Anti Fuzzy HX Ring and Its Level Sub HX Ring

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Abstract **—** In this paper, we define the concept of an anti fuzzy HX ring and define a new algebraic structure of an anti fuzzy sub HX ring of a HX ring. Also we define level sub HX ring of a HX ring and some related properties are investigated. The purpose of this study is to implement the fuzzy set theory and ring theory in anti fuzzy sub HX ring of a HX ring. Characterizations of level subsets of an anti fuzzy sub HX ring of a HX ring are given. We also discussed the relation between a given anti fuzzy sub HX ring of a HX ring and its level sub HX rings and investigate the conditions under which a given HX ring has a properly inclusive chain of sub HX rings. In particular, we formulate how to structure an anti fuzzy sub HX ring of a HX ring by a given chain of sub HX rings.

Keywords — HX ring, anti fuzzy HX ring, level subset, level sub HX ring

I. Introduction

In 1965, Zadeh [13] introduced the concept of fuzzy subset μ of a set X as a function from X into the closed unit interval 0 and 1 and studied their properties. Fuzzy set theory is a useful tool to describe situations in which the data or imprecise or vague and it is applied to logic , set theory, group theory, ring theory, real analysis, measure theory etc. In 1967, Rosenfeld [10] defined the idea of fuzzy subgroups and gave some of its properties. Li Hong Xing [4] introduced the concept of HX group. In 1988, Professor Li Hong Xing [6] proposed the concept of HX ring and derived some of its properties, then Professor Zhong [1,2] gave the structures of HX ring on a class of ring. In this paper we define a new algebraic structure of an fuzzy HX subring of a HX ring and investigate some related properties. In this section we introduce the concept of an anti fuzzy HX ring and define a new algebraic structure of an anti fuzzy HX subring of a fuzzy HX ring and discuss some related properties.

II. Preliminaries

In this section, we site the fundamental definitions that will be used in the sequel. Throughout this paper, $R = (R ,+, \cdot)$ is a Ring, e is the additive identity element of R and xy, we mean x.y.

2.1 Definition [3]

A non-empty set R together with two binary operations '+' and '·' is said to be a ring if the following conditions are satisfied.

For all $a,b,c \in R$,

i. (R ,+) is an abelian group,

ii. (R , ·) is a semi group,

iii. $a (b + c) = ab + ac$ and $(a + b) c = ac + bc$.

2.2 Definition

Let R be a ring. In $2^R - \{\phi\}$, a non-empty set $\vartheta \subset 2^R - \{\phi\}$ with two binary operations ' + ' and '·' is said to be a HX ring on R if ϑ is a ring with respect to the algebraic operation defined by

i. $A + B = \{a + b / a \in A \text{ and } b \in B\}$, which its null element is denoted by Q , and the negative element of A is denoted by – A.

ii. $AB = \{ab / a \in A \text{ and } b \in B\}$

iii. $A (B+C) = AB + AC$ and $(B+ C) A = BA + CA$.

III. Properties of Anti fuzzy HX ring

In this section we study about anti fuzzy HX ring and discuss some related results.

3.1 Definition

Let R be a ring. Let μ be a fuzzy set defined on R. Let $\vartheta \subset 2^R - \{\phi\}$ be a HX ring. A fuzzy set λ_μ of ϑ is called an anti fuzzy HX ring on ϑ or anti fuzzy ring induced by μ if the following conditions are satisfied. For all $A, B \in \vartheta$,

i. $\lambda_\mu (A–B) \leq \max \{ \lambda_\mu (A), \lambda_\mu (B) \}$

ii. $\lambda_\mu (AB) \leq \max \{ \lambda_\mu (A) , \lambda_\mu (B)\}$

where $\lambda_\mu (A) = \min\{ \mu(x) / \text{ for all } x \in A \subseteq R \}$.

3.2 Theorem

Let λ_μ be an anti fuzzy HX subring of a HX ring ϑ. If $\lambda_\mu (A–B) \geq t$, then $\lambda_\mu (A) = \lambda_\mu (B)$, for A and B in ϑ.

Proof

Let A and B in ϑ.

Now, $\lambda_\mu(A) = \lambda_\mu(A-B+B)$
$\leq \max\{\lambda_\mu(A-B), \lambda_\mu(B)\}$
$= \max\{t, \lambda_\mu(B)\}$
$= \lambda_\mu(B)$
$= \lambda_\mu(-B)$
$= \lambda_\mu(-A+A-B)$
$\leq \max\{\lambda_\mu(-A), \lambda_\mu(A-B)\}$
$= \max\{\lambda_\mu(-A), t\}$
$= \lambda_\mu(-A)$
$= \lambda_\mu(A)$.

Therefore, $\lambda^\mu(A) = \lambda^\mu(B)$, for A and B in ϑ.

3.3 Theorem

Let λ_μ be an anti fuzzy HX subring of a HX ring ϑ. If $\lambda_\mu(A-B) \leq t$, then $\lambda_\mu(A) = \lambda_\mu(B)$, for A and B in ϑ.

Proof

Let A and B in ϑ.

Now, $\lambda_\mu(A) = \lambda_\mu(A-B+B)$
$\leq \max\{\lambda_\mu(A-B), \lambda_\mu(B)\}$
$= \max\{t, \lambda_\mu(B)\}$
$= \lambda_\mu(B)$
$= \lambda_\mu(-B)$
$= \lambda_\mu(-A+A-B)$
$\leq \max\{\lambda_\mu(-A), \lambda_\mu(A-B)\}$
$= \max\{\lambda_\mu(-A), t\} = \lambda_\mu(-A)$
$= \lambda_\mu(A)$.

Therefore, $\lambda_\mu(A) = \lambda_\mu(B)$, for A and B in ϑ.

3.4 Theorem

Let $(\vartheta, +, \cdot)$ be a HX ring. If λ_μ is an anti fuzzy subring of ϑ, then $\lambda_\mu(A+B) = \max\{\lambda_\mu(A), \lambda_\mu(B)\}$ with $\lambda_\mu(A) \neq \lambda_\mu(B)$, for each A and B in ϑ.

Proof

Let A , B $\in \vartheta$.

Assume that $\lambda_\mu(A) < \lambda_\mu(B)$

Now, $\lambda_\mu(A) = \lambda_\mu(-A+A+B)$
$\leq \max\{\lambda_\mu(-A), \lambda_\mu(A+B)\}$
$\leq \max\{\lambda_\mu(A), \lambda_\mu(A+B)\}$
$\leq \max\{\lambda_\mu(B), \lambda_\mu(A+B)\}$
$= \lambda_\mu(B)$.

and $\lambda_\mu(B) = \max\{\lambda_\mu(A), \lambda_\mu(A+B)\}$
$= \lambda_\mu(A+B)$.

Therefore, $\lambda_\mu(A+B) = \lambda_\mu(B)$
$= \max\{\lambda^\mu(A), \lambda^\mu(B)\}$, for all A , B$\in\vartheta$.

3.5 Theorem

Let λ_μ be an anti fuzzy HX subring of a HX ring ϑ. If $\lambda_\mu(A) > \lambda_\mu(B)$, for some A , B$\in\vartheta$, then $\lambda_\mu(A+B) = \lambda_\mu(A) = \lambda_\mu(B+A)$, for all A , B$\in\vartheta$.

Proof

Let λ_μ be a fuzzy HX subring of a HX ring ϑ.

Also ,$\lambda_\mu(A) > \lambda_\mu(B)$, for some A , B$\in\vartheta$

$\lambda_\mu(A+B) \leq \max\{\lambda_\mu(A), \lambda_\mu(B)\}$
$= \lambda_\mu(A)$

and $\lambda_\mu(A) = \lambda_\mu(A+B-B)$
$\leq \max\{\lambda_\mu(A+B), \lambda_\mu(-B)\}$
$\leq \max\{\lambda_\mu(A+B), \lambda_\mu(B)\}$
$= \lambda_\mu(A+B)$.

Therefore, $\lambda_\mu(A+B) = \lambda_\mu(A)$, for all A , B$\in\vartheta$.

Hence, $\lambda_\mu(A+B) = \lambda_\mu(A)$
$= \lambda_\mu(A+B)$, for all A and B .

3.6 Theorem

If μ is an anti fuzzy subring of R then the fuzzy set λ_μ is an anti fuzzy HX subring of ϑ .

Proof

Let μ be an anti fuzzy subring on R, and λ_μ be a fuzzy subset on ϑ . For any A, B $\in \vartheta$

i. $\max\{\lambda_\mu(A), \lambda_\mu(B)\}$

$= \max\{\min\{\mu(x) / \text{for all } x \in A \subseteq R\},$
$\min\{\mu(y) / \text{for all } y \in B \subseteq R\}\}$

$= \max\{\mu(x_0), \mu(y_0)\}$, $x_0 \in A$, $y_0 \in B$ and A , B $\subseteq$ R}

$\geq \mu(x_0 - y_0)$, μ is an anti fuzzy subring on R

$= \min\{\mu(x-y) / \text{for all } x \in A, y \in B \text{ and } A-B \subseteq R\}$

$= \lambda_\mu(A-B)$

Therefore, $\lambda_\mu(AB) \leq \max\{\lambda_\mu(A), \lambda_\mu(B)\}$

ii.$\max\{\lambda_\mu(A), \lambda_\mu(B)\}$

$= \max\{\min\{\mu(x) / \text{for all } x \in A \subseteq R\},$
$\min\{\mu(y) / \text{for all } y \in B \subseteq R\}\}$

$= \max\{\mu(x_0), \mu(y_0)\}$, $x_0 \in A$, $y_0 \in B$, A , B $\subseteq$ R }

$\geq \mu(x_0 y_0)$, μ is an anti fuzzy subring on R

$= \min\{\mu(xy) / \text{for all } x \in A, y \in B \text{ and } AB \subseteq R\}$

$= \lambda_\mu(AB)$

Therefore, $\lambda_\mu(AB) \leq \max\{\lambda_\mu(A), \lambda_\mu(B)\}$

Hence, λ_μ is an anti fuzzy HX subring of ϑ .

3.7 Remark : If μ is a fuzzy subset of a ring R and λ_μ be an anti fuzzy HX subring on ϑ , such that $\lambda_\mu(A) = \min\{\mu(x) / \text{for all } x \in A \subseteq R\}$, then μ need not be an anti fuzzy subring of R.

IV. CARTESIAN PRODUCT OF ANTI FUZZY HX SUBRING

4.1 Definition

Let λ_μ and γ_η be two fuzzy subsets of the HX rings ϑ_1 and ϑ_2 then the cartesian product of λ_μ and γ_η is defined as

$(\lambda_\mu \times \gamma_\eta)$ (A, B) = max $\{\lambda_\mu$ (A), γ_η (B)$\}$

for every (A, B) $\in \vartheta_1 \times \vartheta_2$.

4.2 Theorem

Let λ_μ and γ_η be fuzzy subsets of the HX rings ϑ_1 and ϑ_2 respectively. Suppose that Q and Q^1 are identity elements of ϑ_1 and ϑ_2 respectively. If $\lambda_\mu \times \gamma_\eta$ is an anti fuzzy HX sub ring of $\vartheta_1 \times \vartheta_2$ then atleast one of the following statements must hold

i. $\gamma_\eta(Q^1) \leq \lambda_\mu(A)$, for all $A \in \vartheta_1$

ii. $\lambda_\mu(Q) \leq \gamma_\eta(B)$, for all $B \in \vartheta_2$

Proof

Let $\lambda_\mu \times \gamma_\eta$ be a fuzzy HX sub ring of $\vartheta_1 \times \vartheta_2$.By contraposition, suppose that none of the statements (i) and (ii) holds then we can find $A \in \vartheta_1$ and $B \in \vartheta_2$, such that $\lambda_\mu(A) \leq \gamma_\eta(Q^1)$ and $\gamma_\eta(B) \leq \lambda_\mu(Q)$

We have, $(\lambda_\mu \times \gamma_\eta)(A,B) = \max\{\lambda_\mu(A), \gamma_\eta(B)\}$

$< \max\{\gamma_\eta(Q^1), \lambda_\mu(Q)\}$

$= \max\{\lambda_\mu(Q), \gamma_\eta(Q^1)\}$

$= (\lambda_\mu \times \gamma_\eta)(Q, Q^1)$

Thus, $\lambda_\mu \times \gamma_\eta$ is not an anti fuzzy HX sub ring of $\vartheta_1 \times \vartheta_2$.

Hence ,either $\gamma_\eta(Q^1) \leq \lambda_\mu(A)$ for all $A \in \vartheta_1$ or $\lambda_\mu(Q) \leq \gamma_\eta(B)$,for all $B \in \vartheta_2$.

4.3 Theorem

Let λ_μ and γ_η be fuzzy subsets of the HX rings ϑ_1 and ϑ_2 respectively, such that $\lambda_\mu(A) \geq \gamma_\eta(Q^1)$ for all $A \in \vartheta_1$, Q^1 being the identity element of ϑ_2. If $(\lambda_\mu \times \gamma_\eta)$ is an anti fuzzy HX sub ring of $\vartheta_1 \times \vartheta_2$ then λ_μ is an anti fuzzy HX sub ring of ϑ_1.

Proof

Let $\lambda_\mu \times \gamma_\eta$ be an anti fuzzy HX sub ring of $\vartheta_1 \times \vartheta_2$ and $A,B \in \vartheta_1$ then

$(A, Q^1),(B, Q^1) \in \vartheta_1 \times \vartheta_2$.

Now using the property, $\lambda_\mu(A) \geq \gamma_\eta(Q^1)$

for all $A \in \vartheta_1$

i. $\lambda_\mu(A-B) = \max\{\lambda_\mu(A-B), \gamma_\eta(Q^1 - Q^1)\}$

$= (\lambda_\mu \times \gamma_\eta)((A-B), (Q^1 - Q^1))$

$= (\lambda_\mu \times \gamma_\eta)((A, Q^1) - (B, Q^1))$

$\leq \max\{(\lambda_\mu \times \gamma_\eta)(A, Q^1), (\lambda_\mu \times \gamma_\eta)(B, Q^1)\}$

$= \max\{\max\{\lambda_\mu(A), \gamma_\eta(Q^1)\},$
$\max\{\lambda_\mu(B), \gamma_\eta(Q^1)\}\}$

$= \max\{\lambda_\mu(A), \lambda_\mu(B)\}$

ii. $\lambda_\mu(AB) = \max\{\lambda_\mu(AB), \gamma_\eta(Q^1Q^1)\}$

$= (\lambda_\mu \times \gamma_\eta)((AB), (Q^1 Q^1))$

$= (\lambda_\mu \times \gamma_\eta)[(A, Q^1)(B, Q^1)]$

$\leq \max\{(\lambda_\mu \times \gamma_\eta)(A, Q^1),$
$(\lambda_\mu \times \gamma_\eta)(B, Q^1)\}$

$= \max\{\max\{\lambda_\mu(A), \gamma_\eta(Q^1)\},$
$\max\{\lambda_\mu(B), \gamma_\eta(Q^1)\}\}$

$= \max\{\lambda_\mu(A), \lambda_\mu(B)\}$

Hence, λ_μ is an anti fuzzy HX sub ring of ϑ_1.

4.4 Theorem

Let λ_μ and γ_η be fuzzy subsets of the HX rings ϑ_1 and ϑ_2 respectively , such that $\gamma_\eta(A) \geq \lambda_\mu(Q)$ for all $X \in \vartheta_2$, Q being the identity element of ϑ_1. If $\lambda_\mu \times \gamma_\eta$ is a fuzzy HX sub ring of $\vartheta_1 \times \vartheta_2$ then γ_η is a fuzzy HX sub ring of ϑ_2.

Proof

Let $\lambda_\mu \times \gamma_\eta$ be a fuzzy HX sub ring of $\vartheta_1 \times \vartheta_2$ and $A,B \in \vartheta_1$ then $(A, Q),(B, Q) \in \vartheta_1 \times \vartheta_2$.

Now using the property, $\gamma_\eta(A) \geq \lambda_\mu(Q)$ for all $X \in \vartheta_2$

i. $\gamma_\eta(A-B) = \max\{\lambda_\mu(Q-Q), \gamma^\eta(A-B)\}$

$= (\lambda_\mu \times \gamma_\eta)((Q-Q), (A-B))$

$= (\lambda_\mu \times \gamma_\eta)((Q, A) - (Q, B))$

$\leq \max\{(\lambda_\mu \times \gamma_\eta)(Q, A), (\lambda_\mu \times \gamma_\eta)(Q, B)\}$

$= \max\{\max\{\lambda_\mu(Q), \gamma_\eta(A)\},$
$\max\{\lambda_\mu(Q), \gamma_\eta(B)\}\}$

$= \max\{\gamma_\eta(A), \gamma_\eta(B)\}$

ii. $\gamma_\eta(AB) = \max\{\lambda_\mu(QQ), \gamma_\eta(AB)$

$= (\lambda_\mu \times \gamma_\eta)((QQ), (AB))$

$= (\lambda_\mu \times \gamma_\eta)[(Q, A)(Q, B)]$

$\leq \max\{(\lambda_\mu \times \gamma_\eta)(Q, A), (\lambda_\mu \times \gamma_\eta)(Q, B)\}$

$= \max\{\max\{\lambda_\mu(Q), \gamma_\eta(A)\},$
$\max\{\lambda_\mu(Q), \gamma_\eta(B)\}\}$

$= \max\{\gamma_\eta(A), \gamma_\eta(B)\}$

Hence, γ_η is an anti fuzzy HX sub ring of ϑ_2.

4.5 Corollary

Let λ_μ and γ_η be two fuzzy subsets of the HX rings ϑ_1 and ϑ_2 respectively. If $\lambda_\mu \times \gamma_\eta$ is an anti fuzzy HX subring of $\vartheta_1 \times \vartheta_2$, then either λ_μ is an anti fuzzy HX subring of ϑ_1 or γ_η is an anti fuzzy HX subring of ϑ_2.

4.6 Definition

Let μ be a fuzzy subset of R then the anti-strongest fuzzy subset on R is a fuzzy relation on η is μ_η defined by $\mu_\eta(x,y) = \max\{\eta(x), \eta(y)\}$. Let λ_μ be a fuzzy subset of a HX ring ϑ, then the anti-

strongest fuzzy subset on ϑ is a fuzzy relation on λ_μ is γ_η which is defined by
$\gamma_\eta(A, B) = \max\{\lambda_\mu(A), \lambda_\mu(B)\}$, for all A and B in ϑ.

4.7 Theorem

Let λ_μ be a fuzzy subset of a HX ring ϑ and γ_η be the anti-strongest fuzzy relation of ϑ with respect to λ_μ then λ_μ is an anti fuzzy HX subring of ϑ if and only if γ_η is an anti fuzzy HX subring of $\vartheta_1 \times \vartheta_2$.

Proof

Suppose that λ_μ is an anti fuzzy HX subring of ϑ then for any X= (A,B) and Y = (C,D) are in $\vartheta_1 \times \vartheta_2$,We have

$\gamma_\eta(X-Y) = \gamma_\eta((A,B) - (C,D))$
$= \gamma_\eta(A - C , B - D)$
$=\max\{\lambda_\mu(A-C), \lambda_\mu(B-D)\}$
$\le\max\{\max\{\lambda_\mu(A),\lambda_\mu(C)\}, \max\{\lambda_\mu(B),\lambda_\mu(D)\}\}$
$=\max\{\max\{\lambda_\mu(A),\lambda_\mu(B)\}, \max\{\lambda_\mu(C),\lambda_\mu(D)\}\}$
$=\max\{\gamma_\eta(X), \gamma_\eta(Y)\}$
$\gamma_\eta(X-Y) \le \max\{\gamma_\eta(X), \gamma_\eta(Y)\}$
$\gamma_\eta(XY) = \gamma_\eta((A,B)(C,D))$
$=\gamma_\eta(AC, BD)$
$=\max\{\lambda_\mu(AC), \lambda_\mu(BD)\}$
$\le\max\{\max\{\lambda_\mu(A),\lambda_\mu(C)\}, \max\{\lambda_\mu(B),\lambda_\mu(D)\}\}$
$= \max\{\max\{\lambda_\mu(A),\lambda_\mu B)\},\max\{\lambda_\mu(C),\lambda_\mu(D)\}\}$
$=\max\{\gamma_\eta(X), \gamma_\eta(Y)\}$ for all X,Y in $\vartheta \times \vartheta$

Conversely γ_η is a fuzzy HX subring of $\vartheta \times \vartheta$ then for any X= (A,B) and Y = (C,D) are in $\vartheta \times \vartheta$

$\max\{\lambda_\mu(A-C), \lambda_\mu(B-D)\}=\gamma_\eta(A-C, B-D)$
$= \gamma_\eta((A,B) - (C,D))$
$= \gamma_\eta(X-Y)$
$\le \max\{\gamma_\eta(X), \gamma_\eta(Y)\}$
$= \max\{\gamma_\eta(A,B), \gamma_\eta(C,D)\}$
$=\max\{\max\{\lambda_\mu(A),\lambda_\mu(B)\}, \max\{\lambda_\mu(C),\lambda_\mu(D)\}\}$
$\le \max\{\lambda_\mu(A), \lambda_\mu(C)\}$
$\max\{\lambda_\mu(A-C),\lambda_\mu(B-D)\}\le \max\{\lambda_\mu(A), \lambda_\mu(C)\}$
$\max\{\lambda_\mu(A C), \lambda_\mu(BD)\}= \gamma_\eta(AC, BD)$
$= \gamma_\eta((A,B)(C,D))$
$= \gamma_\eta(XY)$
$\le \max\{\gamma_\eta(X), \gamma_\eta(Y)\}$
$=\max\{\max\{\lambda_\mu(A),\lambda_\mu(B)\},\max\{\lambda_\mu(C),\lambda_\mu(D)\}\}$
$\le \max\{\lambda_\mu(A), \lambda_\mu(C)\}$
$= \lambda_\mu(AC)$

Hence λ_μ is an anti fuzzy HX subring of ϑ if and only if γ_η is an anti fuzzy HX subring of $\vartheta_1 \times \vartheta_2$.

V. LEVEL SETS OF AN ANTI FUZZY HX SUBRING

In this section we introduce the concept of level sets of an anti fuzzy HX subring of a HX ring and prove certain properties of these.

5.1 Definition

Let λ_μ be an anti fuzzy HX subring of a HX ring ϑ. For any $t \in [0,1]$, we define the set $L(\lambda_\mu ; t) = \{ A\in\vartheta / \lambda_\mu(A) \le t \}$ is called a lower level subset or a level subset of λ_μ.

5.2 Theorem

Let λ_μ be an anti fuzzy HX subring of a HX ring ϑ then for $t \in [0,1]$, $L(\lambda_\mu ; t)$ is a sub HX ring of ϑ.

Proof

Let λ_μ be an anti fuzzy HX subring of a HX ring ϑ.

For any $A , B \in L(\lambda_\mu; t)$ we have $\lambda_\mu(A) \le t$ and $\lambda_\mu(B) \le t$.

Now, $\lambda_\mu(A-B) \le\max\{\lambda_\mu(A), \lambda_\mu(B)\}$
$\le\max\{t, t\} = t$, for some $t\in [0,1]$
$\lambda_\mu(A-B) \le t$
$\lambda_\mu(AB) \le \max\{\lambda_\mu(A), \lambda_\mu(B)\}$
$\le \max\{t, t\} = t$
$\lambda_\mu(AB) \le t$

Hence, $A - B$, $AB \in L(\lambda_\mu ; t)$.

Hence $L(\lambda_\mu; t)$ is a sub HX ring of a HX ring ϑ.

5.3 Theorem

Let ϑ be a HX ring and λ_μ be a fuzzy subset of ϑ such that $L(\lambda_\mu ; t)$ is a sub HX ring of ϑ for $t\in [0,1]$ then, λ_μ is an anti fuzzy HX subring of ϑ.

Proof

Let $A , B \in\vartheta$,

Let $A \in L(\lambda_\mu ; t_1) \Rightarrow \lambda_\mu(A) \le t_1$
and $B \in L(\lambda^\mu ; t_2) \Rightarrow \lambda^\mu(B) \le t_2$

Suppose $L(\lambda^\mu ; t_1), L(\lambda^\mu ; t_2) \in \vartheta$ then
$A , B \in L(\lambda^\mu ; t_2)$, as $L(\lambda^\mu ; t_2)$ is a subring of ϑ.

$\lambda^\mu(A-B) \le t_2$
$= \max\{t_1, t_2\}$
$= \max\{\lambda^\mu(A) , \lambda^\mu(B)\}$
$\lambda^\mu(A B) \le t_2$
$= \max\{t_1, t_2\}$
$= \max\{\lambda^\mu(A) , \lambda^\mu(B)\}$

Hence λ^μ is an anti fuzzy HX subring of ϑ.

5.4 Theorem

Let λ_μ be an anti fuzzy HX subring of a HX ring .Then two level HX subrings, $L(\lambda_\mu ; t_1)$, $L(\lambda_\mu ; t_2)$ for, $t_1,t_2 \in [0,1]$ and $t_1 , t_2 \geq \lambda_\mu (Q)$ with $t_1 < t_2$ of λ_μ are equal then there is no A in ϑ such that $t_1 < \lambda_\mu (A) \leq t_2$.

Proof

Let $L (\lambda_\mu ; t_1) = L (\lambda_\mu ; t_2)$.
Suppose there exists $A \in \vartheta$ such that $t_1 < \lambda_\mu (A) \leq t_2$, then
$L (\lambda_\mu ; t_2) \subseteq L (\lambda_\mu ; t_1)$. Since, $A \in L (\lambda_\mu ; t_1)$, but $A \notin L (\lambda_\mu ; t_2)$, which contradicts the assumption that, $L (\lambda_\mu ; t_1) = L (\lambda_\mu ; t_2)$.
Hence there is no A in ϑ such that $t_1 < \lambda_\mu (A) \leq t_2$.
Conversely, suppose that there is no A in ϑ such that $t_1 < \lambda_\mu (A) \leq t_2$,
Then by definition $L (\lambda_\mu ; t_2) \subseteq L (\lambda_\mu ; t_1)$.
Let $A \in L (\lambda_\mu ; t_1)$ and there is no A in ϑ such that $t_1 < \lambda_\mu (A) \leq t_2$.
Therefore, $A \in L (\lambda_\mu ; t_2)$ and $L (\lambda_\mu ; t_1) \subseteq L (\lambda_\mu ; t_2)$.
Hence, $L (\lambda_\mu ; t_1) = L (\lambda_\mu ; t_2)$.

5.5 Theorem

A fuzzy subset λ_μ of ϑ is an anti fuzzy HX subring of a HX ring ϑ if and only if the lower level HX subsets $L(\lambda_\mu ; t)$, $t \in$ Image λ_μ , are HX subring of ϑ.

Proof

Assume that λ_μ is an anti fuzzy sub HX subring of ϑ. Let $t \in \text{Im}\, \lambda^\mu$ be arbitrary.
Consider the level subset
$L (\lambda_\mu ; t) = \{ A \in \vartheta \ / \lambda_\mu (A) \leq t \}$
Then, $L(\lambda_\mu ; t) \neq \varphi$
Let $A, B \in L (\lambda_\mu ; t)$.
Then, $\lambda_\mu (A) \leq t$ and $\lambda^\mu (B) \leq t$ and
$\max \{\lambda_\mu (A), \lambda_\mu (B) \} \leq t$
$\Rightarrow \lambda_\mu (A - B) \leq t$ and $\lambda_\mu (AB) \leq t$
$\Rightarrow A - B \in L (\lambda_\mu ; t)$, $AB \in L (\lambda_\mu ; t)$.
Thus ,the level subsets $L (\lambda_\mu ; t)$, $t \in \text{Im}\, \mu$ are HX sub rings of R.
Conversely, assume that the level subsets
$L (\lambda_\mu; t)$, $t \in \text{Im}\, \lambda_\mu$ are HX sub rings of ϑ.
Let $A,B \in \vartheta$ be arbitrary.
Let $\max \{\lambda_\mu (A) , \lambda_\mu (A) \} = t$
then either $\lambda_\mu (A) = t$ and $\lambda_\mu (B) \leq \lambda_\mu (A) = t$
or $\lambda_\mu (B) = t$ and $\lambda_\mu (A) \leq \lambda_\mu (B) = t$
Therefore , $\lambda_\mu (A) \leq t$ and $\lambda_\mu (B) \leq t$
Which implies $A, B \in L (\lambda_\mu ; t)$
Also $A - B, AB \in L (\lambda_\mu ; t)$, since $L (\lambda_\mu ; t)$ is a HX subring of ϑ.
Hence, $\lambda_\mu (A - B) \leq t$ and $\lambda_\mu (AB) \leq t$

Thus λ_μ is an anti fuzzy HX subring of ϑ.

5.6 Theorem

Any sub HX ring H of a HX ring ϑ can be realized as a lower level sub HX ring of some anti fuzzy HX subring of ϑ.

Proof

Let λ_μ be a fuzzy subset and $A \in \vartheta$.
Define, $\lambda_\mu (A) = \begin{cases} 0 & \text{if } A \in H, \text{ where } t \in (0,1] \\ t & \text{if } A \notin H , \end{cases}$
We shall prove that λ_μ is an anti fuzzy HX subring of ϑ.
Let $A , B \in \vartheta$.
i.Suppose A, B $\in$ H, then A+B $\in$ H, A– B $\in$ H and AB $\in$ H.
$\lambda_\mu (A) = 0$, $\lambda_\mu (B) = 0$, $\lambda_\mu (A+B) = 0$, $\lambda_\mu (A-B) = 0$ and $\lambda_\mu (AB) = 0$.
$\lambda_\mu (A-B) \leq \max \{ \lambda_\mu (A), \lambda_\mu (B) \}$
$\lambda_\mu (AB) \leq \max \{ \lambda_\mu (A), \lambda_\mu (B) \}$.
i.Suppose A $\in$ H and B $\notin$ H, then A+ B $\notin$ H, A–B $\notin$ H and AB $\notin$ H.
$\lambda_\mu (A) = 0$, $\lambda_\mu (B) = t$, $\lambda_\mu (A+B) = t$, $\lambda_\mu (A-B) = t$ and $\lambda_\mu (AB) = t$.
$\lambda_\mu (A-B) \leq \max \{ \lambda_\mu (A), \lambda_\mu (B) \}$
$\lambda_\mu (AB) \leq \max \{ \lambda_\mu (A), \lambda_\mu (B) \}$.
i.Suppose A, B $\notin$ H, then A + B $\in$ H or A + B $\notin$ H and AB $\in$ H or AB $\notin$ H.
$\lambda_\mu (A) = t$, $\lambda_\mu (B) = t$, $\lambda_\mu (A-B) =$ t or 0 and $\lambda_\mu (AB) =$ t or 0.
$\lambda_\mu (A-B) \leq \max \{ \lambda_\mu (A), \lambda_\mu (B) \}$
$\lambda_\mu (AB) \leq \max \{ \lambda_\mu (A), \lambda_\mu (B) \}$.
Thus, in all cases, λ_μ is an anti fuzzy HX subring of ϑ.For this anti fuzzy HX subring, $L(\lambda_\mu ; t) = H$.

5.7 Remark

As a consequence of the **Theorem 5.4 and 5.5**, the lower level HX subring of an anti fuzzy HX subring λ_μ of a HX ring ϑ form a chain.Since $\lambda_\mu (Q) \leq \lambda_\mu (A)$ for all A in ϑ and therefore
$L(\lambda_\mu; t_0)$, where $\lambda_\mu \{Q\} = t_0$ is the smallest and we have the chain :$\{Q\}= L(\lambda_\mu ; t_0) \subset L(\lambda_\mu ; t_1) \subset L(\lambda_\mu ; t_2) \subset \ldots \subset L(\lambda_\mu ; t_n) = \vartheta$, where $t_0 < t_1 < t_2 < \ldots < t_n$.

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