Certain finite integrals pertaining to a class of polynomials, Aleph-function of two variables and the multivariable Aleph-function

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ABSTRACT
Our aim is to evaluate four integrals pertaining to the products of Aleph-function of two variables, a class of polynomial $S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s]$ and the multivariable Aleph-function. On account of the most general nature of the function involved herein a very large number of known and new integrals involving multivariable I-function and multivariable H-function follows as particular cases of our main integrals.

KEYWORDS : Aleph-function of several variables, Aleph-function of two variables, finite integral, special function, general class of polynomials.

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1.Introduction and preliminaries.

The function Aleph of several variables generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [4] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We define : $\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\right|$

$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]\ ,[\tau_i(a_{ji};\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{\mathfrak{n}+1,p_i}]:$
$\cdots\cdots\cdots\ ,[\tau_i(b_{ji};\beta_j^{(1)},\cdots,\beta_j^{(r)})_{m+1,q_i}]:$

$[(\mathrm{c}_j^{(1)});\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;[(\mathrm{c}_j^{(r)});\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]$
$[(\mathrm{d}_j^{(1)});\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;[(\mathrm{d}_j^{(r)});\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\Bigg)$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and $$\theta_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

Suppose , as usual , that the parameters

$a_j, j = 1, \cdots, p; b_j, j = 1, \cdots, q;$

$c_j^{(k)}, j = 1, \cdots, n_k; c_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, p_{i^{(k)}};$

$d_j^{(k)}, j = 1, \cdots, m_k; d_{ji^{(k)}}^{(k)}, j = m_k + 1, \cdots, q_{i^{(k)}};$

with $k = 1 \cdots, r, i = 1, \cdots, R$, $i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} + \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)}$$

$$-\tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1 \cdots, r, i = 1, \cdots, R \text{ , } i^{(k)} = 1, \cdots, R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1} \ldots |z_r|^{\alpha_r}), max(|z_1| \ldots |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1} \ldots |z_r|^{\beta_r}), min(|z_1| \ldots |z_r|) \rightarrow \infty$

where, with $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R \ ; \ V = m_1, n_1; \cdots ; m_r, n_r \tag{1.6}$$

$$\mathrm{W} = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i(1)}; R^{(1)}, \cdots , p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i(r)}; R^{(r)} \tag{1.7}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots , \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots , \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.8}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots , \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.9}$$

$$C = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots ,\{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.10}$$

$$D = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots ,\{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.11}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots , z_r) = \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A} : \mathrm{C} \\ \cdot & \\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \\ \mathrm{z}_r & \mathrm{B} : \mathrm{D} \end{array}\right) \tag{1.12}$$

The generalized Aleph-function of two variables is defined by K. Sharma [6], it's an extension of the I-function defined by C.K. Sharma and P.L. Mishra [5] , A. Goyal et al [1 , 2 , 3] , wich itself is a generalisation of G and H-function of two variables. The double Mellin-Barnes integral occuring in this paper will be referred to as the Aleph-function of two variables throughout our present study and will be defined and represented as follows.

$$\aleph(y_1, y_2) = \aleph^{M_1,N_1:M_2,N_2;M_3,N_3}_{P_i^{(1)},Q_i^{(1)},\iota_i;r:P_{i'}^{(2)},Q_{i'}^{(2)},\iota_{i'};r';P_{i''}^{(3)},Q_{i''}^{(3)},\iota_{i''};r''}\left(\begin{array}{c|c} \mathrm{y}_1 & [(\mathrm{u}_j; \mu_j^{(1)}, \mu_j^{(2)}]_{N_1}, [\iota_i(u_{ji}; \mu_{ji}^{(1)}, \mu_{ji}^{(2)})]_{N_1+1,P_i^{(1)}} \\ \mathrm{y}_2 & \cdot\ \cdot\ [\iota_i(v_{ji}; \upsilon_{ji}^{(1)}, \upsilon_{ji}^{(2)})]_{M_1+1,Q_i^{(1)}} \end{array}\right.$$

$$\left.\begin{array}{l} : (\mathrm{a}_j; \alpha_j)_{1,N_2}, [\iota_{i'}(a_{ji'}; \alpha_{ji'})]_{N_2+1,P_{i'}^{(2)}}; (c_j; \gamma_j)_{1,N_3}, [\iota_{i''}(c_{ji''}; \gamma_{ji''})]_{N_3+1,P_{i''}^{(3)}} \\ : (\mathrm{b}_j; \beta_j)_{1,M_2}, [\iota_{i'}(b_{ji'}; \beta_{ji'})]_{M_2+1,Q_{i'}^{(2)}}; (d_j; \delta_j)_{1,M_3}, [\iota_{i''}(d_{ji'}; \delta_{ji''})]_{M_3+1,Q_{i''}^{(3)}} \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \zeta(s, t)\ \ \phi_1(s)\phi_2(s) y_1^t y_2^t \, \mathrm{d}s\mathrm{d}t \tag{1.13}$$

where : $\omega = \sqrt{-1}$

$$\zeta(s, t) = \frac{\prod_{j=1}^{N_1} \Gamma(1 - u_j + \mu_j^{(1)} s + \mu_j^{(2)} t)}{\sum_{i=1}^{r} \iota_i [\prod_{j=M_1+1}^{Q_i^{(1)}} \Gamma(1 - v_{ji} + \upsilon_{ji}^{(1)} s + \upsilon_{ji}^{(2)} t) \prod_{j=N_1+1}^{P_i^{(1)}} \Gamma(u_{ji} - \mu_{ji}^{(1)} s - \mu_{ji}^{(2)} t)]} \tag{1.14}$$

$$\phi_1(s) = \frac{\prod_{j=1}^{M_2} \Gamma(b_j - \beta_j s) \prod_{j=1}^{N_2} \Gamma(1 - a_j + \alpha_j s)}{\sum_{i=1}^{r'} \iota_{i'} [\prod_{j=M_2+1}^{Q_{i'}^{(2)}} \Gamma(1 - b_{ji'} + \beta_{ji'} s) \prod_{j=N_2+1}^{P_{i'}^{(2)}} \Gamma(a_{ji'} - \alpha_{ji'} s)]} \tag{1.15}$$

$$\phi_2(t) = \frac{\prod_{j=1}^{M_3}\Gamma(d_j-\delta_j t)\prod_{j=1}^{N_3}\Gamma(1-c_j+\gamma_j s)}{\sum_{i=1}^{r''}\iota_{i''}[\prod_{j=M_3+1}^{Q_{i''}^{(3)}}\Gamma(1-d_{ji''}+\delta_{ji''}t)\prod_{j=N_3+1}^{P_{i''}^{(3)}}\Gamma(c_{ji''}-\gamma_{ji''}t)]} \tag{1.16}$$

where x and y are not equal to zero. $P_1, P_{i'}^{(1)}, P_{i''}^{(2)}, Q_1, Q_{i'}^{(1)}, Q_{i''}^{(2)}, M_1, N_1, M_2, N_2, M_3, N_3$ are non-negative integers such that $0 \leqslant N_1 \leqslant P_1, 0 \leqslant N_2 \leqslant P_{i'}^{(1)}, 0 \leqslant N_3 \leqslant P_{i''}^{(2)}$,

$Q_i > 0, Q_{i'}^{(1)} > 0, Q_{i''}^{(2)} > 0, ; \iota_i, \iota_{i'}, \iota_{i''} > 0.$

All the $\mu_1 s, \mu_2 s\ \alpha' s, \beta' s, \delta' s, \gamma' s$ are assumed to be positive quantities ; the definition and Aleph-function of two variables given above will however , have a meaning even if some these quantities are zero. The contour L_1 is in the s -plane and run from $-\omega\infty$ to $+\omega\infty$ with loops , if necessary , to ensure that the pole of $\Gamma(b_j-\beta_j s)$, $j=1,\cdots,M_2$ lies to the right and the poles of $\Gamma(1-a_j+\alpha_j s)$, $j=1,\cdots,N_2$; $\Gamma(1-u_j+\mu^{(1)}s+\mu_j^{(2)}t)$; $j=1,\cdots,N_1$ to the left contour.

The contour L_2 is in the t-plane and run from $-\omega\infty$ to $+\omega\infty$ with loops , if necessary , to ensure that the pole of

$\Gamma(d_j-\delta_j t)$;$j=1,\cdots,M_3$ lies to the right and the poles of $\Gamma(1-c_j+\gamma_j t)$; $j=1,\cdots,N_3$; $\Gamma(1-u_j+\mu^{(1)}s+\mu_j^{(2)}t)$; $j=1,\cdots,N_1$ to the left contour.

The existence conditions of (1.13) are given below .

$$U_1 = \iota_{i'}\sum_{j=1}^{P_{i'}^{(2)}}\alpha_{ji'} + \iota_i\sum_{j=1}^{P_i^{(1)}}\alpha_{ji}^{(1)} - \iota_i\sum_{j=1}^{Q_i^{(1)}}\beta_{ji}^{(1)} - \iota_{i'}\sum_{j=1}^{Q_{i'}^{(2)}}\beta_{ji'} < 0 \tag{1.17}$$

$$U_2 = \iota_i\sum_{j=1}^{P_i^{(1)}}\alpha_{ji}^{(2)} + \iota_{i''}\sum_{j=1}^{P_{i''}^{(3)}}\gamma_{ji''} - \iota_i\sum_{j=1}^{Q_i^{(1)}}\beta_{ji}^{(2)} - \iota_{i''}\sum_{j=1}^{Q_{i''}^{(3)}}\delta_{ji''} < 0 \tag{1.18}$$

where $i=1,\cdots,r$;$i'=1,\cdots r'$; $i''=1,\cdots,r''$

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.13) can be obtained by extension of the corresponding conditions for H-function of two variables given by as :

$|arg(y_1)| < A_1\frac{\pi}{2}$ and $|arg(y_2)| < A_2\frac{\pi}{2}$; $i=1,\cdots,r$;$i'=1,\cdots r'$; $i''=1,\cdots,r''$, where :

$$A_1 = \iota_i\sum_{j=N_1+1}^{P_i^{(1)}}\alpha_{ji}^{(1)} - \iota_i\sum_{j=1}^{Q_i^{(1)}}\beta_{ji}^{(1)} + \sum_{j=1}^{m_2}\beta_j - \iota_{i'}\sum_{j=M_2+1}^{Q_{i'}^{(2)}}\beta_{ji'} + \sum_{j=1}^{N_2}\alpha_j - \iota_{i'}\sum_{j=N_2+1}^{P_{i''}^{(2)}}\alpha_{ji'} > 0 \tag{1.19}$$

$$A_2 = \iota_i\sum_{j=N_1+1}^{P_i^{(1)}}\alpha_{ji}^{(2)} - \iota_i\sum_{j=1}^{Q_i^{(1)}}\beta_{ji}^{(2)} + \sum_{j=1}^{M_2}\delta_j - \iota_{i''}\sum_{j=M_3+1}^{Q_{i''}^{(3)}}\delta_{ji''} + \sum_{j=1}^{N_2}\gamma_j - \iota_{i''}\sum_{j=N_3+1}^{P_{i''}^{(3)}}\gamma_{ji''} > 0 \tag{1.20}$$

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(y_1,y_2) = 0(\ |y_1|^{\alpha_1}, |y_2|^{\alpha_r}\ )\ , max(\ |y_1|, |y_2|\ ) \rightarrow 0$

$\aleph(y_1,y_2)\ = 0(\ |y_1|^{\beta_1}, |z_r|^{\beta_r}\ )\ , min(\ |y_1|,|y_2|\ ) \rightarrow \infty$

where : $\alpha_1 = min[Re(b_j/\beta_j)], j=1,\cdots,M_2$ and $\alpha_2 = min[Re(d_j/\delta_j)], j=1,\cdots,M_3$

$\beta_1 = max[Re((a_j-1)/\alpha_j)], j=1,\cdots,N_2$ and $\beta_2 = max[Re((c_j-1)/\gamma_j)], j=1,\cdots,N_3$

Serie representation of Aleph-function of two variables is

$$\aleph^{M_1,N_1:M_2,N_2;M_3,N_3}_{P_i^{(1)},Q_i^{(1)},\iota_i;r:P_{i'}^{(2)},Q_{i'}^{(2)},\iota_{i'};r';P_{i''}^{(3)},Q_{i''}^{(3)},\iota_{i''};r''}\begin{pmatrix}\mathrm{y}_1\\ \mathrm{y}_2\end{pmatrix} = \sum_{G_1,G_2=0}^{\infty}\sum_{g_1=0}^{M_2}\sum_{g_2=0}^{M_3}\frac{(-)^{G_1+G_2}}{\beta_{g_1}G_1!\delta_{g_2}G_2!}$$

$$\times\ \zeta(\eta_{G_1,g_1},\eta_{G_2,g_2})\phi_1(\eta_{G_1,g_1})\ \phi_2(\eta_{G_2,g_2})y_1^{-\eta_{G_1,g_1}}y_2^{-\eta_{G_2,g_2}} \tag{1.21}$$

Where $\zeta(.,.)$, $\phi_1(.)$, $\phi_2(.)$ are given respectively in (1.14), (1.15) and (1.16) and

$$\eta_{G_1,g_1} = \frac{b_{g_1}+G_1}{\beta_{g_1}},\quad \eta_{G_2,g_2} = \frac{d_{g_2}+G_2}{\delta_{g_2}}$$

The generalized polynomials defined by Srivastava [7], is given in the following manner :

$$S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}[y_1,\cdots,y_s] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1,K_1;\cdots;N_s,K_s]y_1^{K_1}\cdots y_s^{K_s} \tag{1.22}$$

Where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_s,K_s]$ are arbitrary constants, real or complex.

2. The main integrals

In the document , we note : $A(\eta_{G_1,g_1},\eta_{G_2,g_2}) = \zeta(\eta_{G_1,g_1},\eta_{G_2,g_2})\phi_1(\eta_{G_1,g_1})\phi_2(\eta_{G_2,g_2})$

and $A_1 = \frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s]$

In this section the following integrals have been derived :

The first integral :

Let $g(x) = \frac{x-s}{x-w}; h(x) = \frac{t-s}{x-w}$

$$\int_s^t (x-s)^{u-1}(t-x)^{v-1}(x-w)^{-u-v}\aleph\begin{pmatrix}\mathrm{y}_1(g(x))^{p_1}(h(x))^{q_1}\\ \mathrm{y}_2(g(x))^{p_2}(h(x))^{q_2}\end{pmatrix}$$

$$S^{P_1,\cdots,P_R}_{Q_1,\cdots,Q_R}[Z_1(g(x))^{H_1}(h(x))^{K_1},\cdots,Z_R(g(x))^{H_R}(h(x))^{K_R}]\ \aleph^{0,\mathfrak{n}:V}_{U:W}\begin{pmatrix}\mathrm{z}_1(g(x))^{h_1}(h(x))^{k_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r(g(x))^{h_r}(h(x))^{k_r}\end{pmatrix}\mathrm{d}x$$

$$= \sum_{G_1,G_2=0}^{\infty}\sum_{g_1=0}^{M_2}\sum_{g_2=0}^{M_3}\frac{(-)^{G_1+G_2}}{\beta_{g_1}G_1!\delta_{g_2}G_2!}A(\eta_{G_1,g_1},\eta_{G_2,g_2})y_1^{\eta_{G_1,g_1}}y_2^{\eta_{G_2,g_2}}\sum_{\alpha_1=0}^{[Q_1/P_1]}\cdots\sum_{\alpha_R=0}^{[Q_R/P_R]}$$

$$A_1\, Z_1^{\alpha_1}\cdots Z_R^{\alpha_R}\,(t-w)^{-u-p_1\eta_{G_1,g_1}-p_2\eta_{G_2,g_2}-H_1\alpha_1-\cdots-H_R\alpha_R}$$

$$(s-w)^{-v-k_1-q_1\eta_{G_1,g_1}-q_2\eta_{G_2,g_2}-K_1\alpha_1-\cdots-K_R\alpha_R}$$

$$(t-s)^{u+v+(h_1+k_1)\eta_{G_1,g_1}+((h_2+k_2)\eta_{G_2,g_2}+(H_1+K_1)\alpha_1+\cdots+(H_R+K_R)\alpha_R-1}$$

$$\aleph_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c} z_1(g'(t))^{h_1}(h'(t))^{k_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r(g'(t))^{h_r}(h(t))^{k_r} \end{array}\middle| \begin{array}{c} (1\text{ -u-p}_1\eta_{G_1,g_1}-p_2\eta_{G_2,g_2}-\sum_{i=1}^R H_i\alpha_i;h_1,\cdots,h_r), \\ \cdots \\ \cdots \\ \cdots\cdots \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{ -v- q}_1\eta_{G_1,g_1}-q_2\eta_{G_2,g_2}-\sum_{i=1}^R K_i\alpha_i;k_1,\cdots,k_r),A:C \\ \cdots \\ \cdots \\ (1\text{-u-v-}(p_1+q_1)\eta_{G_1,g_1}-(p_2+q_2)\eta_{G_2,g_2}-\sum_{i=1}^R(H_i+K_i)\alpha_i;h_1+k_1,\cdots,h_r+k_r),B;D \end{array}\right) \quad (2.1)$$

Where :$U_{21}=p_i+2,q_i+1,\tau_i;R$ and $g'(x)=\dfrac{t-s}{t-w};h'(x)=\dfrac{t-s}{s-w}$

Provided that :

a) $h,k,h',k',h_i,k_i,p_1,q_1,p_2,q_2>0,i=1,\cdots,r$, k is an integer

b) $Re[u+p_1\min\limits_{1\leqslant j\leqslant M_2}\dfrac{b_j}{\beta_j}+p_2\min\limits_{1\leqslant j\leqslant M_3}\dfrac{d_j}{\delta_j}+\sum\limits_{i=1}^r h_i\min\limits_{1\leqslant j\leqslant m_i}\dfrac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

c) $Re[v+q_1\min\limits_{1\leqslant j\leqslant M_2}\dfrac{b_j}{\beta_j}+q_2\min\limits_{1\leqslant j\leqslant M_3}\dfrac{d_j}{\delta_j}+\sum\limits_{i=1}^r h_i\min\limits_{1\leqslant j\leqslant m_i}\dfrac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

d) $|argz_k|<\dfrac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.5)

e) $|arg(y_1)|<A_1\dfrac{\pi}{2}$ and $|arg(y_2)|<A_2\dfrac{\pi}{2}$ where A_1 and A_2 are given respectively in (1.19) and (1.20)

The second integral :

$$\int_0^t x^u\,(t-x)^{v-1}S_{Q_1,\cdots,Q_R}^{P_1,\cdots,P_R}[Z_1(t-x)^{K_1}x^{H_1},\cdots,Z_R(t-x)^{K_R}x^{H_R}]$$

$$\aleph\begin{pmatrix} y_1x^{p_1}(1-x)^{q_1} \\ y_2x^{p_2}(1-x)^{q_2} \end{pmatrix}\aleph_{U:W}^{0,\mathfrak{n}:V}\begin{pmatrix} z_1x^{h_1}(t-x)^{k_1} \\ \cdot \\ \cdot \\ \cdot \\ z_rx^{h_r}(t-x)^{k_r} \end{pmatrix}\mathrm{d}x$$

$$=\sum_{G_1,G_2=0}^{\infty}\sum_{g_1=0}^{M_2}\sum_{g_2=0}^{M_3}\frac{(-)^{G_1+G_2}}{\beta_{g_1}G_1!\delta_{g_2}G_2!}\,A(\eta_{G_1,g_1},\eta_{G_2,g_2})y_1^{\eta_{G_1,g_1}}y_2^{\eta_{G_2,g_2}}\sum_{\alpha_1=0}^{[Q_1/P_1]}\cdots\sum_{\alpha_R=0}^{[Q_R/P_R]}$$

$$A_1 Z_1^{\alpha_1} \cdots Z_R^{\alpha_R} \ (t-s)^{u+v-1+(p_1+q_1)\eta_{G_1,g_1}+(p_2+q_2)\eta_{G_2,g_2}+(H_1+K_1)\alpha_1+\cdots+(H_R+K_R)\alpha_R}$$

$$\aleph_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c} \mathrm{z}_1 t^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r t^{h_r+k_r} \end{array}\right| \begin{array}{c} (1\text{ -u-p}_1\eta_{G_1,g_1}-p_2\eta_{G_2,g_2}-\sum_{i=1}^R H_i\alpha_i;h_1,\cdots,h_r), \\ \cdots \\ \cdots \\ \cdots\cdots \end{array}$$

$$\left.\begin{array}{c} (1\text{ -v- q}_1\eta_{G_1,g_1}-q_2\eta_{G_2,g_2}-\sum_{i=1}^R K_i\alpha_i;k_1,\cdots,k_r),A:C \\ \cdots \\ \cdots \\ (1\text{-u-v-(p}_1+q_1)\eta_{G_1,g_1}-(p_2+q_2)\eta_{G_2,g_2}-\sum_{i=1}^R(H_i+K_i)\alpha_i;h_1+k_1,\cdots,h_r+k_r),B;D \end{array}\right) \quad (2.2)$$

Where :$U_{21}=p_i+2,q_i+1,\tau_i;R$

Provided that :

a) $h,k,h',k',h_i,k_i,p_1,q_1,p_2,q_2>0, i=1,\cdots,r$, k is an integer

b) $Re[u+p_1\min\limits_{1\leqslant j\leqslant M_2}\frac{b_j}{\beta_j}+p_2\min\limits_{1\leqslant j\leqslant M_3}\frac{d_j}{\delta_j}+\sum\limits_{i=1}^r h_i\min\limits_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

c) $Re[v+q_1\min\limits_{1\leqslant j\leqslant M_2}\frac{b_j}{\beta_j}+q_2\min\limits_{1\leqslant j\leqslant M_3}\frac{d_j}{\delta_j}+\sum\limits_{i=1}^r h_i\min\limits_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

d) $|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.5)

e) $|arg(y_1)|<A_1\frac{\pi}{2}$ and $|arg(y_2)|<A_2\frac{\pi}{2}$ where A_1 and A_2 are given respectively in (1.19) and (1.20)

The third integral :

$$\int_0^1 x^u\,(1-x)^{v-1}{}_2F_1[\alpha,\beta;u;x]S_{Q_1,\cdots,Q_R}^{P_1,\cdots,P_R}[Z_1(1-x)^{H_1},\cdots,Z_R(1-x)^{H_R}]$$

$$\aleph\begin{pmatrix}\mathrm{y}_1(1-x)^p\\ \mathrm{y}_2(1-x)^q\end{pmatrix}\aleph_{U:W}^{0,\mathfrak{n}:V}\begin{pmatrix}\mathrm{z}_1(1-x)^{h_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r(1-x)^{h_r}\end{pmatrix}\mathrm{d}x$$

$$=\sum_{G_1,G_2=0}^{\infty}\sum_{g_1=0}^{M_2}\sum_{g_2=0}^{M_3}\frac{(-)^{G_1+G_2}}{\beta_{g_1}G_1!\delta_{g_2}G_2!}\ A(\eta_{G_1,g_1},\eta_{G_2,g_2})\ \ y_1^{\eta_{G_1,g_1}}y_2^{\eta_{G_2,g_2}}\sum_{\alpha_1=0}^{[Q_1/P_1]}\cdots\sum_{\alpha_R=0}^{[Q_R/P_R]}$$

$$A_1 Z_1^{\alpha_1} \cdots Z_R^{\alpha_R} \aleph_{U_{22}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c|c} z_1 & (1\text{ -u -v-p}\eta_{G_1,g_1}+\alpha+\beta-\sum_{i=1}^R H_i\alpha_i;h_1,\cdots,h_r), \\ \cdot & \cdots \\ \cdot & \cdots \\ \cdot & (1\text{ -u-v -q}\eta_{G_2,g_2}+\alpha-\sum_{i=1}^R H_i\alpha_i;h_1,\cdots,h_r), \\ z_r & \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{ -v -p}\eta_{G_1,g_1}-\sum_{i=1}^R H_i\alpha_i;h_1,\cdots,h_r),A:C \\ \cdots \\ \cdots \\ (1\text{-u-v- q}\eta_{G_2,g_2}+\beta-\sum_{i=1}^R H_i\alpha_i;h_1,\cdots,h_r),B;D \end{array}\right) \quad (2.3)$$

Where :$U_{22}=p_i+2,q_i+2,\tau_i;R$

Provided that :

a) $h,k,h',k',h_i,k_i,p_1,q_1,p_2,q_2>0,i=1,\cdots,r$, k is an integer

b) $Re[u+p_1\min\limits_{1\leqslant j\leqslant M_2}\frac{b_j}{\beta_j}+p_2\min\limits_{1\leqslant j\leqslant M_3}\frac{d_j}{\delta_j}+\sum_{i=1}^r h_i\min\limits_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

c) $Re[v+q_1\min\limits_{1\leqslant j\leqslant M_2}\frac{b_j}{\beta_j}+q_2\min\limits_{1\leqslant j\leqslant M_3}\frac{d_j}{\delta_j}+\sum_{i=1}^r h_i\min\limits_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

d) $|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.5)

e) $|arg(y_1)|<A_1\frac{\pi}{2}$ and $|arg(y_2)|<A_2\frac{\pi}{2}$ where A_1 and A_2 are given respectively in (1.19) and (1.20)

f) $Re(u+v+\alpha+\beta)>0,Re(u)>0$

The fourth integral :

$$\int_{-1}^1(1+x)^{u-1}(1-x)^{v-1}P_w^{(\alpha',\beta')}(1-st(1-x)/2)\ \aleph\left(\begin{array}{c} y_1(1+x)^{p_1}(1-x)^{q_1} \\ y_2(1+x)^{p_2}(1-x)^{q_2}\end{array}\right)$$

$$S_{Q_1,\cdots,Q_R}^{P_1,\cdots,P_R}[Z_1(1+x)^{H_1}(1-x)^{K_1},\cdots,Z_R(1+x)^{H_R}(1-x)^{K_R}]$$

$$\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} z_1(1+x)^{h_1}(1-x)^{k_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r(1-x)^{h_r}(1-x)^{k_r}\end{array}\right)\mathrm{d}x$$

$$=\sum_{G_1,G_2=0}^{\infty}\sum_{g_1=0}^{M_2}\sum_{g_2=0}^{M_3}\frac{(-)^{G_1+G_2}}{\beta_{g_1}G_1!\delta_{g_2}G_2!}A(\eta_{G_1,g_1},\eta_{G_2,g_2})y_1^{\eta_{G_1,g_1}}y_2^{\eta_{G_2,g_2}}\sum_{\alpha_1=0}^{[Q_1/P_1]}\cdots\sum_{\alpha_R=0}^{[Q_R/P_R]}$$

$$A_1\ Z_1^{\alpha_1}\cdots Z_R^{\alpha_R}\ \frac{2^{u+v+(p_1+q_1)\eta_{G_1,g_1}+(p_2+q_2)\eta_{G_2,g_2}+(H_1+k_1)\alpha_1+\cdots+(H_R+K_R)\alpha_R-1}(\alpha+1;w)}{w!}$$

$$\sum_{R=0}^{w}\frac{(-w:R)(1+\alpha'+\beta'+w;R)}{R!(\alpha'+1;R)}(st/2)^R$$

$$\aleph_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c|c} z_1 2^{h_1+k_1} & (1-u-p_1\eta_{G_1,g_1}-p_2\eta_{G_2,g_2}-\sum_{i=1}^{R}H_i\alpha_i;h_1,\cdots,h_r), \\ \cdot & \cdots \\ \cdot & \cdots \\ \cdot & \cdots\cdots \\ z_r 2^{h_r+k_r} & \end{array}\right.$$

$$\left.\begin{array}{c} (1-v-q_1\eta_{G_1,g_1}-q_2\eta_{G_2,g_2}-\sum_{i=1}^{R}K_i\alpha_i;k_1,\cdots,k_r),A:C \\ \cdots \\ \cdots \\ (1-u-v-(p_1+q_1)\eta_{G_1,g_1}-(p_2+q_2)\eta_{G_2,g_2}-\sum_{i=1}^{R}(H_i+K_i)\alpha_i;h_1+k_1,\cdots,h_r+k_r),B;D \end{array}\right) \quad (2.4)$$

Where :$U_{21}=p_i+2,q_i+1,\tau_i;R$;

Provided that :

a) $h,k,h',k',h_i,k_i,p_1,q_1,p_2,q_2>0, i=1,\cdots,r$, k is an integer

b) $Re[u+p_1\min\limits_{1\leqslant j\leqslant M_2}\frac{b_j}{\beta_j}+p_2\min\limits_{1\leqslant j\leqslant M_3}\frac{d_j}{\delta_j}+\sum_{i=1}^{r}h_i\min\limits_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

c) $Re[v+q_1\min\limits_{1\leqslant j\leqslant M_2}\frac{b_j}{\beta_j}+q_2\min\limits_{1\leqslant j\leqslant M_3}\frac{d_j}{\delta_j}+\sum_{i=1}^{r}h_i\min\limits_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

d) $|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.5)

e) $|arg(y_1)|<A_1\frac{\pi}{2}$ and $|arg(y_2)|<A_2\frac{\pi}{2}$ where A_1 and A_2 are given respectively in (1.19) and (1.20)

Proof :

To establish the integral (2.1), express the generalized polynomials , occuring on the L.H.S. In the series form given by (1.22),the Aleph-function of two variables in serie form given by (1.21) and the multivariable Aleph-function involving there in terms of Mellin|-Barnes contour integral by (1.1). Now interchange the order of summation and integration (which is permissible under the conditions stated), so that the L.H.S of (2.1) say I assume the following from after a little simplification :

$$\text{I} = = \sum_{G_1,G_2=0}^{\infty}\sum_{g_1=0}^{M_2}\sum_{g_2=0}^{M_3}\sum_{\alpha_1=0}^{[Q_1/P_1]}\cdots\sum_{\alpha_R=0}^{[Q_R/P_R]}\frac{(-)^{G_1+G_2}}{\beta_{g_1}G_1!\delta_{g_2}G_2!}A(\eta_{G_1,g_1},\eta_{G_2,g_2})\,y_1^{\eta_{G_1,g_1}}y_2^{\eta_{G_2,g_2}}$$

$$A_1 Z_1^{\alpha_1} \cdots Z_R^{\alpha_R} \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1} \theta_k(s_k) \ z_k^{s_k}$$

$$\Big[\int_s^t (x-s)^{u+p_1\eta_{G_1,g_1}+p_2\eta_{G_2,g_2}+H_1\alpha_1+\cdots+H_R\alpha_R+h_1s_1+\cdots+h_rs_r-1}$$

$$(t-x)^{v+q_1\eta_{G_1,g_1}+q_2\eta_{G_2,g_2}+K_1\alpha_1+\cdots+K_R\alpha_R+k_1s_1+\cdots+k_rs_r-1}$$

$$(x-w)^{-u-v-(p_1+q_1)\eta_{G_1,g_1}-(p_2+q_2)\eta_{G_2,g_2}-(H_1+K_1)\alpha_1-\cdots-(H_R+K_R)\alpha_R-(h_1+k_1)s_1-\cdots-(h_r+k_r)s_r-1}$$

$$\mathrm{dx}\mathrm{d}s_1 \cdots \mathrm{d}s_r \Big] \tag{2.5}$$

On evaluating the inner integral occuring on the R.H.S. Of (2.5), we get after simplification :

$$\mathrm{I} = (t-s)^{u+v}(t-w)^{-u}(s-w)^{-v} \sum_{G_1,G_2=0}^{\infty} \sum_{g_1=0}^{M_2} \sum_{g_2=0}^{M_3} \sum_{\alpha_1=0}^{[Q_1/P_1]} \cdots \sum_{\alpha_R=0}^{[Q_R/P_R]} \frac{(-)^{G_1+G_2}}{\beta_{g_1} G_1! \delta_{g_2} G_2!} y_1^{\eta_{G_1,g_1}} y_2^{\eta_{G_2,g_2}}$$

$$A_1 \, Z_1^{\alpha_1} \cdots Z_R^{\alpha_R} \quad A(\eta_{G_1,g_1}, \eta_{G_2,g_2}) \ \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) \, z_k^{s_k}$$

$$[g(t)]^{(H_1+K_1)\alpha_1+\cdots+(H_R+K_R)\alpha_R} [g(t)]^{(h_1+k_1)s_1+\cdots+(h_r+k_r)s_r}$$

$$\Gamma(u + (p_1\eta_{G_1,g_1} + p_2\eta_{G_2,g_2} + H_1\alpha_1 + \cdots + H_R\alpha_R + h_1s_1 + \cdots + h_rs_r)$$

$$\times\Gamma(v + q_1\eta_{G_1,g_1} + q_2\eta_{G_2,g_2} + k_1\alpha_1 + \cdots + k_R\alpha_R + k_1s_1 + \cdots + k_rs_r)$$

$$\times[\,\Gamma(u + v + (p_1+q_1)\eta_{G_1,g_1} + (p_2+q_2)\eta_{G_2,g_2} + (H_1+K_1)\alpha_1 + \cdots + (H_R+K_R)\alpha_R +$$

$$+(h_1+k_1)s_1 + \cdots + (h_r+k_r)s_r)]^{-1}\mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{2.6}$$

where $g(t) = \dfrac{t-s}{t-w}$

Finally, on reinterpreting the Mellin-Barnes contour integral in the R.H.S. Of (2.6) in term of the multivariable Aleph-function given by (1.11), we arrive at the desired result. The proofs of the integrals (2.2), (2.3) and (2.4) can be developed on similar method.

3. Particular cases

a) If $\iota_i = \iota_{i^{(1)}} = \cdots = \iota_{i^{(r)}} = 1$, then the Aleph-function of several variables degenere to the I-function of several variables defined by Sharma and Ahmad [4].

$$\text{a)} \int_s^t (x-s)^{u-1}(t-x)^{v-1}(x-w)^{-u-v} \ \aleph\begin{pmatrix} \mathrm{y}_1(g(x))^{p_1}(h(x))^{q_1} \\ \mathrm{y}_2(g(x))^{p_2}(h(x))^{q_2} \end{pmatrix}$$

$$S_{Q_1,\cdots,Q_R}^{P_1,\cdots,P_R}[y_1(g(x))^{H_1}(h(x))^{K_1}, \cdots, y_R(g(x))^{H_R}(h(x))^{K_R}] \ I_{U:W}^{0,\mathfrak{n}:V} \begin{pmatrix} \mathrm{z}_1(g(x))^{h_1}(h(x))^{k_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r(g(x))^{h_r}(h(x))^{k_r} \end{pmatrix} \mathrm{d}x$$

$$= \sum_{G_1,G_2=0}^{\infty}\sum_{g_1=0}^{M_2}\sum_{g_2=0}^{M_3}\sum_{\alpha_1=0}^{[Q_1/P_1]}\cdots\sum_{\alpha_R=0}^{[Q_R/P_R]}\frac{(-)^{G_1+G_2}}{\beta_{g_1}G_1!\delta_{g_2}G_2!}A(\eta_{G_1,g_1},\eta_{G_2,g_2})\,y_1^{\eta_{G_1,g_1}}y_2^{\eta_{G_2,g_2}}$$

$$A_1\,y_1^{\alpha_1}\cdots y_R^{\alpha_R}\,(s-w)^{-v-q_1\eta_{G_1,g_1}-q_2\eta_{G_2,g_2}-K_1\alpha_1-\cdots-K_R\alpha_R}$$

$$(t-w)^{-u-p_1\eta_{G_1,g_1}-p_2\eta_{G_2,g_2}-H_1\alpha_1-\cdots-H_R\alpha_R}$$

$$(t-s)^{u+v+(p_1+q_1)\eta_{G_1,g_1}+(p_2+q_2)\eta_{G_2,g_2}+(H_1+K_1)\alpha_1+\cdots+(H_R+K_R)\alpha_R}$$

$$I_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c|c} \mathrm{z}_1(g'(t))^{h_1}(h'(t))^{k_1} & (1-u-p_1\eta_{G_1,g_1}-p_2\eta_{G_2,g_2}-\sum_{i=1}^R H_i\alpha_i;h_1,\cdots,h_r), \\ \cdot & \cdots \\ \cdot & \cdots \\ \cdot & \cdots\cdots \\ \mathrm{z}_r(g'(t))^{h_r}(h(t))^{k_r} & \end{array}\right.$$

$$\left.\begin{array}{c} (1-v-q_1\eta_{G_1,g_1}-q_2\eta_{G_2,g_2}-\sum_{i=1}^R K_i\alpha_i;k_1,\cdots,k_r),A:C \\ \cdots \\ \cdots \\ (1-u-v-(p_1+q_1)\eta_{G_1,g_1}-(p_2+q_2)\eta_{G_2,g_2}-\sum_{i=1}^R(H_i+K_i)\alpha_i;h_1+k_1,\cdots,h_r+k_r),B;D \end{array}\right) \quad (3.1)$$

Where $g(x)=\dfrac{x-s}{x-w}$ and $U_{21}=p_i+2,q_i+1;R$

b) $\int_0^t x^u\,(t-x)^{v-1}S_{Q_1,\cdots,Q_R}^{P_1,\cdots,P_R}[y_1(t-x)^{K_1}x^{R_1},\cdots,y_R(t-x)^{K_R}x^{H_R}]$

$$\aleph\begin{pmatrix}\mathrm{y}_1x^{p_1}(t-x)^{q_1}\\ \mathrm{y}_2x^{p_2}(t-x)^{q_2}\end{pmatrix} I_{U:W}^{0,\mathfrak{n}:V}\begin{pmatrix}\mathrm{z}_1x^{h_1}(t-x)^{k_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_rx^{h_r}(t-x)^{k_r}\end{pmatrix}\mathrm{d}x$$

$$= \sum_{G_1,G_2=0}^{\infty}\sum_{g_1=0}^{M_2}\sum_{g_2=0}^{M_3}\sum_{\alpha_1=0}^{[Q_1/P_1]}\cdots\sum_{\alpha_R=0}^{[Q_R/P_R]}\frac{(-)^{G_1+G_2}}{\beta_{g_1}G_1!\delta_{g_2}G_2!}A(\eta_{G_1,g_1},\eta_{G_2,g_2})y_1^{\eta_{G_1,g_1}}y_2^{\eta_{G_2,g_2}}$$

$$A_1y_1^{\alpha_1}\cdots y_R^{\alpha_R}(t-s)^{u+v-1+(p_1+q_1)\eta_{G_1,g_1}+(p_2+q_2)\eta_{G_2,g_2}+(H_1+K_1)\alpha_1+\cdots+(H_R+K_R)\alpha_R}$$

$$I_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c|c} \mathrm{z}_1t^{h_1+k_1} & (1-u-p_1\eta_{G_1,g_1}-p_2\eta_{G_2,g_2}-\sum_{i=1}^R H_i\alpha_i;h_1,\cdots,h_r), \\ \cdot & \cdots \\ \cdot & \cdots \\ \cdot & \cdots\cdots \\ \mathrm{z}_rt^{h_r+k_r} & \end{array}\right.$$

$$\left.\begin{array}{c} (1-v-q_1\eta_{G_1,g_1}-q_2\eta_{G_2,g_2}-\sum_{i=1}^R K_i\alpha_i;k_1,\cdots,k_r),A:C \\ \cdots \\ \cdots \\ (1-u-v-(p_1+q_1)\eta_{G_1,g_1}-(p_2+q_2)\eta_{G_2,g_2}-\sum_{i=1}^R(H_i+K_i)\alpha_i;h_1+k_1,\cdots,h_r+k_r),B;D \end{array}\right) \quad (3.2)$$

Where : $U_{21} = p_i + 2, q_i + 1; R$

c) $$\int_0^1 x^u\,(1-x)^{v-1}{}_2F_1[\alpha,\beta;u;x]S^{P_1,\cdots,P_R}_{Q_1,\cdots,Q_R}[y_1(1-x)^{H_1},\cdots,y_R(1-x)^{H_R}]$$

$$\aleph\begin{pmatrix}\mathrm{y}_1(1-x)^p\\ \mathrm{y}_2(1-x)^q\end{pmatrix} I\begin{pmatrix}\mathrm{z}_1(1-x)^{h_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r(1-x)^{h_r}\end{pmatrix}\mathrm{d}x$$

$$=\sum_{G_1,G_2=0}^{\infty}\sum_{g_1=0}^{M_2}\sum_{g_2=0}^{M_3}\sum_{\alpha_1=0}^{[Q_1/P_1]}\cdots\sum_{\alpha_R=0}^{[Q_R/P_R]}\frac{(-)^{G_1+G_2}}{\beta_{g_1}G_1!\delta_{g_2}G_2!}A(\eta_{G_1,g_1},\eta_{G_2,g_2})y_1^{\eta_{G_1,g_1}}y_2^{\eta_{G_2,g_2}}$$

$$A_1y_1^{\alpha_1}\cdots y_R^{\alpha_R}\ I^{0,\mathfrak{n}+2:V}_{U_{22}:W}\left(\begin{array}{c|c}\mathrm{z}_1 & (1\text{ -u -v-p}\eta_{G_1,g_1}+\alpha+\beta-\sum_{i=1}^R H_i\alpha_i;h_1,\cdots,h_r),\\ \cdot & \cdots\\ \cdot & \cdots\\ \cdot & \\ \mathrm{z}_r & (1\text{ -u-v -q}\eta_{G_2,g_2}+\alpha-\sum_{i=1}^R H_i\alpha_i;h_1,\cdots,h_r),\end{array}\right.$$

$$\left.\begin{array}{c}(1\text{ -v -p}\eta_{G_1,g_1}-\sum_{i=1}^R H_i\alpha_i;h_1,\cdots,h_r),A:C\\ \cdots\\ \cdots\\ (1\text{-u-v- q}\eta_{G_2,g_2}+\beta-\sum_{i=1}^R H_i\alpha_i;h_1,\cdots,h_r),B;D\end{array}\right) \tag{3.3}$$

Where : $U_{22} = p_i + 2, q_i + 2; R$

d) $$\int_{-1}^1(1+x)^{u-1}(1-x)^{v-1}P_w^{(\alpha',\beta')}(1-st(1-x)/2)\ \ \aleph\begin{pmatrix}\mathrm{y}_1(1+x)^{p_1}(1-x)^{q_1}\\ \mathrm{y}_2(1+x)^{p_2}(1-x)^{q_2}\end{pmatrix}$$

$$S^{P_1,\cdots,P_R}_{Q_1,\cdots,Q_R}[y_1(1+x)^{H_1}(1-x)^{K_1},\cdots,y_R(1+x)^{H_R}(1-x)^{K_R}]$$

$$I^{0,\mathfrak{n}:V}_{U:W}\begin{pmatrix}\mathrm{z}_1(1+x)^{h_1}(1-x)^{k_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r(1-x)^{h_r}(1-x)^{k_r}\end{pmatrix}\mathrm{d}x$$

$$=\sum_{G_1,G_2=0}^{\infty}\sum_{g_1=0}^{M_2}\sum_{g_2=0}^{M_3}\sum_{\alpha_1=0}^{[Q_1/P_1]}\cdots\sum_{\alpha_R=0}^{[Q_R/P_R]}\frac{(-)^{G_1+G_2}}{\beta_{g_1}G_1!\delta_{g_2}G_2!}A(\eta_{G_1,g_1},\eta_{G_2,g_2})y_1^{\eta_{G_1,g_1}}y_2^{\eta_{G_2,g_2}}$$

$$A_1 y_1^{\alpha_1}\cdots y_R^{\alpha_R} \quad \frac{2^{u+v+(p_1+q_1)\eta_{G_1,g_1}+(p_2+q_2)\eta_{G_2,g_2}+(H_1+k_1)\alpha_1+\cdots+(H_R+K_R)\alpha_R-1}(\alpha+1;w)}{w!}$$

$$\sum_{R=0}^{w}\frac{(-w:R)(1+\alpha'+\beta'+w;R)}{R!(\alpha'+1;R)}(st/2)^R$$

$$I_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c} \mathrm{z}_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r 2^{h_r+k_r}\end{array}\middle| \begin{array}{c}(1\text{ -u-p}_1\eta_{G_1,g_1}-p_2\eta_{G_2,g_2}-\sum_{i=1}^R H_i\alpha_i;h_1,\cdots,h_r), \\ \cdots \\ \cdots \\ \cdots\cdots\end{array}\right.$$

$$\left.\begin{array}{c}(1\text{ -v- q}_1\eta_{G_1,g_1}-q_2\eta_{G_2,g_2}-\sum_{i=1}^R K_i\alpha_i;k_1,\cdots,k_r),A:C \\ \cdots \\ \cdots \\ (1\text{-u-v-}(\mathrm{p}_1+q_1)\eta_{G_1,g_1}-(p_2+q_2)\eta_{G_2,g_2}-\sum_{i=1}^R(H_i+K_i)\alpha_i;h_1+k_1,\cdots,h_r+k_r),B;D\end{array}\right) \quad (3.4)$$

Where :$U_{21}=p_i+2,q_i+1;R$

b) If $R=R^{(1)}=\cdots=R^{(r)}=1$, the multivariable I-function degenere to the multivariable H-function , see Srivastava et al [8] , we obtain :

a) $$\int_s^t (x-s)^{u-1}(t-x)^{v-1}(x-w)^{-u-v} \ \aleph\left(\begin{array}{c}\mathrm{y}_1(g(x))^{p_1}(h(x))^{q_1} \\ \mathrm{y}_2(g(x))^{p_2}(h(x))^{q_2}\end{array}\right)$$

$$S_{Q_1,\cdots,Q_R}^{P_1,\cdots,P_R}[y_1(g(x))^{H_1}(h(x))^{K_1},\cdots,y_R(g(x))^{H_R}(h(x))^{K_R}] \ H_{p,q:V}^{0,\mathfrak{n}:U}\left(\begin{array}{c}\mathrm{z}_1(g(x))^{h_1}(h(x))^{k_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r(g(x))^{h_r}(h(x))^{k_r}\end{array}\right)\mathrm{d}x$$

$$=\sum_{G_1,G_2=0}^{\infty}\sum_{g_1=0}^{M_2}\sum_{g_2=0}^{M_3}\sum_{\alpha_1=0}^{[Q_1/P_1]}\cdots\sum_{\alpha_R=0}^{[Q_R/P_R]}\frac{(-)^{G_1+G_2}}{\beta_{g_1}G_1!\delta_{g_2}G_2!}A(\eta_{G_1,g_1},\eta_{G_2,g_2})y_1^{\eta_{G_1,g_1}}y_2^{\eta_{G_2,g_2}}$$

$$A_1\, y_1^{\alpha_1}\cdots y_R^{\alpha_R}\ (s-w)^{-v-q_1\eta_{G_1,g_1}-q_2\eta_{G_2,g_2}-K_1\alpha_1-\cdots-K_R\alpha_R}$$

$$(t-w)^{-u-p_1\eta_{G_1,g_1}-p_2\eta_{G_2,g_2}-H_1\alpha_1-\cdots-H_R\alpha_R}$$

$$(t-s)^{u+v+(p_1+q_1)\eta_{G_1,g_1}+(p_2+q_2)\eta_{G_2,g_2}+(H_1+K_1)\alpha_1+\cdots+(H_R+K_R)\alpha_R}$$

$$H_{p+2,q+1;:V}^{0,\mathfrak{n}+2:U}\left(\begin{array}{c}\mathrm{z}_1(g'(t))^{h_1}(h'(t))^{k_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r(g'(t))^{h_r}(h(t))^{k_r}\end{array}\middle|\begin{array}{c}(1\text{ -u-p}_1\eta_{G_1,g_1}-p_2\eta_{G_2,g_2}-\sum_{i=1}^R H_i\alpha_i;h_1,\cdots,h_r), \\ \cdots \\ \cdots \\ \cdots\cdots\end{array}\right.$$

$$\left.\begin{matrix} (1\text{ -v- } q_1\eta_{G_1,g_1} - q_2\eta_{G_2,g_2} - \sum_{i=1}^R K_i\alpha_i; k_1,\cdots,k_r), A : C \\ \cdots \\ \cdots \\ (1\text{-u-v-}(p_1+q_1)\eta_{G_1,g_1} - (p_2+q_2)\eta_{G_2,g_2} - \sum_{i=1}^R (H_i+K_i)\alpha_i; h_1+k_1,\cdots,h_r+k_r), B; D \end{matrix}\right) \quad (3.5)$$

Where $g(x) = \dfrac{x-s}{x-w}$

b) $\int_0^t x^u\,(t-x)^{v-1} S_{Q_1,\cdots,Q_R}^{P_1,\cdots,P_R}[y_1(t-x)^{K_1}x^{H_1},\cdots,y_R(t-x)^{K_R}x^{H_R}]$

$$\aleph\begin{pmatrix} y_1x^{p_1}(t-x)^{q_1} \\ y_2x^{p_2}(t-x)^{q_2}\end{pmatrix} H_{p,q:V}^{0,\mathfrak{n}:U}\begin{pmatrix} z_1x^{h_1}(t-x)^{k_1} \\ \cdot \\ \cdot \\ \cdot \\ z_rx^{h_r}(t-x)^{k_r}\end{pmatrix}\mathrm{d}x$$

$$= \sum_{G_1,G_2=0}^{\infty}\sum_{g_1=0}^{M_2}\sum_{g_2=0}^{M_3}\sum_{\alpha_1=0}^{[Q_1/P_1]}\cdots\sum_{\alpha_R=0}^{[Q_R/P_R]}\frac{(-)^{G_1+G_2}}{\beta_{g_1}G_1!\delta_{g_2}G_2!}A(\eta_{G_1,g_1},\eta_{G_2,g_2})y_1^{\eta_{G_1,g_1}}y_2^{\eta_{G_2,g_2}}$$

$$A_1\,y_1^{\alpha_1}\cdots y_R^{\alpha_R}(t-s)^{u+v-1+(p_1+q_1)\eta_{G_1,g_1}+(p_2+q_2)\eta_{G_2,g_2}+(H_1+K_1)\alpha_1+\cdots+(H_R+K_R)\alpha_R}$$

$$H_{p+2,q+1;:V}^{0,\mathfrak{n}+2:U}\left(\begin{matrix} z_1t^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ z_rt^{h_r+k_r}\end{matrix}\ \middle|\ \begin{matrix}(1\text{ -u-}p_1\eta_{G_1,g_1} - p_2\eta_{G_2,g_2} - \sum_{i=1}^R H_i\alpha_i; h_1,\cdots,h_r), \\ \cdots \\ \cdots \\ \cdots\cdots \end{matrix}\right.$$

$$\left.\begin{matrix} (1\text{ -v- } q_1\eta_{G_1,g_1} - q_2\eta_{G_2,g_2} - \sum_{i=1}^R K_i\alpha_i; k_1,\cdots,k_r), A : C \\ \cdots \\ \cdots \\ (1\text{-u-v-}(p_1+q_1)\eta_{G_1,g_1} - (p_2+q_2)\eta_{G_2,g_2} - \sum_{i=1}^R (H_i+K_i)\alpha_i; h_1+k_1,\cdots,h_r+k_r), B; D \end{matrix}\right) \quad (3.6)$$

c) $\int_0^1 x^u\,(1-x)^{v-1}{}_2F_1[\alpha,\beta;u;x] S_{Q_1,\cdots,Q_R}^{P_1,\cdots,P_R}[y_1(1-x)^{H_1},\cdots,y_R(1-x)^{H_R}]$

$$\aleph\begin{pmatrix} y_1(1-x)^p \\ y_2(1-x)^q\end{pmatrix} H_{p,q:V}^{0,\mathfrak{n}:U}\begin{pmatrix} z_1(1-x)^{h_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r(1-x)^{h_r}\end{pmatrix}\mathrm{d}x$$

$$= \sum_{G_1,G_2=0}^{\infty} \sum_{g_1=0}^{M_2} \sum_{g_2=0}^{M_3} \sum_{\alpha_1=0}^{[Q_1/P_1]} \cdots \sum_{\alpha_R=0}^{[Q_R/P_R]} \frac{(-)^{G_1+G_2}}{\beta_{g_1} G_1! \delta_{g_2} G_2!} A(\eta_{G_1,g_1}, \eta_{G_2,g_2}) \; y_1^{\eta_{G_1,g_1}} y_2^{\eta_{G_2,g_2}}$$

$$A_1 y_1^{\alpha_1} \cdots y_R^{\alpha_R} H^{0,\mathfrak{n}+2:U}_{p+2,q+1;V} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array} \middle| \begin{array}{c} (1\text{ -u -v-p}\eta_{G_1,g_1} + \alpha + \beta - \sum_{i=1}^R H_i \alpha_i; h_1, \cdots, h_r), \\ \cdots \\ \cdots \\ (1\text{ -u-v -q}\eta_{G_2,g_2} + \alpha - \sum_{i=1}^R H_i \alpha_i; h_1, \cdots, h_r), \end{array} \right.$$

$$\left. \begin{array}{c} (1\text{ -v -p}\eta_{G_1,g_1} - \sum_{i=1}^R H_i \alpha_i; h_1, \cdots, h_r), A : C \\ \cdots \\ \cdots \\ (1\text{-u-v- q}\eta_{G_2,g_2} + \beta - \sum_{i=1}^R H_i \alpha_i; h_1, \cdots, h_r), B; D \end{array} \right) \tag{3.7}$$

d) $\int_{-1}^{1} (1+x)^{u-1}(1-x)^{v-1} P_w^{(\alpha',\beta')}(1 - st(1-x)/2) \;\; \aleph \begin{pmatrix} y_1(1+x)^{p_1}(1-x)^{q_1} \\ y_2(1+x)^{p_2}(1-x)^{q_2} \end{pmatrix}$

$$S^{P_1,\cdots,P_R}_{Q_1,\cdots,Q_R}[y_1(1+x)^{H_1}(1-x)^{K_1}, \cdots, y_R(1+x)^{H_R}(1-x)^{K_R}]$$

$$H^{0,\mathfrak{n}:U}_{p,q:V} \begin{pmatrix} z_1(1+x)^{h_1}(1-x)^{k_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r(1-x)^{h_r}(1-x)^{k_r} \end{pmatrix} \mathrm{d}x$$

$$= \sum_{G_1,G_2=0}^{\infty} \sum_{g_1=0}^{M_2} \sum_{g_2=0}^{M_3} \sum_{\alpha_1=0}^{[Q_1/P_1]} \cdots \sum_{\alpha_R=0}^{[Q_R/P_R]} \frac{(-)^{G_1+G_2}}{\beta_{g_1} G_1! \delta_{g_2} G_2!} A(\eta_{G_1,g_1}, \eta_{G_2,g_2}) y_1^{\eta_{G_1,g_1}} y_2^{\eta_{G_2,g_2}}$$

$$A_1 y_1^{\alpha_1} \cdots y_R^{\alpha_R} \quad \frac{2^{u+v+(p_1+q_1)\eta_{G_1,g_1} + (p_2+q_2)\eta_{G_2,g_2} + (H_1+k_1)\alpha_1 + \cdots + (H_R+K_R)\alpha_R - 1}(\alpha+1; w)}{w!}$$

$$\sum_{R=0}^{w} \frac{(-w:R)(1+\alpha'+\beta'+w;R)}{R!(\alpha'+1;R)} (st/2)^R$$

$$H^{0,\mathfrak{n}+2:U}_{p+2,q+1;V} \left(\begin{array}{c} z_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r 2^{h_r+k_r} \end{array} \middle| \begin{array}{c} (1\text{ -u-p}_1\eta_{G_1,g_1} - p_2\eta_{G_2,g_2} - \sum_{i=1}^R H_i \alpha_i; h_1, \cdots, h_r), \\ \cdots \\ \cdots \\ \cdots\cdots\cdot \end{array} \right.$$

$$\left.\begin{array}{c}(1\text{ -v- }q_1\eta_{G_1,g_1}-q_2\eta_{G_2,g_2}-\sum_{i=1}^R K_i\alpha_i;k_1,\cdots,k_r),A:C\\ \cdots\\ \cdots\\ (1\text{-u-v-}(p_1+q_1)\eta_{G_1,g_1}-(p_2+q_2)\eta_{G_2,g_2}-\sum_{i=1}^R(H_i+K_i)\alpha_i;h_1+k_1,\cdots,h_r+k_r),B;D\end{array}\right)\quad(3.8)$$

4.Conclusion

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions such as I-function of several variables defined by Sharma and Ahmad [4] , multivariable H-function , see Srivastava et al [8], the Aleph-function of two variables defined by K.sharma [6], the I-function of two variables defined by Goyal and Agrawal [1,2,3] , ,and the h-function of two variables , see Srivastava et al [8].

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