Finite integrals pertaining to a product of special functions and multivariable Aleph-functions II

Frédéric Ayant

*Teacher in High School , France

ABSTRACT
An attempt has been made to establish an integral tranformation concerning the M-series, a class of polynomials of several variables and two multivariable Aleph-functions. The result established in this paper are of general nature and hence encompass several particular cases.

Keywords :Multivariable Aleph-function, general class of polynomials, M-serie.

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1.Introduction and preliminaries.

The function Aleph of several variables generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [1] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We define : $$\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};\cdots;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}}\left(\begin{array}{c|}\mathrm{y}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{y}_r\end{array}\right.$$

$$\begin{array}{l}[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]\ ,[\tau_i(a_{ji};\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{\mathfrak{n}+1,p_i}]:\\ \cdots\cdots\cdots\cdots\cdots\ ,[\tau_i(b_{ji};\beta_j^{(1)},\cdots,\beta_j^{(r)})_{m+1,q_i}]:\end{array}$$

$$\left.\begin{array}{l}[(\mathrm{c}_j^{(1)});\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;[(\mathrm{c}_j^{(r)});\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]\\ [(\mathrm{d}_j^{(1)});\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;[(\mathrm{d}_j^{(r)});\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^r\theta_k(s_k)y_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^r\alpha_j^{(k)}s_k)}{\sum_{i=1}^R[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^r\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^r\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and $$\theta_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

Suppose , as usual , that the parameters

$a_j, j=1,\cdots,p; b_j, j=1,\cdots,q;$

$c_j^{(k)}, j = 1, \cdots, n_k; c_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, p_{i^{(k)}};$

$d_j^{(k)}, j = 1, \cdots, m_k; d_{ji^{(k)}}^{(k)}, j = m_k + 1, \cdots, q_{i^{(k)}};$

with $k = 1 \cdots, r, i = 1, \cdots, R$, $i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} + \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)}$$

$$-\tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1 \cdots, r, i = 1, \cdots, R \text{ , } i^{(k)} = 1, \cdots, R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(y_1, \cdots, y_r) = 0(|y_1|^{\alpha_1} \dots |y_r|^{\alpha_r}), max(|y_1| \dots |y_r|) \to 0$

$\aleph(y_1, \cdots, y_r) = 0(|y_1|^{\beta_1} \dots |y_r|^{\beta_r}), min(|y_1| \dots |y_r|) \to \infty$

where, with $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)} / \delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$$

Serie representation of Aleph-function of several variables is given by

$$\aleph(y_1,\cdots,y_r)=\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}\psi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})$$

$$\times\ \theta_1(\eta_{G_1,g_1})\cdots\theta_r(\eta_{G_r,g_r})y_1^{-\eta_{G_1,g_1}}\cdots y_r^{-\eta_{G_r,g_r}} \tag{1.6}$$

Where $\psi(.,\cdots,.),\theta_i(.),i=1,\cdots,r$ are given respectively in (1.2), (1.3) and

$$\eta_{G_1,g_1}=\frac{d_{g_1}^{(1)}+G_1}{\delta_{g_1}^{(1)}},\cdots,\ \eta_{G_r,g_r}=\frac{d_{g_r}^{(r)}+G_r}{\delta_{g_r}^{(r)}}$$

which is valid under the conditions $\delta_{g_i}^{(i)}[d_j^i+p_i]\neq\delta_j^{(i)}[d_{g_i}^i+G_i]$ (1.7)

for $j\neq m_i,m_i=1,\cdots\eta_{G_i,g_i};p_i,n_i=0,1,2,\cdots,;y_i\neq 0,i=1,\cdots,r$ (1.8)

Consider the Aleph-function of s variables

$$\aleph(z_1,\cdots,z_s)=\aleph^{0,N:M_1,N_1,\cdots,M_s,N_s}_{P_i,Q_i,\iota_i;r:P_{i^{(1)}},Q_{i^{(1)}},\iota_{i^{(1)}};r^{(1)};\cdots;P_{i^{(s)}},Q_{i^{(s)}};\iota_{i^{(s)}};r^{(s)}}\left(\begin{array}{c|}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_s\end{array}\right.$$

$$\begin{array}{ll}[(\mathrm{u}_j;\mu_j^{(1)},\cdots,\mu_j^{(r)})_{1,N}] & ,[\iota_i(u_{ji};\mu_j^{(1)},\cdots,\mu_j^{(r)})_{N+1,P_i}]:\\ \cdots\cdots\cdots\cdots & ,[\iota_i(v_{ji};\upsilon_j^{(1)},\cdots,\upsilon_j^{(r)})_{M+1,Q_i}]:\end{array}$$

$$\left.\begin{array}{l}[(\mathrm{a}_j^{(1)});\alpha_j^{(1)})_{1,N_1}],[\iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)};\alpha_{ji^{(1)}}^{(1)})_{N_1+1,P_i^{(1)}}];\cdots;[(\mathrm{a}_j^{(s)});\alpha_j^{(s)})_{1,N_s}],[\iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)};\alpha_{ji^{(s)}}^{(s)})_{N_s+1,P_i^{(s)}}]\\ [(\mathrm{b}_j^{(1)});\beta_j^{(1)})_{1,M_1}],[\iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)};\beta_{ji^{(1)}}^{(1)})_{M_1+1,Q_i^{(1)}}];\cdots;[(\mathrm{b}_j^{(s)});\beta_j^{(s)})_{1,M_s}],[\iota_{i^{(s)}}(b_{ji^{(s)}}^{(s)};\beta_{ji^{(s)}}^{(s)})_{M_s+1,Q_i^{(s)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^s}\int_{L_1}\cdots\int_{L_r}\zeta(t_1,\cdots,t_s)\prod_{k=1}^{s}\phi_k(t_k)z_k^{t_k}\,\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.9}$$

with $\omega=\sqrt{-1}$

$$\zeta(t_1,\cdots,t_s)=\frac{\prod_{j=1}^{N}\Gamma(1-u_j+\sum_{k=1}^{s}\mu_j^{(k)}t_k)}{\sum_{i=1}^{r'}[\iota_i\prod_{j=N+1}^{P_i}\Gamma(u_{ji}-\sum_{k=1}^{s}\mu_{ji}^{(k)}t_k)\prod_{j=1}^{Q_i}\Gamma(1-v_{ji}+\sum_{k=1}^{s}\upsilon_{ji}^{(k)}t_k)]} \tag{1.10}$$

$$\text{and}\ \ \phi_k(t_k)=\frac{\prod_{j=1}^{M_k}\Gamma(b_j^{(k)}-\beta_j^{(k)}t_k)\prod_{j=1}^{N_k}\Gamma(1-a_j^{(k)}+\alpha_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{r^{(k)}}[\iota_{i^{(k)}}\prod_{j=M_k+1}^{Q_{i^{(k)}}}\Gamma(1-b_{ji^{(k)}}^{(k)}+\beta_{ji^{(k)}}^{(k)}t_k)\prod_{j=N_k+1}^{P_{i^{(k)}}}\Gamma(a_{ji^{(k)}}^{(k)}-\alpha_{ji^{(k)}}^{(k)}s_k)]} \tag{1.11}$$

Suppose , as usual , that the parameters

$u_j,j=1,\cdots,P;v_j,j=1,\cdots,Q;$

$a_j^{(k)}, j = 1, \cdots, N_k; a_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, P_{i^{(k)}};$

$b_{ji^{(k)}}^{(k)}, j = m_k + 1, \cdots, Q_{i^{(k)}}; b_j^{(k)}, j = 1, \cdots, M_k;$

with $k = 1 \cdots, s, i = 1, \cdots, r', i^{(k)} = 1, \cdots, r^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{N} \mu_j^{(k)} + \iota_i \sum_{j=N+1}^{P_i} \mu_{ji}^{(k)} + \sum_{j=1}^{N_k} \alpha_j^{(k)} + \iota_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \alpha_{ji^{(k)}}^{(k)} - \iota_i \sum_{j=1}^{Q_i} \upsilon_{ji}^{(k)} - \sum_{j=1}^{M_k} \beta_j^{(k)}$$

$$-\iota_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.12}$$

The reals numbers τ_i are positives for $i = 1, \cdots, r$, $\iota_{i^{(k)}}$ are positives for $i^{(k)} = 1 \cdots r^{(k)}$

The contour L_k is in the t_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(b_j^{(k)} - \beta_j^{(k)} t_k)$ with $j = 1$ to M_k are separated from those of $\Gamma(1 - u_j + \sum_{i=1}^{s} \mu_j^{(k)} t_k)$with $j = 1$ to N and $\Gamma(1 - a_j^{(k)} + \alpha_j^{(k)} t_k)$ with $j = 1$ to N_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k| < \frac{1}{2} B_i^{(k)} \pi$, where

$$B_i^{(k)} = \sum_{j=1}^{N} \mu_j^{(k)} - \iota_i \sum_{j=N+1}^{P_i} \mu_{ji}^{(k)} - \iota_i \sum_{j=1}^{Q_i} \upsilon_{ji}^{(k)} + \sum_{j=1}^{N_k} \alpha_j^{(k)} - \iota_{i^{(k)}} \sum_{j=N_k+1}^{P_{i^{(k)}}} \alpha_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{M_k} \beta_j^{(k)} - \iota_{i^{(k)}} \sum_{j=M_k+1}^{q_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} > 0, \text{ with } k = 1 \cdots, s, i = 1, \cdots, r, i^{(k)} = 1, \cdots, r^{(k)} \tag{1.13}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_s) = 0(|z_1|^{\alpha'_1} \dots |z_s|^{\alpha'_s}), max(|z_1| \dots |z_s|) \to 0$

$\aleph(z_1, \cdots, z_s) = 0(|z_1|^{\beta'_1} \dots |z_s|^{\beta'_s}), min(|z_1| \dots |z_s|) \to \infty$

where, with $k = 1, \cdots, z : \alpha'_k = min[Re(b_j^{(k)} / \beta_j^{(k)})], j = 1, \cdots, M_k$ and

$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, N_k$

We will use these following notations in this paper

$$U = P_i, Q_i, \iota_i; r' ; \; V = M_1, N_1; \cdots ; M_s, N_s \quad (1.15)$$

$$\mathrm{W} = P_{i^{(1)}}, Q_{i^{(1)}}, \iota_{i(1)}; r^{(1)}, \cdots , P_{i^{(r)}}, Q_{i^{(r)}}, \iota_{i(s)}; r^{(s)} \quad (1.16)$$

$$A = \{(u_j; \mu_j^{(1)}, \cdots , \mu_j^{(s)})_{1,N}\}, \{\iota_i(u_{ji}; \mu_{ji}^{(1)}, \cdots , \mu_{ji}^{(s)})_{N+1,P_i}\} \quad (1.17)$$

$$B = \{\iota_i(v_{ji}; v_{ji}^{(1)}, \cdots , v_{ji}^{(s)})_{M+1,Q_i}\} \quad (1.18)$$

$$C = (a_j^{(1)}; \alpha_j^{(1)})_{1,N_1}, \iota_{i^{(1)}}(a_{ji^{(1)}}^{(1)}; \alpha_{ji^{(1)}}^{(1)})_{N_1+1,P_{i^{(1)}}}, \cdots , (a_j^{(s)}; \alpha_j^{(s)})_{1,N_s}, \iota_{i^{(s)}}(a_{ji^{(s)}}^{(s)}; \alpha_{ji^{(s)}}^{(s)})_{N_s+1,P_{i^{(s)}}} \quad (1.19)$$

$$D = (b_j^{(1)}; \beta_j^{(1)})_{1,M_1}, \iota_{i^{(1)}}(b_{ji^{(1)}}^{(1)}; \beta_{ji^{(1)}}^{(1)})_{M_1+1,Q_{i^{(1)}}}, \cdots , (b_j^{(s)}; \beta_j^{(s)})_{1,M_s}, \iota_{i^{(s)}}(\beta_{ji^{(s)}}^{(s)}; \beta_{ji^{(s)}}^{(s)})_{M_s+1,Q_{i^{(s)}}} \quad (1.20)$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots , z_s) = \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A : C} \\ \cdot & \cdots \\ \cdot & \mathrm{B : D} \\ \mathrm{z}_s & \end{array}\right) \quad (1.21)$$

The generalized polynomials defined by Srivastava [6], is given in the following manner :

$$S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[y_1, \cdots , y_u] = \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_u=0}^{[N_u/\mathfrak{M}_u]} \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \cdots \frac{(-N_u)_{\mathfrak{M}_u K_u}}{K_u!} \quad (1.22)$$

The M-serie is defined, see Sharma [3].

$${}_{p'}M_{q'}^{\alpha}(y) = \sum_{s'=0}^{\infty} \frac{[(a_{p'})]_{s'}}{[(b_{q'})]_{s'}} \frac{y^{s'}}{\Gamma(\alpha s' + 1)} \quad (1.23)$$

Here $\alpha \in \mathbb{C}, Re(\alpha) > 0$. $[(a_{p'})]_{s'} = (a_1)_{s'} \cdots (a_{p'})_{s'}; [(b_{q'})]_{s'} = (b_1)_{s'} \cdots (b_{q'})_{s'}$.
The serie (1.23) converge if $p' \leqslant q' and \, |y| < 1$.

In the document , we note :

$$G(\eta_{G_1,g_1}, \cdots , \eta_{G_r,g_r}) = \phi(\eta_{G_1,g_1}, \cdots , \eta_{G_r,g_r})\theta_1(\eta_{G_1,g_1}) \cdots \theta_r(\eta_{G_r,g_r}) \quad (1.24)$$

$$A_1 = \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \cdots \frac{(-N_u)_{\mathfrak{M}_u K_u}}{K_u!} A[N_1, K_1; \cdots ; N_u, K_u] \quad (1.25)$$

2. Formulas

Formula 1

$${}_4F_3\left(\begin{array}{c} \mathrm{a,\ b,\ (m+d)/2,\ (m+d+1)/2} \\ \cdots \\ \mathrm{a+b,\ m,\ d} \end{array} ; 4x(1-x)\right) = \sum_{k=0}^{\infty} \frac{(m+d-1)_k}{(a+b)_k} m_k x^k$$

where m_k is given by the following relation, see Slater [4]

$$ {}_2F_1(a,b;m;x)\,{}_2F_1(a,b;d;x) = \sum_{k=0}^{\infty} m_k x^k \tag{2.1}$$

Formula 2

$$\int_0^1 x^h \aleph(y_1 x^{h_1'},\cdots,y_r x^{h_r'})\,\aleph(z_1 x^{h_1},\cdots,z_s x^{h_s})\,{}_{p'}M_{q'}^{\alpha}(\tau x^l)\,S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[\tau_1 x_1^{l_1},\cdots,\tau_u x_u^{l_u}]\,\mathrm{d}x$$

$$= \sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_u=0}^{[N_u/\mathfrak{M}_u]}\sum_{L=0}^{\infty} A_1 G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\frac{[(a_{p'})]_L}{[(b_{q'})]_L}\frac{\tau^L}{\Gamma(\alpha L+1)}$$

$$\tau_1^{K_1}\cdots\tau_u^{K_u}\,\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}y_1^{\eta_{G_1,g_1}}\cdots y_r^{\eta_{G_r,g_r}}$$

$$\aleph_{U_{11}:W}^{0,N+1:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_s \end{array}\middle| \begin{array}{c} (\text{-h-}\sum_{i=1}^{r} h_i'\eta_{G_i,g_i} - \sum_{i=1}^{u} K_i l_i - Ll; h_1,\cdots,h_s),\ \mathrm{A}:\mathrm{C} \\ \cdots \\ (\text{-h-1-}\sum_{i=1}^{r} h_i'\eta_{G_i,g_i} - \sum_{i=1}^{u} K_i l_i - Ll; h_1,\cdots,h_s),\mathrm{B}:\mathrm{D} \end{array}\right) \tag{2.2}$$

Where : $U_{11} = P_i + 1, Q_i + 1, \iota_i; r'$

provided :

a) $h_i' > 0, i = 1,\cdots,r\,;\ h_i > 0, i = 1,\cdots,s\ ; p' \leqslant q' and\, |\tau| < 1$

b) $Re[h + \sum_{i=1}^{r} h_i' \min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}} + \sum_{i=1}^{s} h_i \min_{1\leqslant j\leqslant M_i}\frac{b_j^{(i)}}{\beta_j^{(i)}}] > 0$

d)$|arg z_k| < \frac{1}{2}B_i^{(k)}\pi$, where $B_i^{(k)}$ is given in (1.13)

Proof of (2.2)

To establish the finite integral (2.2), express the generalized class of polynomials $S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}$ in several variables occuring on the L.H.S in the series form given by (1.22), the M-function in the serie given by (1.23), the Aleph-function of r variables in serie form given by (1.6) and the Aleph-function of s variables involving there in terms of Mellin-Barnes contour integral by (1.9). We interchange the order of summation and integration (which is permissible under the conditions stated). Now evaluating the x-integral, after simplifications and on reinterpreting the Mellin-Barnes contour integral, we get the desired result.

3. Main Result

We establish a general finite integral transformation

If ${}_2F_1(a,b;m;x)\,{}_2F_1(a,b;d;x) = \sum_{k=0}^{\infty} m_k x^k$, then

$$\int_0^1 {}_4F_3\left(\begin{matrix}\text{a, b, (m+d)/2, (m+d+1)/2} \\ \cdots \\ \text{a+b, m, d}\end{matrix} ; 4x(1-x)\right) \aleph(y_1 x^{h'_1},\cdots,y_r x^{h'_r})\,\aleph(z_1 x^{h_1},\cdots,z_s x^{h_s})$$

$${}_{p'}M_{q'}^{\alpha}(\tau x^l) S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[\tau_1 x_1^{l_1},\cdots,\tau_u x_u^{l_u}]\mathrm{d}x$$

$$= \sum_{k=0}^{\infty}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_u=0}^{[N_u/\mathfrak{M}_u]}\sum_{L=0}^{\infty} A_1 G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\frac{[(a_{p'})]_L}{[(b_{q'})]_L}\frac{\tau^L}{\Gamma(\alpha L+1)}$$

$$\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}\frac{(m+d-1)_k}{(a+b)_k} m_k\ \tau_1^{K_1}\cdots\tau_u^{K_u}\, y_1^{\eta_{G_1,g_1}}\cdots y_r^{\eta_{G_r,g_r}}$$

$$\aleph_{U_{11}:W}^{0,N+1:V}\left(\begin{matrix} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_s\end{matrix}\ \middle|\ \begin{matrix}(\text{-k-}\sum_{i=1}^r h'_i\eta_{G_i,g_i}-\sum_{i=1}^u K_i l_i - Ll; h_1,\cdots,h_s),\quad \mathrm{A}:\mathrm{C} \\ \cdots \\ (\text{-k-1-}\sum_{i=1}^r h'_i\eta_{G_i,g_i}-\sum_{i=1}^u K_i l_i - Ll; h_1,\cdots,h_s)\ ,\mathrm{B}:\mathrm{D}\end{matrix}\right) \tag{3.1}$$

Where : $U_{11} = P_i+1, Q_i+1, \iota_i; r'$

provided :

a) $h'_i > 0, i=1,\cdots,r\,;\ h_i>0, i=1,\cdots,s\ ; p' \leqslant q' and\,|\tau|<1$

b) $Re[h+\sum_{i=1}^r h'_i \min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}+\sum_{i=1}^s h_i\min_{1\leqslant j\leqslant M_i}\frac{b_j^{(i)}}{\beta_j^{(i)}}]>0$

d)$|arg z_k| < \frac{1}{2}B_i^{(k)}\pi$, where $B_i^{(k)}$ is given in (1.13)

Proof of (3.1)

Multiplying both sides of (2.1) by $S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[\tau_1 x_1^{l_1},\cdots,\tau_u x_u^{l_u}]\ \aleph(y_1 x^{h'_1},\cdots,y_r x^{h'_r})_{p'}M_{q'}^{\alpha}(\tau x^l)$ $\aleph(z_1 x^{h_1},\cdots,z_s x^{h_s})$ and integrating it with respect to x from 0 to 1. Evaluating the right side thus obtained by interchanging the order of integration ans summations (which is justified due to a absolute convergence of the integral involved in the process) and then integrating the inner integral with the help of the result (2.2). We get the equation (3.1).

4. Particular cases

a) If $p_i=q_i=n=0$ and $P_i=Q_i=N=0$ then the Aleph-function of r variables degenere to product of r Aleph-functions of one variable and the Aleph-function of s variables degenere to product of s Aleph-functions of one variable, and we the following result.

If ${}_2F_1(a,b;m;x){}_2F_1(a,b;d;x)=\sum_{k=0}^{\infty} m_k x^k$, then

$$\int_0^1 {}_4F_3\left(\begin{matrix}\text{a, b, (m+d)/2, (m+d+1)/2} \\ \cdots \\ \text{a+b, m, d}\end{matrix} ; 4x(1-x)\right){}_{p'}M_{q'}^{\alpha}(\tau x^l)\ S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[\tau_1 x_1^{l_1},\cdots,\tau_u x_u^{l_u}]$$

$$\prod_{a=1}^{r} \aleph_{p_{i(a)},q_{i(a)},\tau_{i(a)};R^{(a)}}^{m_a,n_a}\left(y_a x^{h'_a}\right) \prod_{b=1}^{s} \aleph_{P_{i(b)},Q_{i(b)},\iota_{i(b)};r^{(b)}}^{M_b,N_b}\left(z_b x^{h_b}\right) \mathrm{dt}$$

$$= \sum_{k=0}^{\infty} \sum_{G_1,\cdots,G_r=0}^{\infty} \sum_{g_1=0}^{m_1} \cdots \sum_{g_r=0}^{m_r} \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_u=0}^{[N_u/\mathfrak{M}_u]} \sum_{L=0}^{\infty} A_1\, G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r}) \frac{(m+d-1)_k}{(a+b)_k} m_k$$

$$\frac{[(a_{p'})]_L}{[(b_{q'})]_L} \frac{\tau^L \qquad (-)^{G_1+\cdots+G_r}}{\Gamma(\alpha L+1)\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!} \quad \tau_1^{K_1} \cdots \tau_u^{K_u} y_1^{\eta_{G_1,g_1}} \cdots y_r^{\eta_{G_r,g_r}}$$

$$\aleph_{1,1:W}^{0,1:V}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_s \end{array} \middle| \begin{array}{l} (\text{-k-}\sum_{i=1}^{r} h'_i \eta_{G_i,g_i} - \sum_{i=1}^{u} K_i l_i - Ll; h_1,\cdots,h_s) \quad : \mathrm{C} \\ \cdots \\ (\text{-k-1-}\sum_{i=1}^{r} h'_i \eta_{G_i,g_i} - \sum_{i=1}^{u} K_i l_i - Ll; h_1,\cdots,h_s) \ : \mathrm{D} \end{array}\right) \tag{4.1}$$

Where $G'(\eta_{G_1,g_1},\cdots\eta_{G_r,g_r}) = \theta_1(\eta_{G_1,g_1})\cdots\theta_r(\eta_{G_r,g_r}), \theta_i(.), i=1,\cdots,r$ is given respectively in (1.2)

b) If $\iota_i = \iota_{i(1)} = \cdots = \iota_{i(s)} = 1$,and $\tau_i = \tau_{i(1)} = \cdots = \tau_{i(r)} = 1$ then the multivariable Aleph-function degenere to the multivariable I-function defined by Sharma et al [1]. And we have the following result.

If ${}_2F_1(a,b;m;x){}_2F_1(a,b;d;x) = \sum_{k=0}^{\infty} m_k x^k$, then

$$\int_0^1 {}_4F_3\left(\begin{array}{c} \text{a, b, (m+d)/2, (m+d+1)/2} \\ \cdots \\ \text{a+b, m, d} \end{array} ; 4x(1-x)\right) I(y_1 x^{h'_1},\cdots,y_r x^{h'_r})\, I(z_1 x^{h_1},\cdots,z_s x^{h_s})$$

$${}_{p'}M_{q'}^{\alpha}(\tau x^l) \ \ S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[\tau_1 x_1^{l_1},\cdots,\tau_u x_u^{l_u}] \ \mathrm{d}x$$

$$= \sum_{k=0}^{\infty} \sum_{G_1,\cdots,G_r=0}^{\infty} \sum_{g_1=0}^{m_1} \cdots \sum_{g_r=0}^{m_r} \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_u=0}^{[N_u/\mathfrak{M}_u]} \sum_{L=0}^{\infty} A_1 G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r}) \frac{[(a_{p'})]_L}{[(b_{q'})]_L} \frac{\tau^L}{\Gamma(\alpha L+1)}$$

$$\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!} \frac{(m+d-1)_k}{(a+b)_k} m_k \ \tau_1^{K_1} \cdots \tau_u^{K_u} y_1^{\eta_{G_1,g_1}} \cdots y_r^{\eta_{G_r,g_r}}$$

$$I_{U_{11}:W}^{0,N+1:V}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_s \end{array} \middle| \begin{array}{l} (\text{-k-}\sum_{i=1}^{r} h'_i \eta_{G_i,g_i} - \sum_{i=1}^{u} K_i l_i - Ll; h_1,\cdots,h_s), \quad \mathrm{A}_1 : C_1 \\ \cdots \\ (\text{-k-1-}\sum_{i=1}^{r} h'_i \eta_{G_i,g_i} - \sum_{i=1}^{u} K_i l_i - Ll; h_1,\cdots,h_s) \ ,\mathrm{B}_1 : D_1 \end{array}\right) \tag{4.2}$$

Where : $U_{11} = P_i + 1, Q_i + 1; r'$

$G_1(\eta_{G_1,g_1},\cdots\eta_{G_r,g_r}) = G(\eta_{G_1,g_1},\cdots\eta_{G_r,g_r})_{\tau=\tau_{i(1)}=\cdots,\tau_{i(r)}=1}$

$A_1 = A_{\iota=\iota_{i(1)}=\cdots=\iota_{i(s)}=1}$; $B_1 = B_{\iota=\iota_{i(1)}=\cdots=\iota_{i(s)}=1}$

$C_1 = C_{\iota=\iota_{i(1)}=\cdots=\iota_{i(s)}=1}$; $D_1 = D_{\iota=\iota_{i(1)}=\cdots=\iota_{i(s)}=1}$

provided :

a) $h'_i > 0, i = 1, \cdots, r\,;\ h_i > 0, i = 1, \cdots, s\ ; p' \leqslant q' and\, |\tau| < 1$

b) $Re[h + \sum_{i=1}^{r} h'_i \min_{1\leqslant j\leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}} + \sum_{i=1}^{s} h_i \min_{1\leqslant j\leqslant M_i} \frac{b_j^{(i)}}{\beta_j^{(i)}}] > 0$

c)$|argz_k| < \frac{1}{2} B'^{(k)}_i \pi$,

where $B'^{(k)}_i = \sum_{j=1}^{N} \mu_j^{(k)} - \sum_{j=N+1}^{P_i} \mu_{ji}^{(k)} - \sum_{j=1}^{Q_i} \upsilon_{ji}^{(k)} + \sum_{j=1}^{N_k} \alpha_j^{(k)} - \sum_{j=N_k+1}^{P_{i^{(k)}}} \alpha_{ji^{(k)}}^{(k)}$

$+ \sum_{j=1}^{M_k} \beta_j^{(k)} - \sum_{j=M_k+1}^{q_{i^{(k)}}} \beta_{ji^{(k)}}^{(k)} > 0,$ with $k = 1 \cdots, s, i = 1, \cdots, r\ , i^{(k)} = 1, \cdots, r^{(k)}$

c) If $\iota_i = \iota_{i^{(1)}} = \cdots = \iota_{i^{(s)}} = 1$ and $r = r^{(1)} = \cdots = r^{(s)} = 1$, then the multivariable Aleph-function degenere to the multivariable H-function defined by Srivastava et al [8]. And we have the following result.

If ${}_2F_1(a, b; m; x) {}_2F_1(a, b; d; x) = \sum_{k=0}^{\infty} m_k x^k$, then

$$\int_0^1 {}_4F_3 \left(\begin{array}{c} \text{a, b, (m+d)/2, (m+d+1)/2} \\ \cdots \\ \text{a+b, m, d} \end{array} ; 4x(1-x) \right) \aleph(y_1 x^{h'_1}, \cdots, y_r x^{h'_r})\, H(z_1 x^{h_1}, \cdots, z_s x^{h_s})$$

$${}_{p'}M_{q'}^{\alpha}(\tau x^l)\ \ S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[\tau_1 x_1^{l_1}, \cdots, \tau_u x_u^{l_u}]\ \ dx$$

$$= \sum_{k=0}^{\infty} \sum_{G_1,\cdots,G_r=0}^{\infty} \sum_{g_1=0}^{m_1} \cdots \sum_{g_r=0}^{m_r} \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_u=0}^{[N_u/\mathfrak{M}_u]} \sum_{L=0}^{\infty} A_1 G(\eta_{G_1,g_1}, \cdots, \eta_{G_r,g_r}) \frac{[(a_{p'})]_L}{[(b_{q'})]_L} \frac{\tau^L}{\Gamma(\alpha L + 1)}$$

$$\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!} \frac{(m+d-1)_k}{(a+b)_k} m_k\ \tau_1^{K_1} \cdots \tau_u^{K_u}\, y_1^{\eta_{G_1,g_1}} \cdots y_r^{\eta_{G_r,g_r}}$$

$$H_{P+1,Q+1:W}^{0,N+1:V} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_s \end{array} \middle| \begin{array}{l} (\text{-k-}\sum_{i=1}^{r} h'_i \eta_{G_i,g_i} - \sum_{i=1}^{u} K_i l_i - Ll; h_1, \cdots, h_s),\quad \text{A' : C'} \\ \cdots \\ (\text{-k-1-}\sum_{i=1}^{r} h'_i \eta_{G_i,g_i} - \sum_{i=1}^{u} K_i l_i - Ll; h_1, \cdots, h_s)\ \text{,B' : D'} \end{array} \right) \tag{4.3}$$

provided :

a) $h'_i > 0, i = 1, \cdots, r\,;\ h_i > 0, i = 1, \cdots, s\ ; p' \leqslant q' and\, |\tau| < 1$

b) $Re[h + \sum_{i=1}^{r} h'_i \min_{1\leqslant j\leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}} + \sum_{i=1}^{s} h_i \min_{1\leqslant j\leqslant M_i} \frac{b_j^{(i)}}{\beta_j^{(i)}}] > 0$

c) $|argz_k| < \frac{1}{2} B_i \pi$, $k = 1, \cdots, s$

where $B_i = \sum_{j=1}^{N}\mu_j^{(i)} - \sum_{j=N+1}^{P}\mu_j^{(i)} - \sum_{j=1}^{Q}\upsilon_j^{(i)} + \sum_{j=1}^{N_i}\alpha_j^{(i)} - \sum_{j=N_i+1}^{P_i}\alpha_j^{(i)} + \sum_{j=1}^{M_i}\beta_j^{(i)} - \sum_{j=M_i+1}^{Q_i}\beta_j^{(i)} > 0$

d) If $r = s = 2$, we obtain two Aleph-functions of two variables defined by K. Sharma [2].

If ${}_2F_1(a,b;m;x){}_2F_1(a,b;d;x) = \sum_{k=0}^{\infty} m_k x^k$, then

$$\int_0^1 {}_4F_3\left(\begin{matrix}\text{a, b, (m+d)/2, (m+d+1)/2} \\ \cdots \\ \text{a+b, m, d}\end{matrix};4x(1-x)\right)\aleph(y_1x^{h'_1},y_2x^{h'_2})\,\aleph(z_1x^{h_1},z_2x^{h_2})$$

$${}_{p'}M_{q'}^{\alpha}(\tau x^l)S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[\tau_1x_1^{l_1},\cdots,\tau_ux_u^{l_u}]\ \mathrm{d}x$$

$$= \sum_{k=0}^{\infty}\sum_{G_1,G_2=0}^{\infty}\sum_{g_1=0}^{m_1}\sum_{g_2=0}^{m_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_u=0}^{[N_u/\mathfrak{M}_u]}\sum_{L=0}^{\infty}A_1G(\eta_{G_1,g_1},\eta_{G_2,g_2})\frac{[(a_{p'})]_L}{[(b_{q'})]_L}\frac{\tau^L}{\Gamma(\alpha L+1)}$$

$$\frac{(-)^{G_1+G_2}}{\delta_{g_1}G_1!\delta_{g_2}G_2!}\frac{(m+d-1)_k}{(a+b)_k}m_k\ \tau_1^{K_1}\cdots\tau_u^{K_u}\,y_1^{\eta_{G_1,g_1}}\cdots y_r^{\eta_{G_r,g_r}}$$

$$\aleph_{U_{11}:W}^{0,N+1:V}\left(\begin{matrix}\text{z}_1\\ \cdot\\ \text{z}_2\end{matrix}\ \middle|\ \begin{matrix}(\text{-k-}\sum_{i=1}^{2}h'_i\eta_{G_i,g_i}-\sum_{i=1}^{u}K_il_i-Ll;h_1,h_2),\ \text{A : C}\\ \cdots\\ (\text{-k-1-}\sum_{i=1}^{2}h'_i\eta_{G_i,g_i}-\sum_{i=1}^{u}K_il_i-Ll;h_1,h_2),\text{B : D}\end{matrix}\right) \tag{4.4}$$

Where : $U_{11} = P_i+1, Q_i+1, \iota_i; r'$ and $G(\eta_{G_1,g_1},\eta_{G_2,g_2}) = \phi(\eta_{G_1,g_1},\eta_{G_2,g_2})\theta_1(\eta_{G_1,g_1})\,\theta_2(\eta_{G_2,g_2})$

provided :

a) $h'_i > 0, i = 1,\cdots,r\,;\ h_i > 0, i = 1,\cdots,s\ ; p' \leqslant q' and\,|\tau| < 1$

b) $Re[h + \sum_{i=1}^{2} h'_i \min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}} + \sum_{i=1}^{2} h_i \min_{1\leqslant j\leqslant M_i}\frac{b_j^{(i)}}{\beta_j^{(i)}}] > 0$

c) $|arg(y_1)| < A_1\frac{\pi}{2}$ and $|arg(y_2)| < A_2\frac{\pi}{2}$; where : $i = 1,2$;$i' = 1,2$; $i'' = 1,2$ with

$$A_1 = \iota_i\sum_{j=N+1}^{P_i}\alpha_{ji}^{(1)} - \iota_i\sum_{j=1}^{Q_i}\beta_{ji}^{(1)} + \sum_{j=1}^{M_1}\beta_j - \iota_{i'}\sum_{j=M_1+1}^{Q_{i'}^{(1)}}\beta_{ji'} + \sum_{j=1}^{N_1}\alpha_j - \iota_{i'}\sum_{j=N_1+1}^{P_{i''}^{(1)}}\alpha_{ji'} > 0$$

$$A_2 = \iota_i\sum_{j=N+1}^{P_i}\alpha_{ji}^{(1)} - \iota_i\sum_{j=1}^{Q_i}\beta_{ji}^{(2)} + \sum_{j=1}^{M_1}\delta_j - \iota_{i''}\sum_{j=M_2+1}^{Q_{i''}^{(2)}}\delta_{ji''} + \sum_{j=1}^{N_2}\gamma_j - \iota_{i''}\sum_{j=N_2+1}^{P_{i''}^{(2)}}\gamma_{ji''} > 0$$

e) If $r = s = 1$, we obtain two Aleph-functions of one variable defined by Südland [9]. We have

If ${}_2F_1(a,b;m;x){}_2F_1(a,b;d;x) = \sum_{k=0}^{\infty} m_k x^k$, then

$$\int_0^1 {}_4F_3\left(\begin{matrix}\text{a, b, (m+d)/2, (m+d+1)/2} \\ \cdots \\ \text{a+b, m, d}\end{matrix} ; 4x(1-x)\right) \aleph(yx^{h'})\, \aleph(zx^{h})$$

$${}_{p'}M_{q'}^{\alpha}(\tau x^l)\ \ S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[\tau_1 x_1^{l_1},\cdots,\tau_u x_u^{l_u}]\ \ \mathrm{d}x = \sum_{k=0}^{\infty}\sum_{G=1}^{m}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_u=0}^{[N_u/\mathfrak{M}_u]}\sum_{L=0}^{\infty} A_1 G(\eta_{G,g})$$

$$\frac{[(a_{p'})]_L}{[(b_{q'})]_L}\frac{\tau^L}{\Gamma(\alpha L+1)}\ \frac{(m+d-1)_k}{(a+b)_k} m_k\ \tau_1^{K_1}\cdots\tau_u^{K_u}\, y_1^{\eta_{G_1,g_1}}\cdots y_r^{\eta_{G_r,g_r}}$$

$$\aleph_{P_i+1,Q_i+1,c_i;r}^{M,N+1}\left(z\ \middle|\ \begin{matrix}(\text{-k-h'}\eta_{G,g}-\sum_{i=1}^{u}K_i l_i - Ll; h),\ \ (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ \cdots \qquad\qquad \cdots \\ (\text{-k-1-h'}\eta_{G,g}-\sum_{i=1}^{u}K_i l_i - Ll; h), (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right) \qquad (4.5)$$

Where $G(\eta_{G,g}) = \dfrac{(-)^G\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{B_g G!}$ $\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)$ is defined by Südland [10]

Provided :

a) $h>0, h'>0, ; p' \leqslant q' and\, |\tau|<1, Re(\rho)>0$

b) $Re[\sigma + k' \min\limits_{1\leqslant j\leqslant m}\dfrac{d_j}{\delta_j} + k \min\limits_{1\leqslant j\leqslant M}\dfrac{b_j}{\beta_j}] > -1$

c) $|argz| < \dfrac{1}{2}\pi\Omega$ Where $\Omega = \sum_{j=1}^{M}\beta_j + \sum_{j=1}^{N}\alpha_j - c_i(\sum_{j=M+1}^{Q_i}\beta_{ji} + \sum_{j=N+1}^{P_i}\alpha_{ji}) > 0$

f) If $\tau_2=\cdots=\tau_u=0$, then the class of polynomials $S_{N_1,\cdots,N_u}^{M_1,\cdots,M_u}(\tau_1,\cdots,\tau_u)$ defined of (1.14) degenere to the class of polynomials $S_N^M(\tau)$ defined by Srivastava [5]and we have.

If ${}_2F_1(a,b;m;x){}_2F_1(a,b;d;x) = \sum_{k=0}^{\infty} m_k x^k$, then

$$\int_0^1 {}_4F_3\left(\begin{matrix}\text{a, b, (m+d)/2, (m+d+1)/2} \\ \cdots \\ \text{a+b, m, d}\end{matrix} ; 4x(1-x)\right) \aleph(y_1 x^{h_1'},\cdots,y_r x^{h_r'})\, \aleph(z_1 x^{h_1},\cdots,z_s x^{h_s})$$

$${}_{p'}M_{q'}^{\alpha}(\tau x^l)\ \ S_{N_1}^{\mathfrak{M}_1}[\tau_1 x_1^{l_1}]\ \ \mathrm{d}x$$

$$= \sum_{k=0}^{\infty}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}\sum_{K_1=0}^{[N1/\mathfrak{M}_1]}\sum_{L=0}^{\infty} A_1 G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\ \frac{(m+d-1)_k}{(a+b)_k} m_k$$

$$\frac{[(a_{p'})]_L}{[(b_{q'})]_L}\frac{\tau^L}{\Gamma(\alpha L+1)}\ \frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}\tau_1^{K_1}\, y_1^{\eta_{G_1,g_1}}\cdots y_r^{\eta_{G_r,g_r}}$$

$$\aleph_{U_{11}:W}^{0,N+1:V}\left(\begin{array}{c|c} z_1 & (\text{-k-}\sum_{i=1}^{r} h_i'\eta_{G_i,g_i} - K_1l_1 - Ll; h_1,\cdots,h_s), \quad A:C \\ \cdot & \cdots \\ \cdot & \\ z_s & (\text{-k-1-}\sum_{i=1}^{r} h_i'\eta_{G_i,g_i} - K_1l_1 - Ll; h_1,\cdots,h_s)\ ,B:D \end{array}\right) \quad (4.6)$$

Where : $U_{11} = P_i + 1, Q_i + 1, \iota_i; r'$

provided :

a) $h_i' > 0, i = 1,\cdots,r;\ h_i > 0, i = 1,\cdots,s\ ; p' \leqslant q' and\,|\tau| < 1$

b) $Re[h + \sum_{i=1}^{r} h_i' \min_{1\leqslant j\leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}} + \sum_{i=1}^{s} h_i \min_{1\leqslant j\leqslant M_i} \frac{b_j^{(i)}}{\beta_j^{(i)}}] > 0$

d)$|argz_k| < \frac{1}{2}B_i^{(k)}\pi$, where $B_i^{(k)}$ is given in (1.13)

g) Letting $m = d = b$ in (3.1), we get the following integral.

$$\int_0^1 {}_2F_1\left(\begin{array}{c} \text{a, m-1/2} \\ ;4x(1-x) \\ \text{a+m} \end{array}\right)\aleph(y_1x^{h_1'},\cdots,y_rx^{h_r'})\,\aleph(z_1x^{h_1},\cdots,z_sx^{h_s})\ {}_{p'}M_{q'}^{\alpha}(\tau x^l)$$

$$S_{N_1,\cdots,N_u}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_u}[\tau_1x_1^{l_1},\cdots,\tau_ux_u^{l_u}]\,\mathrm{d}x = \sum_{k=0}^{\infty}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{m_1}\cdots\sum_{g_r=0}^{m_r}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_u=0}^{[N_u/\mathfrak{M}_u]}\sum_{L=0}^{\infty}A_1$$

$$G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\frac{[(a_{p'})]_L}{[(b_{q'})]_L}\frac{\tau^L}{\Gamma(\alpha L+1)}\frac{(2m-1)_k(2a)_k}{(a+m)_kk!}\ \tau_1^{K_1}\cdots\tau_u^{K_u}\,y_1^{\eta_{G_1,g_1}}\cdots y_r^{\eta_{G_r,g_r}}$$

$$\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}\aleph_{U_{11}:W}^{0,N+1:V}\left(\begin{array}{c|c} z_1 & (\text{-k-}\sum_{i=1}^{r} h_i'\eta_{G_i,g_i} - \sum_{i=1}^{u}K_il_i - Ll; h_1,\cdots,h_s),\ A:C \\ \cdot & \cdots \\ \cdot & \\ z_s & (\text{-k-1-}\sum_{i=1}^{r} h_i'\eta_{G_i,g_i} - \sum_{i=1}^{u}K_il_i - Ll; h_1,\cdots,h_s),B:D \end{array}\right) \quad (4.7)$$

Where : $U_{11} = P_i + 1, Q_i + 1, \iota_i; r'$

provided :

a) $h_i' > 0, i = 1,\cdots,r;\ h_i > 0, i = 1,\cdots,s\ ; p' \leqslant q' and\,|\tau| < 1$

b) $Re[h + \sum_{i=1}^{r} h_i' \min_{1\leqslant j\leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}} + \sum_{i=1}^{s} h_i \min_{1\leqslant j\leqslant M_i} \frac{b_j^{(i)}}{\beta_j^{(i)}}] > 0$

d)$|argz_k| < \frac{1}{2}B_i^{(k)}\pi$, where $B_i^{(k)}$ is given in (1.13)

5. Conclusion

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions such as , multivariable H-function , defined by Srivastava et al [8] , the Aleph-function of two variables defined by K.sharma [2].

REFERENCES

[1] Sharma C.K.and Ahmad S.S.: On the multivariable I-function. Acta ciencia Indica Math , 1994 vol 20,no2, p 113-116.

[2] Sharma K. On the integral representation and applications of the generalized function of two variables , International Journal of Mathematical Engineering and Sciences , Vol 3 , issue1 (2014) , page1-13.

[3] Sharma M. Fractional integration and fractional differentiation of the M-series, Fractional calculus appl. Anal. Vol11(2), 2008, p.188-191.

[4} Slater L.J. Generalized hypergeometric functions, Cambridge University press (1966).

[5] Srivastava H.M., A contour integral involving Fox's H-function. Indian J.Math. 14(1972), page1-6.

[6] Srivastava H.M. A multilinear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial, Pacific. J. Math. 177(1985), page183-191.

[7] Srivastava H.M. and Daoust M.C. Certain generalized Neumann expansions associated with Kampé de Fériet function. Nederl. Akad. Wetensch. Proc. Ser. A72 = Indag. Math, 31, (1969), p 449-457.

[8] H.M. Srivastava And R.Panda. Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24(1975), p.119-137.

[9] Südland N.; Baumann, B. and Nonnenmacher T.F. , Open problem : who knows about the Aleph-functions? Fract. Calc. Appl. Anal., 1(4) (1998): 401-402.

Personal adress : 411 Avenue Joseph Raynaud

Le parc Fleuri , Bat B

83140 , Six-Fours les plages

Tel : 06-83-12-49-68

Department : VAR

Country : FRANCE