma} \leq -qQ + \frac{1}{(\gamma+1)^{\gamma+1}} \frac{r(z^\Delta)^{\gamma+1}}{z^\gamma}.
$$

Integrating from t_3 to t, we get

$$
\int_{t_3}^{t} [Q(s)q(s) - \frac{1}{(\gamma+1)^{\gamma+1}} \frac{r(z^\Delta(s))^{\gamma+1}}{z^\gamma(s)}] \Delta s \leq w(t_3) - w(t) \leq w(t_3).
$$

Taking the limit suprimum of both sides as $t \to \infty$, we get a contradiction with condition (3.1). This completes the proof.

Theorem 3.2. Assume that the conditions $H_1 - H_4$ hold and $\beta \geq \gamma > 0$. Also, assume that there exists a positive rd-continuous Δ-differentiable function $z(t)$ such that for a constant $b > 0$,

$$\lim_{t\to\infty} sup \int_{t_0}^{t} [Q_1(s)q(s) - \frac{1}{(\gamma+1)^{(\gamma+1)}} \frac{r(s)(z^\Delta(s))^{\gamma+1}}{z^\gamma(s)}]\Delta s = \infty, \tag{3.11}$$

where

$$Q_1(s) = \frac{z(s)}{m^\beta(\delta(s))}[1 - \frac{p(\delta(s))}{m(\tau(\delta(s)))}]^\beta(\psi(t,t_2))^\beta b^{\beta-\gamma}.$$

Then, every solution of Eq. (1.4) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Suppose that $y(t)$ is a nonoscillatory solution of Eq. (1.4) and proceeding as in the proof of Theorem (3.1), we get

$$w^\Delta(t) \leq z(\frac{-q}{m^\beta(\delta(t))})[1 - \frac{p(\delta(t))}{m(\tau(\delta(t)))}]^\beta(\psi(t,t_2))^\beta x^{\beta-\gamma} + w^\sigma \frac{z^\Delta}{z^\sigma} - (r(x^\Delta)^\gamma)^\sigma \frac{z(x^\gamma)^\Delta}{x^\gamma(x^\sigma)^\gamma}. \tag{3.12}$$

Since $x^\Delta > 0$, we have

$$x(t_1) \leq x(t) \text{ for } t \geq t_1.$$

Then, there exists a suitable constant $b > 0$ such that

$$b \leq x(t) \text{ for } t \geq t_1.$$

Hence, inequality (3.12) becomes

$$\begin{aligned} w^\Delta(t) &\leq z(\frac{-q}{m^\beta(\delta(t))})[1 - \frac{p(\delta(t))}{m(\tau(\delta(t)))}]^\beta(\psi(t,t_2))^\beta b^{\beta-\gamma} + w^\sigma \frac{z^\Delta}{z^\sigma} - (r(x^\Delta)^\gamma)^\sigma \frac{z(x^\gamma)^\Delta}{x^\gamma(x^\sigma)^\gamma} \\ &\leq -q(t)Q_1(t) + w^\sigma \frac{z^\Delta}{z^\sigma} - (r(x^\Delta)^\gamma)^\sigma \frac{z(x^\gamma)^\Delta}{x^\gamma(x^\sigma)^\gamma}. \end{aligned}$$

The remainder of the proof is similar to that of Theorem (3.1). So, it is omitted.

Define,

$$D = \{(t,s) \in \mathbb{T}: t \geq s \geq 0\},$$

and

$$\mathbb{H}_* = \{H(t,s) \in C^1(D,R^+): H(t,t) = 0, H(t,s) > 0 \text{ for } t > s \geq 0\}.$$

Theorem 3.3. Assume that the conditions $H_1 - H_4$ hold and $\gamma \geq \beta > 0$. Also, assume that there exist a positive rd-continuous Δ-differentiable function $z(t)$ and a function $H \in \mathbb{H}_*$ such that for a constant $b > 0$,

$$\lim_{t\to\infty} sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} [H(t,s)q(s)Q(s) - \frac{C^{\gamma+1}(t,s)r(s)(z(\sigma(s)))^{\gamma+1}}{(\gamma+1)^{\gamma+1}H^\gamma(t,s)z^\gamma(s)}]\Delta s = \infty, \tag{3.13}$$

where

$$Q(s) = \frac{z(s)}{m^\beta(\delta(s))}[1 - \frac{p(\delta(s))}{m\left(\tau(\delta(s))\right)}]^\beta \frac{(\psi(t,t_2))^\beta}{b^{\gamma-\beta}(A(t,t_1))^{\gamma-\beta}}$$

and

$$C(t,s) = H_S^\Delta(t,s) + H(t,s)\frac{z^\Delta(s)}{z(\sigma(s))}.$$

Then, every solution of Eq. (1.4) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Suppose that $y(t)$ is a nonoscillatory solution of Eq. (1.4) and proceeding as in the proof of Theorem (3.1), we get

$$w^{\Delta}(t) \le -q(t)Q(t) + w^{\sigma}\frac{z^{\Delta}}{z^{\sigma}} - \frac{\gamma z}{r^{\frac{1}{\gamma}}(z^{\sigma})^{\frac{\gamma+1}{\gamma}}}(w^{\sigma})^{\frac{\gamma+1}{\gamma}}. \tag{3.14}$$

From inequality (3.14), we find that for a function $H \in \mathbb{H}_*$ and all $t \ge t_3$, we have

$$\int_{t_3}^{t} H(t,s)q(s)Q(s)\Delta s \le$$

$$-\int_{t_3}^{t} H(t,s)w^{\Delta}(s)\Delta s + \int_{t_3}^{t} H(t,s)w^{\sigma}(s)\frac{z^{\Delta}}{z^{\sigma}}\Delta s - \int_{t_3}^{t} H(t,s)\frac{\gamma z}{r^{\frac{1}{\gamma}}(z^{\sigma})^{\frac{\gamma+1}{\gamma}}}(w^{\sigma})^{\frac{\gamma+1}{\gamma}}\Delta s. \tag{3.15}$$

Integrating by parts, we get

$$-\int_{t_3}^{t} H(t,s)w^{\Delta}(s)\Delta s = -H(t,s)w(s)|_{t_3}^{t} + \int_{t_3}^{t} H_s^{\Delta}(t,s)w^{\sigma}(s)\Delta s =$$

$$H(t,t_3)w(t_3) + \int_{t_3}^{t} H_s^{\Delta}(t,s)w^{\sigma}(s)\Delta s. \tag{3.16}$$

From inequality (3.15) and Eq. (3.16), we have

$$\begin{aligned}\int_{t_3}^{t} H(t,s)q(s)Q(s)\Delta s &\le H(t,t_3)w(t_3) + \int_{t_3}^{t} H_s^{\Delta}(t,s)w^{\sigma}(s)\Delta s + \int_{t_3}^{t} H(t,s)w^{\sigma}(s)\frac{z^{\Delta}}{z^{\sigma}}\Delta s \\ &\quad - \int_{t_3}^{t} H(t,s)\frac{\gamma z}{r^{\frac{1}{\gamma}}(z^{\sigma})^{\frac{\gamma+1}{\gamma}}}(w^{\sigma})^{\frac{\gamma+1}{\gamma}}\Delta s \\ &\le H(t,t_3)w(t_3) + \int_{t_3}^{t}\left[H_s^{\Delta}(t,s) + H(t,s)\frac{z^{\Delta}}{z^{\sigma}}\right]w^{\sigma}(s)\Delta s \\ &\quad - \int_{t_3}^{t} H(t,s)\frac{\gamma z}{r^{\frac{1}{\gamma}}(z^{\sigma})^{\frac{\gamma+1}{\gamma}}}(w^{\sigma})^{\frac{\gamma+1}{\gamma}}\Delta s.\end{aligned} \tag{3.17}$$

Taking $B = [H_s^{\Delta}(t,s) + H(t,s)\frac{z^{\Delta}}{z^{\sigma}}]$, $A = H(t,s)\frac{\gamma z}{r^{\frac{1}{\gamma}}(z^{\sigma})^{\frac{\gamma+1}{\gamma}}} > 0$ and $u = w^{\sigma}$, then (3.17) becomes

$$\int_{t_3}^{t} H(t,s)q(s)Q(s)\Delta s \le H(t,t_3)w(t_3) + \int_{t_3}^{t}\left[Bu - Au^{\frac{\gamma+1}{\gamma}}\right]\Delta s.$$

Using Lemma (2.3), we have

$$\begin{aligned}\int_{t_3}^{t} H(t,s)q(s)Q(s)\Delta s &\le H(t,t_3)w(t_3) + \int_{t_3}^{t}\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^{\gamma}}\Delta s \\ &\le H(t,t_3)w(t_3) + \int_{t_3}^{t}\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}[H_s^{\Delta}(t,s) + H(t,s)\frac{z^{\Delta}}{z^{\sigma}}]^{\gamma+1}\frac{r(z^{\sigma})^{\gamma+1}}{H^{\gamma}(t,s)z^{\gamma}\gamma^{\gamma}}\Delta s \\ &\le H(t,t_3)w(t_3) + \int_{t_3}^{t}\frac{[C(t,s)]^{\gamma+1}r(z^{\sigma})^{\gamma+1}}{H^{\gamma}(t,s)z^{\gamma}(\gamma+1)^{\gamma+1}}\Delta s.\end{aligned}$$

Hence,

$$\frac{1}{H(t,t_2)}\int_{t_3}^{t}[H(t,s)q(s)Q(s)-\frac{[C(t,s)]^{\gamma+1}r(z^{\sigma})^{\gamma+1}}{H^{\gamma}(t,s)z^{\gamma}(\gamma+1)^{\gamma+1}}]\Delta s\le w(t_3). \tag{3.18}$$

Taking the limit suprimum of (3.18) as $t\to\infty$, we get a contradiction to condition (3.13).

This completes the proof.

Theorem 3.4. Assume that the conditions H_1-H_4 hold and $\beta\ge\gamma>0$. Also, assume that there exist a positive rd-continuous Δ-differentiable function $z(t)$ and a function $H\in\mathbb{H}_*$ such that for a constant $b>0$,

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)q(s)Q_1(s)-\frac{C^{\gamma+1}(t,s)r(s)(z(\sigma(s)))^{\gamma+1}}{(\gamma+1)^{\gamma+1}H^{\gamma}(t,s)z^{\gamma}(s)}\right]\Delta s=\infty. \tag{3.19}$$

Where $Q_1(s)=\frac{z(s)}{m^{\beta}(\delta(s))}[1-\frac{p(\delta(s))}{m(\tau(\delta(s)))}]^{\beta}(\psi(t,t_2))^{\beta}b^{\beta-\gamma}$

and

$C(t,s)=H_S^{\Delta}(t,s)+H(t,s)\frac{z^{\Delta}(s)}{z(\sigma(s))}$.

Then, every solution of Eq. (1.4) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

Proof. The proof is similar to that of Theorem (3.3). So, it is omitted.

Theorem 3.5. Assume that the conditions H_1-H_4 hold, $\gamma\ge1$, $\beta\ge0$ and $\gamma\ge\beta$. Also, assume that there exists a positive rd-continuous Δ-differentiable function $z(t)$ such that for a constant $b>0$,

$$\lim_{t\to\infty}\sup\int_{t_0}^{t}\left[Q_2(s)q(s)-\frac{r^{\sigma}(s)(z^{\Delta}(s))^2}{4C_1(s)z(s)}\right]\Delta s=\infty. \tag{3.20}$$

Where $Q_2(s)=\frac{z(s)}{m^{\beta}(\delta(s))}[1-\frac{p(\delta(s))}{m(\tau(\delta(s)))}]^{\beta}\frac{(\psi(s,t_2))^{\beta}}{b^{\gamma-\beta}(A(\sigma(s),t_1))^{\gamma-\beta}}$

and

$$C_1(s)=\frac{\gamma(A(s,t_1))^{\gamma-1}}{r^{\frac{1}{\gamma}}(s)}.$$

Then, every solution of Eq. (1.4) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose that $y(t)$ is a nonoscillatory solution of Eq. (1.4).Then by H_1-H_4, there exists t_1 sufficiently large such that $y(t)>0$, $y(\tau(t))>0$ and $y(\delta(t))>0$. Define the function $w(t)$ by the Riccati substitution

$$w(t):=\frac{z(t)r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}\text{ for }t\ge t_1. \tag{3.21}$$

Using Eq. (1.7), we get

$$w(t)>0,$$

and

$$w^{\Delta}(t)=z^{\Delta}[\frac{r(x^{\Delta})^{\gamma}}{x^{\gamma}}]^{\sigma}+z[\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}x^{\gamma}-r(x^{\Delta})^{\gamma}(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}].$$

Since $x^{\Delta}>0$ and $(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}<0$, then

$$w^{\Delta}(t)\le\frac{z^{\Delta}}{z^{\sigma}}w^{\sigma}+\frac{z(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}}{(x^{\sigma})^{\gamma}}-\frac{zr^{\sigma}((x^{\Delta})^{\sigma})^{\gamma}(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}. \tag{3.22}$$

Using Remark (2.1), inequality (3.22) becomes

$$w^{\Delta}(t) \le z(\frac{-q}{m^{\beta}(\delta(t))})[1-\frac{p(\delta(t))}{m(\tau(\delta(t)))}]^{\beta}\frac{(x(\delta(t)))^{\beta}}{(x^{\sigma})^{\gamma}}+w^{\sigma}\frac{z^{\Delta}}{z^{\sigma}}-\frac{zr^{\sigma}((x^{\Delta})^{\sigma})^{\gamma}(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}.$$

Using Lemma (2.2) and inequality (3.7), we find that for $t \ge t_2$,

$$w^{\Delta}(t) \le z(\frac{-q}{m^{\beta}(\delta(t))})[1-\frac{p(\delta(t))}{m(\tau(\delta(t)))}]^{\beta}(\psi(t,t_2))^{\beta}\frac{1}{(x^{\sigma})^{\gamma-\beta}}+w^{\sigma}\frac{z^{\Delta}}{z^{\sigma}}-\frac{zr^{\sigma}((x^{\Delta})^{\sigma})^{\gamma}\gamma x^{\gamma-1}(t)x^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}. \qquad (3.23)$$

From Lemma (2.1), we have

$$x(t)-x(t_1)=\int_{t_1}^{t}\frac{(r(s)(x^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{(r(s))^{\frac{1}{\gamma}}}\Delta(s) \le r^{\frac{1}{\gamma}}(t_1)x^{\Delta}(t_1)A(t,t_1).$$

Therefore, there exists $T \in [t_1,\infty)_{\mathbb{T}}$ and a suitable constant $b>0$ such that

$$x(t) \le bA(t,t_1) \quad \text{for} \quad t \in [T,\infty)_{\mathbb{T}},$$

$$x(\sigma(t)) \le bA(\sigma(t),t_1) \text{for} \quad t \in [T,\infty)_{\mathbb{T}}.$$

Since $\gamma \ge \beta$, then

$$\frac{-1}{x^{\gamma-\beta}(\sigma(t))} \le \frac{-1}{b^{\gamma-\beta}(A(\sigma(t),t_1))^{\gamma-\beta}} \text{ for } t \in [T,\infty)_{\mathbb{T}}. \qquad (3.24)$$

Taking $t_3 = max\{t_2,T\}$, then from inequalities (3.23) and (3.24) we find that for every $t \in [t_3,\infty)_{\mathbb{T}}$,

$$w^{\Delta}(t) \le -q(t)Q_2(t)+w^{\sigma}\frac{z^{\Delta}}{z^{\sigma}}-\frac{\gamma z}{r^{\sigma}(z^{\sigma})^2}[\frac{z^{\sigma}(t)r^{\sigma}((x^{\Delta})^{\sigma})^{\gamma}}{x^{\gamma}(\sigma(t))}]^2\frac{(x^{\sigma})^{\gamma}x^{\Delta}}{x((x^{\Delta})^{\sigma})^{\gamma}}. \qquad (3.25)$$

Since $x^{\Delta}>0$ and $(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}<0$, then

$$w^{\Delta}(t) \le -q(t)Q_2(t)+w^{\sigma}\frac{z^{\Delta}}{z^{\sigma}}-\frac{\gamma z}{r(z^{\sigma})^2}(w^{\sigma})^2(\frac{x}{x^{\Delta}})^{\gamma-1}. \qquad (3.26)$$

From Lemma (2.1), we have

$$x(t)>x(t)-x(t_1)=\int_{t_1}^{t}\frac{(r(s)(x^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{(r(s))^{\frac{1}{\gamma}}}\Delta(s) \ge r^{\frac{1}{\gamma}}(t)x^{\Delta}(t)A(t,t_1).$$

Hence,

$$\frac{x^{\Delta}(t)}{x(t)} < \frac{1}{r^{\frac{1}{\gamma}}(t)A(t,t_1)}. \qquad (3.27)$$

From inequalities (3.28) and (3.27), we get

$$w^{\Delta}(t) \le -q(t)Q_2(t)+w^{\sigma}\frac{z^{\Delta}}{z^{\sigma}}-\frac{\gamma z}{r(z^{\sigma})^2}(w^{\sigma})^2 r^{\frac{\gamma-1}{\gamma}}(t)(A(t,t_1))^{\gamma-1} \le -q(t)Q_2(t)+w^{\sigma}\frac{z^{\Delta}}{z^{\sigma}}-$$

$$\frac{zC_1(t)}{(z^{\sigma})^2}(w^{\sigma})^2. \qquad (3.28)$$

Therefore,

$$\begin{aligned} w^{\Delta}(t) &\le -q(t)Q_2(t)-zC_1(t)[(\frac{w^{\sigma}}{z^{\sigma}})^2-\frac{z^{\Delta}}{zC_1(t)}\frac{w^{\sigma}}{z^{\sigma}}] \\ &\le -q(t)Q_2(t)-zC_1(t)[(\frac{w^{\sigma}}{z^{\sigma}}-\frac{z^{\Delta}}{2zC_1(t)})^2-\frac{(z^{\Delta})^2}{4z^2C_1^2(t)}]. \end{aligned}$$

Hence,

$$w^{\Delta}(t) \le -q(t)Q_2(t)+\frac{(z^{\Delta})^2}{4zC_1(t)}.$$

Integrating from t_3 to t, we get

$$\int_{t_3}^{t} [Q_2(s)q(s) - \frac{r^{\sigma}(s)(z^{\Delta}(s))^2}{4C_1(s)z(s)}]\Delta s \leq w(t_3) - w(t) \leq w(t_3), \tag{3.29}$$

taking the limit suprimum of (3.29) as $t \to \infty$, we get a contradiction to condition (3.20).

This completes the proof.

Corollary 3.1. Assume that the conditions $H_1 - H_4$ hold, $\gamma \geq 1$, $\beta > 0$ and $\gamma \geq \beta$. Also, assume that there exists a positive *rd*-continuous Δ-differentiable function $z(t)$ such that for a constant $b > 0$,

$$\lim_{t\to\infty} sup \int_{t_3}^{t} [Q_3(s)q(s) - (z^{\Delta})^2 \frac{r^{\frac{1}{\gamma}}}{\gamma} \frac{1}{(A(t,t_1))^{\gamma-1}}]\Delta s = \infty, \tag{3.30}$$

where

$$Q_3(s) = \frac{(z^2)^{\sigma}}{m^{\beta}(\delta(s))}[1 - \frac{p(\delta(s))}{m(\tau(\delta(s)))}]^{\beta}(\psi(t,t_2))^{\beta}(z^2)^{\sigma}\frac{1}{b^{\gamma-\beta}(A(\sigma(t),t_1))^{\gamma-\beta}}.$$

Then, every solution of Eq. (1.4) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose that $y(t)$ is a nonoscillatory solution of Eq. (1.4). Then by $H_1 - H_4$, there exists t_1 sufficiently large such that $y(t) > 0$ and $y(\tau(t)) > 0$ and $y(\delta(t)) > 0$. Define the function $w(t)$ by the Riccati substitution

$$w(t) := \frac{z^2(t)r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)} > 0 \quad \text{for } t \geq t_1, \tag{3.31}$$

where $x(t)$ is given by Eq. (1.7). Then

$$w^{\Delta}(t) = [r(x^{\Delta})^{\gamma}]^{\Delta}(\frac{z^2}{x^{\gamma}})^{\sigma} + r(x^{\Delta})^{\gamma}(\frac{z^2}{x^{\gamma}})^{\Delta}.$$

Using Remark (2.1), we have

$$w^{\Delta}(t) \leq (\frac{-q}{m^{\beta}(\delta(t))})[1 - \frac{p(\delta(t))}{m(\tau(\delta(t)))}]^{\beta}(x(\delta(t)))^{\beta}\frac{(z^2)^{\sigma}}{(x^{\sigma})^{\gamma}} + r(x^{\Delta})^{\gamma}(\frac{z^2}{x^{\gamma}})^{\Delta}.$$

Using Lemma (2.2), then for $t \geq t_2$, we have

$$w^{\Delta}(t) \leq (\frac{-q}{m^{\beta}(\delta(t))})[1 - \frac{p(\delta(t))}{m(\tau(\delta(t)))}]^{\beta}(\psi(t,t_2))^{\beta}(z^2)^{\sigma}\frac{1}{(x^{\sigma})^{\gamma-\beta}} + r(x^{\Delta})^{\gamma}(\frac{z^2}{x^{\gamma}})^{\Delta}.$$

From inequality(3.24), we find that for $t \geq t_3$

$$w^{\Delta}(t) \leq (\frac{-q}{m^{\beta}(\delta(t))})[1 - \frac{p(\delta(t))}{m(\tau(\delta(t)))}]^{\beta}(\psi(t,t_2))^{\beta}(z^2)^{\sigma}\frac{1}{b^{\gamma-\beta}(A(\sigma(t),t_1))^{\gamma-\beta}} + r(x^{\Delta})^{\gamma}(\frac{z^2}{x^{\gamma}})^{\Delta},$$

where $t_3 = max\{T, t_2\}$. Hence,

$$w^{\Delta}(t) \leq -q(t)Q_3(t) + \frac{r(x^{\Delta})^{\gamma}}{(x^{\gamma})^{\Delta}}(x^{\gamma})^{\Delta}(\frac{z^2}{x^{\gamma}})^{\Delta}.$$

Using Lemma (2.4), we obtain

$$w^{\Delta}(t) \leq -q(t)Q_3(t) + \frac{r(x^{\Delta})^{\gamma}}{(x^{\gamma})^{\Delta}}[(z^{\Delta})^2 - x^{\gamma}(x^{\sigma})^{\gamma}((\frac{z}{x^{\gamma}})^{\Delta})^2] \qquad \leq -q(t)Q_3(t) + \frac{r(x^{\Delta})^{\gamma}}{(x^{\gamma})^{\Delta}}(z^{\Delta})^2.$$

From inequality (3.7), we have

$$w^{\Delta}(t) \leq -q(t)Q_3(t) + \frac{r(x^{\Delta})^{\gamma}}{\gamma x^{\gamma} x^{\Delta}}(z^{\Delta})^2 \leq -q(t)Q_3(t) + \frac{r}{\gamma}(\frac{x^{\Delta}}{x})^{\gamma-1}(z^{\Delta})^2.$$

Using inequality (3.27), we get

$$w^{\Delta}(t) \leq -q(t)Q_3(t) + (z^{\Delta})^2 \frac{r^{\frac{1}{\gamma}}}{\gamma} \frac{1}{(A(t,t_1))^{\gamma-1}}.$$

Integrating from t_3 to t, we get

$$\int_{t_3}^{t} [Q_3(s)q(s) - (z^{\Delta})^2 \frac{r^{\frac{1}{\gamma}}}{\gamma} \frac{1}{(A(s,t_1))^{\gamma-1}}]\Delta s \leq w(t_3) - w(t) \leq w(t_3).$$

Taking the limit suprimum as $t \to \infty$, we get a contradiction to condition (3.30). This completes the proof.

Theorem 3.6. Assume that the conditions $H_1 - H_4$ hold, $0 < \beta \leq \gamma < 1$ and

$$\lim_{t\to\infty} sup \int_{t_0}^{t} \frac{q(s)}{m^{\beta}(\delta(s))} [1 - \frac{p(\delta(s))}{m\left(\tau(\delta(s))\right)}]^{\beta} (A(\delta(s),t_1))^{\beta^2} (\psi(s,t_2))^{\beta} \Delta s = \infty. \tag{3.32}$$

Then every solution of Eq. (1.4) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose that $y(t)$ is a nonoscillatory solution of Eq. (1.4). Then by $H_1 - H_4$, there exists t_1 sufficiently large such that $y(t) > 0$, $y(\tau(t)) > 0$ and$y(\delta(t)) > 0$. Taking $\phi(t) = r(x^{\Delta})^{\gamma}$ where $x(t)$ is defined by Eq. (1.7), then

$$\phi > 0 \text{ and } \phi^{\Delta} = (r(x^{\Delta})^{\gamma})^{\Delta} \leq 0.$$

Therefore, Eq. (1.8) becomes

$$\phi^{\Delta}(t) + f(t, y(\delta(t))) = 0. \tag{3.33}$$

Using Remark (2.1), we have

$$0 \geq (r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + \frac{q(t)}{m^{\beta}(\delta(t))} [1 - \frac{p(\delta(t))}{m(\tau(\delta(t)))}]^{\beta} (x(\delta(t)))^{\beta}.$$

Hence,

$$0 \geq \phi^{\Delta} + \frac{q(t)}{m^{\beta}(\delta(t))} [1 - \frac{p(\delta(t))}{m(\tau(\delta(t)))}]^{\beta} (x(\delta(t)))^{\beta},$$

therefore,

$$0 \geq \frac{\phi^{\Delta}}{(\phi^{\sigma})^{\gamma}} + \frac{q(t)}{m^{\beta}(\delta(t))} [1 - \frac{p(\delta(t))}{m(\tau(\delta(t)))}]^{\beta} \frac{(x(\delta(t)))^{\beta}}{(\phi^{\sigma})^{\gamma}}. \tag{3.34}$$

Using $0 < \gamma < 1$ and Keller's chain rule, we get

$$\begin{aligned}(\phi^{1-\gamma})^{\Delta} &= (1-\gamma)[\int_0^1 (\phi(t) + h\mu(t)\phi^{\Delta}(t))^{-\gamma} dh]\phi^{\Delta}(t) \\ &\leq (1-\gamma)[\int_0^1 ((1-h)\phi(\sigma(t)) + h\phi(\sigma(t)))^{-\gamma} dh]\phi^{\Delta}(t).\end{aligned}$$

Hence,

$$\frac{\phi^{\Delta}}{(\phi^{\sigma})^{\gamma}} \geq \frac{(\phi^{1-\gamma})^{\Delta}}{1-\gamma}. \tag{3.35}$$

Now, we find an estimation for $\frac{(x(\delta(t)))^{\beta}}{(\phi^{\sigma})^{\gamma}}$.

Since, $\phi(t) = r(x^{\Delta})^{\gamma}$, then

$$\frac{(x(\delta(t)))^{\beta}}{(\phi^{\sigma})^{\gamma}} = \frac{(x(\delta(t)))^{\beta}}{[r^{\sigma}((x^{\Delta})^{\sigma})^{\gamma}]^{\gamma}}. \tag{3.36}$$

Since, $(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} < 0$, we have

$$\frac{1}{x^{\Delta}(\sigma(t))} \geq \frac{(r^{\sigma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(\delta(t))x^{\Delta}(\delta(t))}. \tag{3.37}$$

From Eq. (3.36) and inequality (3.37), we get

$$\frac{(x(\delta(t)))^{\beta}}{(\phi^{\sigma})^{\gamma}} \geq \frac{(x(\delta(t)))^{\beta}}{r^{\gamma}(\delta(t))(x^{\Delta}(\delta(t)))^{\gamma^2}}. \tag{3.38}$$

Using inequality (3.27), we obtain

$$\frac{(x(\delta(t)))^{\beta}}{(\phi^{\sigma})^{\gamma}} \geq \frac{(A(\delta(t),t))^{\gamma^2}}{(x(\delta(t)))^{\gamma^2-\beta^2}}(x(\delta(t)))^{\beta}. \tag{3.39}$$

From inequality (3.4) and Lemma (2.2), we get

$$\begin{aligned} \frac{(x(\delta(t)))^{\beta}}{(\phi^{\sigma})^{\gamma}} &\geq \frac{(A(\delta(t),t_1))^{\beta^2}}{b^{\gamma^2-\beta^2}}(x(\delta(t)))^{\beta} \\ &\geq \frac{(A(\delta(t),t_1))^{\beta^2}}{b^{\gamma^2-\beta^2}}(\psi(t,t_2))^{\beta}(x(\sigma(t)))^{\beta}. \end{aligned} \tag{3.40}$$

Since, $x^{\Delta} > 0$, then

$$x(t_1) \leq x(t) \text{for } t \geq t_1.$$

Therefore, there exists a constant $d > 0$ such that $x(\sigma(t)) > d$ for $t \geq t_1$. Hence,

$$\frac{(x(\delta(t)))^{\beta}}{(\phi^{\sigma})^{\gamma}} \geq \frac{(A(\delta(t),t_1))^{\beta^2}}{b^{\gamma^2-\beta^2}}(\psi(t,t_2))^{\beta}d^{\beta}. \tag{3.41}$$

From inequalities (3.34), (3.35) and (3.41), we obtain

$$0 \geq \frac{(\phi^{1-\gamma})^{\Delta}}{1-\gamma} + \frac{q(t)}{m^{\beta}(\delta(t))}[1 - \frac{p(\delta(t))}{m(\tau(\delta(t)))}]^{\beta} \frac{(A(\delta(t),t_1))^{\beta^2}}{b^{\gamma^2-\beta^2}}(\psi(t,t_2))^{\beta}d^{\beta}. \tag{3.42}$$

Integrating from t_3 to t, we get

$$\begin{aligned} \int_{t_3}^{t} \frac{q(s)}{m^{\beta}(\delta(s))}[1 - \frac{p(\delta(s))}{m(\tau(\delta(s)))}]^{\beta}(A(\delta(t),t_1))^{\beta^2}(\psi(t,t_2))^{\beta}\Delta s \quad &\leq \frac{-b^{\gamma^2-\beta^2}}{(1-\gamma)d^{\beta}}\int_{t_3}^{t}(\phi^{1-\gamma})^{\Delta}(s)\Delta s \\ &\leq \frac{-b^{\gamma^2-\beta^2}}{(1-\gamma)d^{\beta}}[\phi^{1-\gamma}(t_3) - \phi^{1-\gamma}(t)] \\ &\leq \frac{-b^{\gamma^2-\beta^2}}{(1-\gamma)d^{\beta}}\phi^{1-\gamma}(t_3), \end{aligned} \tag{3.43}$$

taking the limit suprimum of (3.43) as $t \to \infty$, we get a contradiction to condition (3.32). This completes the proof.

Theorem 3.7. Assume that the conditions $H_1 - H_4$ hold. Let either $(\gamma \geq 1, \beta > 0 \text{ and } \gamma \geq \beta)$or $(0 < \beta \leq \gamma < 1)$. Also, assume that there exists a positive *rd*-continuous Δ-differentiable and increasing function $z(t)$ such that for a constant $b > 0$,

$$\lim_{t\to\infty} sup \int_{t_0}^{t} [q(s)Q(s) - z^{\Delta}(s)r(s)(\tfrac{2}{s})^{\gamma}]\Delta s = \infty. \tag{3.44}$$

Where$Q(s) = \frac{z(s)}{m^{\beta}(\delta(s))}[1 - \frac{p(\delta(s))}{m(\tau(\delta(s)))}]^{\beta} \frac{(\psi(s,t_2))^{\beta}}{b^{\gamma-\beta}(A(s,t_1))^{\gamma-\beta}}$.

Then, every solution of Eq. (1.4) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Suppose that $y(t)$ is a non oscillatory solution of Eq. (1.4).Then by $H_1 - H_4$ there exists t_1

sufficiently large such that $y(t)>0$, $y(\tau(t))>0$ and $y(\delta(t))>0$. Define the function $w(t)$ by the Riccati substitution

$$w(t):=\frac{z(t)r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)} \text{ for } t\geq t_1, \tag{3.45}$$

where $x(t)$ is given by Eq. (1.7).Then $w(t)>0$ and

$$w^{\Delta}(t)=[zr(x^{\Delta})^{\gamma}]^{\Delta}\frac{1}{x^{\gamma}}+[zr(x^{\Delta})^{\gamma}]^{\sigma}(\frac{1}{x^{\gamma}})^{\Delta}.$$

Since,

$$(\frac{1}{x^{\gamma}})^{\Delta}=\frac{-(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\gamma})^{\sigma}}<0,$$

then,

$$w^{\Delta}(t)\leq z^{\Delta}(r(x^{\Delta})^{\gamma})^{\sigma}\frac{1}{x^{\gamma}}+z(r(x^{\Delta})^{\gamma})^{\Delta}\frac{1}{x^{\gamma}}.$$

Using Remark (2.1) and $z^{\Delta}>0$, we get

$$w^{\Delta}(t)\leq z^{\Delta}(r(x^{\Delta})^{\gamma})^{\sigma}\frac{1}{x^{\gamma}}-\frac{z(t)}{m^{\beta}(\delta(t))}[1-\frac{p(\delta(t))}{m\left(\tau(\delta(t))\right)}]^{\beta}\frac{(x(\delta(t)))^{\beta}}{x^{\gamma}(t)} \text{ for } t\geq t_1.$$

By Lemma (2.2), Remarks (3.5) and (3.27), we obtain

$$w^{\Delta}(t)\leq\frac{z^{\Delta}}{(A(t,t_1))^{\gamma}}-q(t)Q(t).$$

Integrating from t_3 to t, we get

$$\int_{t_3}^{t}\left[q(s)Q(s)-\frac{z^{\Delta}}{(A(t,t_1))^{\gamma}}\right]\Delta s\leq w(t_3)-w(t)\leq w(t_3). \tag{3.46}$$

Taking the limit suprimum of (3.46) as $t\to\infty$, we get a contradiction to condition (3.44).

This completes the proof.

Case(2): when condition (1.6) holds.

Theorem 3.8. Assume that the conditions H_1-H_4 hold, $\gamma>0$ and $\beta>0$. Also, assume that $lim_{t\to\infty}p(t)>0$ and there exists a positive rd-continuous Δ-differentiable function $z(t)$ such that for a constant $b>0$, we have one of the following:

(i) $\gamma\geq\beta$, condition (3.1) holds, and

$$\int_{t_2}^{\infty}\frac{1}{r^{\frac{1}{\gamma}}(s)}(\int_{t_2}^{s}\frac{q(u)}{m^{\beta}(\delta(u))}[1-\frac{p(\delta(u))}{m(\tau(\delta(u)))}]^{\beta}\Delta u)^{\frac{1}{\gamma}}\Delta s=\infty. \tag{3.47}$$

(ii) $\beta\geq\gamma$, conditions (3.11) and (3.47) hold.

Then, every solution of Eq. (1.4) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$ or tends to zero.

Proof. (i) Suppose that $y(t)$ is a non oscillatory solution of Eq. (1.4). Without loss of generality, we assume that $y(t)$ is eventually positive solution of Eq. (1.4). Then there exists $t_1\in[t_0,\infty)_{\mathbb{T}}$ sufficiently large such that $y(t)>0$, $y(\tau(t))>0$ and $y(\delta(t))>0$.

Since,

$$x(t):=m(t)y(t)+p(t)y(\tau(t)),$$

then $x(t)>0$.

From Eq. (1.8), we have

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} < 0.$$

Hence, $r(t)(x^{\Delta}(t))^{\gamma}$ is strictly decreasing on $[t_0,\infty)_{\mathbb{T}}$ and eventually of one sign. Consequently, $x^{\Delta}(t)$ is eventually nonnegative or eventually negative.

If $x^{\Delta}(t) \geq 0$, the proof is similar to that of Theorem (3.1) and hence it is omitted.

Next, assume that $x^{\Delta}(t)$ is eventually negative. Since $x(t) > 0$, then $\lim_{t\to\infty} x(t) = a \geq 0$.

Now, we want to show that $a = 0$. If not, then $x^{\beta}(\delta(t)) \to a^{\beta}$ as $t \to \infty$. Then there exists $t_2 \in [t_0,\infty)_{\mathbb{T}}$ such that

$$x^{\beta}\big(\delta(t)\big) \geq a^{\beta} \text{for } t \geq t_2.$$

From Remark (2.1), we have

$$\begin{aligned}(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} &\leq \frac{-q(t)}{m^{\beta}(\delta(t))}[1-\frac{p(\delta(t))}{m(\tau(\delta(t)))}]^{\beta}(x(\delta(t)))^{\beta}\\ &\leq \frac{-q(t)}{m^{\beta}(\delta(t))}[1-\frac{p(\delta(t))}{m(\tau(\delta(t)))}]^{\beta}a^{\beta}.\end{aligned}$$

Integrating from t_2 to t, we get

$$r(t)(x^{\Delta}(t))^{\gamma} \leq r(t)(x^{\Delta}(t))^{\gamma} - r(t_2)(x^{\Delta}(t_2))^{\gamma} - a^{\beta}\int_{t_2}^{t}\frac{q(s)}{m^{\beta}\big(\delta(s)\big)}[1-\frac{p\big(\delta(s)\big)}{m\Big(\tau\big(\delta(s)\big)\Big)}]^{\beta}\Delta s.$$

Hence,

$$x^{\Delta}(t) \leq -a^{\frac{\beta}{\gamma}}(\frac{1}{r(t)}\int_{t_2}^{t}\frac{q(s)}{m^{\beta}(\delta(s))}[1-\frac{p(\delta(s))}{m(\tau(\delta(s)))}]^{\beta}\Delta s)^{\frac{1}{\gamma}}.$$

Integrating from t_2 to t, we get

$$x(t) \leq x(t_2) - a^{\frac{\beta}{\gamma}}\int_{t_2}^{t}\frac{1}{r^{\frac{1}{\gamma}}(t)}(\int_{t_2}^{s}\frac{q(u)}{m^{\beta}\big(\delta(u)\big)}[1-\frac{p\big(\delta(u)\big)}{m\Big(\tau\big(\delta(u)\big)\Big)}]^{\beta}\Delta u)^{\frac{1}{\gamma}}\Delta s.$$

Consequently, condition (3.47) implies that $x(t)$ is eventually negative, which is a contradiction.

Therefore, $\lim_{t\to\infty} x(t) = 0$. From (1.7), we have

$$0 < m(t)y(t) \leq x(t),$$

which implies that $\lim_{t\to\infty} y(t) = 0$. This completes the proof.

(ii)The proof is similar to that of case (i) and hence it is omitted.

Theorem 3.9. Assume that the conditions $H_1 - H_4$ hold,$\gamma > 0$ and $\beta > 0$. Also, assume that there exists a positive rd-continuous Δ-differentiable function $\mathrm{z(t)}$ such that for a constant $b > 0$, we have one of the following:

(i) $\gamma \geq \beta$, condition (3.1) holds and

$$\int_{t_2}^{\infty}(\frac{1}{r(z)}(\int_{t_2}^{z}\frac{q(s)}{m^{\beta}(\delta(s))}[1-\frac{p(\delta(s))}{m(\tau(\delta(s)))}]^{\beta}K^{\beta}(\delta(s))\Delta s)^{\frac{1}{\gamma}}\Delta z = \infty, \tag{3.48}$$

where,

$$K(t) := \int_t^{\infty} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \Delta s.$$

(ii) $\beta \geq \gamma$, conditions (3.11) and (3.48) hold.

Then, every solution of Eq. (1.4) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Proof. (i) Suppose that $y(t)$ is a non oscillatory solution of Eq. (1.4). Without loss of generality, we assume that $y(t)$ is eventually positive solution of Eq. (1.4). Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ sufficiently large such that $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\delta(t)) > 0$.

Since,

$$x(t) := m(t)y(t) + p(t)y(\tau(t)).$$

Then, $x(t) > 0$.

From Eq. (1.8), we have

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} < 0.$$

Hence, $r(t)(x^{\Delta}(t))^{\gamma}$ is strictly decreasing on $[t_0, \infty)_{\mathbb{T}}$ and eventually of one sign. Consequently, $x^{\Delta}(t)$ is eventually nonnegative or eventually negative.

In case of $x^{\Delta}(t) \geq 0$, the proof is similar to that of Theorem (3.1) and hence it is omitted.

Next, assume that $x^{\Delta}(t)$ is eventually negative. Since,

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} < 0 \text{ for } t_2 \in [t_0, \infty)_{\mathbb{T}}.$$

Then, for $s \geq t \geq t_2$ we have

$$r(s)(x^{\Delta}(s))^{\gamma} \leq r(t)(x^{\Delta}(t))^{\gamma}.$$

Hence,

$$x^{\Delta}(s) \leq \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} r^{\frac{1}{\gamma}}(t) x^{\Delta}(t).$$

Integrating on s from t to z $(z \geq t)$, we get

$$x(z) - x(t) \leq r^{\frac{1}{\gamma}}(t) x^{\Delta}(t) \int_t^{z} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \Delta s.$$

Then,

$$-x(t) \leq r^{\frac{1}{\gamma}}(t) x^{\Delta}(t) \int_t^{z} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \Delta s.$$

As $z \to \infty$, we get

$$x(t) \geq r^{\frac{1}{\gamma}}(t)(-x^{\Delta}(t)) \int_t^{\infty} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \Delta s. \tag{3.49}$$

Then,

$$x(t) \geq K(t) r^{\frac{1}{\gamma}}(t)(-x^{\Delta}(t)).$$

Where,

$$K(t) := \int_t^{\infty} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \Delta s.$$

Since,

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} < 0 \text{ for } t_2 \in [t_0, \infty)_{\mathbb{T}}.$$

Then, for $t \geq t_2$, we have

$$-r(t)(x^{\Delta}(t))^{\gamma} \geq -r(t_2)(x^{\Delta}(t_2))^{\gamma}. \tag{3.50}$$

From equations (3.49) and (3.50), we get

$$x(t) \geq CK(t), \tag{3.51}$$

where,

$$C := -r^{\frac{1}{\gamma}}(t_2)x^{\Delta}(t_2) > 0.$$

From Remark (2.1), we have

$$\begin{aligned}(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} &\leq \frac{-q(t)}{m^{\beta}(\delta(t))}[1 - \frac{p(\delta(t))}{m(\tau(\delta(t)))}]^{\beta}(x(\delta(t)))^{\beta} \\ &\leq \frac{-q(t)}{m^{\beta}(\delta(t))}[1 - \frac{p(\delta(t))}{m(\tau(\delta(t)))}]^{\beta}C^{\beta}K^{\beta}(\delta(t)).\end{aligned}$$

Integrating from t_2 to t, we get

$$\begin{aligned}r(t)(x^{\Delta}(t))^{\gamma} &\leq r(t_2)(x^{\Delta}(t_2))^{\gamma} - C^{\beta}\int_{t_2}^{t}\frac{-q(s)}{m^{\beta}(\delta(s))}[1 - \frac{p(\delta(s))}{m(\tau(\delta(s)))}]^{\beta}K^{\beta}(\delta(s))\Delta s \\ &< -C^{\beta}\int_{t_2}^{t}\frac{-q(s)}{m^{\beta}(\delta(s))}[1 - \frac{p(\delta(s))}{m(\tau(\delta(s)))}]^{\beta}K^{\beta}(\delta(s))\Delta s.\end{aligned}$$

Hence,

$$x^{\Delta}(t) < -C^{\frac{\beta}{\gamma}}(\frac{1}{r(t)}\int_{t_2}^{t}\frac{-q(s)}{m^{\beta}(\delta(s))}[1 - \frac{p(\delta(s))}{m(\tau(\delta(s)))}]^{\beta}K^{\beta}(\delta(s))\Delta s)^{\frac{1}{\gamma}}.$$

Integrating from t_2 to t, we get

$$x(t) < x(t_2) - C^{\frac{\beta}{\gamma}}\int_{t_2}^{t}(\frac{1}{r(z)}\int_{t_2}^{z}\frac{-q(s)}{m^{\beta}(\delta(s))}[1 - \frac{p(\delta(s))}{m(\tau(\delta(s)))}]^{\beta}K^{\beta}(\delta(s))\Delta s)^{\frac{1}{\gamma}}\Delta z.$$

Consequently, condition (3.48) implies that $x(t)$ is eventually negative, which is a contradiction. This completes the proof.

(ii)The proof is similar to that of case (i) and hence it is omitted.

4 Examples

In this section, we give some examples of second order neutral delay dynamic equations which cannot be studied by the previous known criteria of oscillation and illustrate our results.

The following two examples illustrate Theorem (3.1).

Example 4.1 Consider the equation

$$(((2(t+1)y(t) + ty(t-1))^{\Delta})^{\frac{5}{3}})^{\Delta} + \frac{\lambda\sigma(t)}{t}y(t) = 0, \; t \in [t_0, \infty)_{\mathbb{T}}. \tag{4.1}$$

Here, $\beta = 1$, $\gamma = \frac{5}{3}$, $r(t) = 1$, $m(t) = 2(t+1)$, $p(t) = t$, $\tau(t) = t - 1$ and $\delta(t) = t$.

Hence, $\gamma > \beta$, $\int_{t_0}^{\infty} (\frac{1}{r(t)})^{\frac{1}{\gamma}} \Delta t = \infty$ and $m(\tau(t)) > p(t)$.

Also,

$$A(s, t_1) = \int_{t_1}^{s} (\frac{1}{r(t)})^{\frac{1}{\gamma}} \Delta t = s - t_1,$$

and

$$\psi(s, t_2) = \frac{\int_{t_2}^{\delta(s)} (\frac{1}{r(t)})^{\frac{1}{\gamma}} \Delta t}{\int_{t_2}^{\sigma(s)} (\frac{1}{r(t)})^{\frac{1}{\gamma}} \Delta t} = \frac{s - t_2}{\sigma(s) - t_2}.$$

Choosing$z = 2s$, then $z^{\Delta} = 2$ and

$$\lim_{t\to\infty} sup \int_{t_0}^{t} [Q(s)q(s) - \frac{1}{(\gamma+1)^{(\gamma+1)}} \frac{r(s)(z^{\Delta}(s))^{\gamma+1}}{z^{\gamma}(s)}]\Delta s$$

$$= \lim_{t\to\infty} sup \int_{t_0}^{t} [\frac{\lambda\sigma(s)(s - t_2)}{2b^{\frac{2}{3}}(s+1)(\sigma(s) - t_2)(s - t_1)^{\frac{2}{3}}} - (\frac{3}{8})^{\frac{8}{3}} \frac{1}{2s^{\frac{5}{3}}}]\Delta s$$

$$> \lim_{t\to\infty} sup \int_{t_0}^{t} [\frac{\lambda(s - t_2)}{2b^{\frac{2}{3}}(s+1)s^{\frac{2}{3}}} - (\frac{3}{8})^{\frac{8}{3}} \frac{1}{2s^{\frac{5}{3}}}]\Delta s$$

$$> \lim_{t\to\infty} sup \int_{t_0}^{t} [\frac{\lambda}{2b^{\frac{2}{3}}}(\frac{1}{s^{\frac{2}{3}}} - \frac{(1+t_2)}{(s+1)s^{\frac{2}{3}}}) - (\frac{3}{8})^{\frac{8}{3}} \frac{1}{2s^{\frac{5}{3}}}]\Delta s$$

$$> \lim_{t\to\infty} sup \int_{t_0}^{t} [\frac{\lambda}{2b^{\frac{2}{3}}}(\frac{1}{s^{\frac{2}{3}}} - \frac{(1+t_2)}{s^{\frac{5}{3}}}) - (\frac{3}{8})^{\frac{8}{3}} \frac{1}{2s^{\frac{5}{3}}}]\Delta s = \infty, \text{if } \lambda > 0.$$

Hence, by Theorem (3.1) every solution of Eq. (4.1) is oscillatory if $\lambda > 0$.

Example 4.2 Consider the equation

$$(((3\sqrt{t+1}y(t) + \sqrt{t}y(t-1))^{\Delta})^{\frac{3}{7}})^{\Delta} + \lambda y^{\frac{1}{7}}(t) = 0, \ t \in \mathbb{T}, \tag{4.2}$$

where $\mathbb{T} = [1, \infty)$. Here, $\beta = \frac{1}{7}$, $\gamma = \frac{3}{7}$, $r(t) = 1$, $m(t) = 3\sqrt{t+1}$,$p(t) = \sqrt{t}$,$\tau(t) = t - 1$ and $\delta(t) = t$.

Hence, $\gamma > \beta$, $\int_{t_0}^{\infty} (\frac{1}{r(t)})^{\frac{1}{\gamma}} \Delta t = \infty$ and $m(\tau(t)) > p(t)$.

Also,

$$A(s, t_1) = \int_{t_1}^{s} (\frac{1}{r(t)})^{\frac{1}{\gamma}} \Delta t = s - t_1,$$

and

$$\psi(s, t_2) = \frac{\int_{t_2}^{\delta(s)} (\frac{1}{r(t)})^{\frac{1}{\gamma}} \Delta t}{\int_{t_2}^{\sigma(s)} (\frac{1}{r(t)})^{\frac{1}{\gamma}} \Delta t} = \frac{s - t_2}{\sigma(s) - t_2} = 1.$$

Choosing $z = 1$, then $z^{\Delta} = 0$ and

$$\limsup_{t\to\infty}\int_{t_0}^{t}[Q(s)q(s)-\frac{1}{(\gamma+1)^{(\gamma+1)}}\frac{r(s)(z^{\Delta}(s))^{\gamma+1}}{z^{\gamma}(s)}]\Delta s$$
$$=\limsup_{t\to\infty}\int_{t_0}^{t}(\frac{2}{9})^{\frac{1}{7}}\frac{\lambda}{b^{\frac{1}{7}}(s+1)^{\frac{1}{14}}(s-t_1)^{\frac{2}{7}}}\Delta s$$
$$>\limsup_{t\to\infty}\int_{t_0}^{t}(\frac{2}{9})^{\frac{1}{7}}\frac{\lambda}{b^{\frac{1}{7}}(s+1)^{\frac{1}{14}}s^{\frac{2}{7}}}\Delta s=\infty,\text{if } \lambda>0.$$

Hence, by Theorem (3.1) every solution of Eq. (4.2) is oscillatory if $\lambda>0$.

Remark 4.1 The results of [6, 7, 10, 11] can not be applied to equations 4.1 and 4.2.But according to Theorem (3.1), those equations are oscillatory.

The following two examples illustrate Theorem (3.2).

Example 4.3 Consider the equation

$$(((2(t+2)y(t)+ty(t-2))^{\Delta})^{\frac{1}{7}})^{\Delta}+\frac{\lambda\sigma(t)}{t}y(t)=0,\ t\in[t_0,\infty)_{\mathbb{T}}. \tag{4.3}$$

Here, $\beta=1,\ \gamma=\frac{1}{7}, r(t)=1,\ m(t)=2(t+2), p(t)=t,\ \tau(t)=t-2$ and $\delta(t)=t$.

Hence, $\beta>\gamma,\ \int_{t_0}^{\infty}(\frac{1}{r(t)})^{\frac{1}{\gamma}}\Delta t=\infty$ and $m(\tau(t))>p(t)$.

Also,

$$\psi(s,t_2)=\frac{\int_{t_2}^{\delta(s)}(\frac{1}{r(t)})^{\frac{1}{\gamma}}\Delta t}{\int_{t_2}^{\sigma(s)}(\frac{1}{r(t)})^{\frac{1}{\gamma}}\Delta t}=\frac{s-t_2}{\sigma(s)-t_2}.$$

Choosing$z=1$, then $z^{\Delta}=0$ and

$$\limsup_{t\to\infty}\int_{t_0}^{t}[Q_1(s)q(s)-\frac{1}{(\gamma+1)^{(\gamma+1)}}\frac{r(s)(z^{\Delta}(s))^{\gamma+1}}{z^{\gamma}(s)}]\Delta s$$
$$=\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\lambda b^{\frac{6}{7}}\sigma(s)(s-t_2)}{4s(s+2)(\sigma(s)-t_2)}\Delta s$$
$$>\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\lambda b^{\frac{6}{7}}}{4}\left[\frac{1}{(s+2)}-\frac{t_2}{s(s+2)}\right]\Delta s=\infty,\text{if } \lambda>0.$$

Hence, by Theorem (3.2) every solution of Eq. (4.3) is oscillatory if $\lambda>0$.

Example 4.4 Consider the equation

$$(\frac{1}{t+\sigma(t)}((2(t+1)^{\frac{3}{2}}y(t)+t^{\frac{3}{2}}y(t-2))^{\Delta})^{\frac{3}{5}})^{\Delta}+\frac{\lambda\sigma^2(t)}{t}y(t)=0,\ t\in[t_0,\infty)_{\mathbb{T}}. \tag{4.4}$$

Here, $\beta=1,\ \gamma=\frac{3}{5}, r(t)=\frac{1}{t+\sigma(t)},\ m(t)=2(t+1)^{\frac{3}{2}}, p(t)=t^{\frac{3}{2}}, \tau(t)=t-1$ and $\delta(t)=t$.

Hence, $\beta>\gamma,\ \int_{t_0}^{\infty}(\frac{1}{r(t)})^{\frac{1}{\gamma}}\Delta t=\infty$ and $m(\tau(t))>p(t)$.

Also,

$$\psi(s,t_2)=\frac{\int_{t_2}^{\delta(s)}(\frac{1}{r(t)})^{\frac{1}{\gamma}}\Delta t}{\int_{t_2}^{\sigma(s)}(\frac{1}{r(t)})^{\frac{1}{\gamma}}\Delta t}=\frac{s^2-t_2^2}{\sigma^2(s)-t_2^2}.$$

We choose $z=1$, then $z^{\Delta}=0$ and

$$\limsup_{t\to\infty}\int_{t_0}^{t}[Q_1(s)q(s)-\frac{1}{(\gamma+1)^{(\gamma+1)}}\frac{r(s)(z^{\Delta}(s))^{\gamma+1}}{z^{\gamma}(s)}]\Delta s$$

$$=\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\lambda b^{\frac{2}{5}}\sigma^2(s)(s^2-t_2^2)}{4s(s+1)^{\frac{3}{2}}(\sigma^2(s)-t_2^2)}\Delta s$$

$$>\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\lambda b^{\frac{2}{5}}}{4}[\frac{s^2-t_2^2}{s(s+1)^{\frac{3}{2}}}]\Delta s$$

$$>\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\lambda b^{\frac{2}{5}}}{4}\left[\frac{1}{(s+1)^{\frac{1}{2}}}-\frac{t_2+1}{(s+1)^{\frac{3}{2}}}\right]\Delta s=\infty,\text{if } \lambda>0.$$

Hence, by Theorem (3.2) every solution of Eq. (4.4) is oscillatory if $\lambda>0$.

Remark 4.2 The results of [6, 7, 10, 11] can not be applied to equations 4.3 and 4.4, but according to Theorem (3.2), those equations are oscillatory.

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