Finite integrals pertaining to a product of special functions and multivariable Aleph-functions I

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ABSTRACT
The main object of this document is to obtain integral transformation using certain product of multivariable Aleph-function with a general class of polynomials and special functions. The result established in this paper are of general nature and hence encompass several particular cases.

Keywords :Multivariable Aleph-function, Aleph-function, Generalized Lauricella function, general class of polynomials, M-serie.

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1. Introduction and preliminaries.

The Aleph- function , introduced by Südland [10] et al , however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z)=\aleph_{P_i,Q_i,c_i;r}^{M,N}\left(z\ \middle|\ \begin{matrix}(a_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\(b_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right)$$

$$=\frac{1}{2\pi\omega}\int_L \Omega_{P_i,Q_i,c_i;r}^{M,N}(s)z^{-s}\mathrm{d}s \tag{1.1}$$

for all z different to 0 and

$$\Omega_{P_i,Q_i,c_i;r}^{M,N}(s)=\frac{\prod_{j=1}^{M}\Gamma(b_j+B_js)\prod_{j=1}^{N}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}c_i\prod_{j=N+1}^{P_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}-B_{ji}s)} \tag{1.2}$$

With :

$$|argz|<\frac{1}{2}\pi\Omega \quad \text{Where } \Omega=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}\alpha_j-c_i\Big(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji}\Big)>0 \quad \text{with } i=1,\cdots,r$$

For convergence conditions and other details of Aleph-function , see Südland et al [10].

Serie representation of Aleph-function is given by Chaurasia et al [2].

$$\aleph_{P_i,Q_i,c_i;r}^{M,N}(z)=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{B_Gg!}z^{-s} \tag{1.3}$$

With $s=\eta_{G,g}=\dfrac{b_G+g}{B_G}, P_i<Q_i, |z|<1$ and $\Omega_{P_i,Q_i,c_i;r}^{M,N}(s)$ is given in (1.2) (1.4)

The generalized polynomials defined by Srivastava [7], is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1,K_1;\cdots;N_s,K_s]y_1^{K_1}\cdots y_s^{K_s} \tag{1.5}$$

Where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_s,K_s]$ are arbitrary constants, real or complex.

In the present paper, we use the following notation

$$A_1 = \frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s] \tag{1.6}$$

Let $F\begin{pmatrix} x_1 \\ \cdot\cdot\cdot \\ x_r \end{pmatrix}$ denote the generalized Lauricella function of several complex variables defined by Srivastava et al [7].

We have : $$F\begin{pmatrix} x_1 \\ \cdot\cdot\cdot \\ x_r \end{pmatrix} = \sum_{k_1,\cdots,k_r=0}^{\infty} A(k_1,\cdots,k_r)\frac{x_1^{k_1}\cdots x_r^{k_r}}{k_1!\cdots k_r!} \tag{1.7}$$

where : $$A(k_1,\cdots,k_r) = \frac{\prod_{j=1}^{A}(a_j)_{k_1\theta'_j+\cdots+k_r\theta_j^{(r)}}\prod_{j=1}^{B'}(b'_j)_{k_1\phi'_j}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(r)})_{k_r\phi_j^{(r)}}}{\prod_{j=1}^{C}(c_j)_{k_1\epsilon'_j+\cdots+k_r\epsilon_j^{(r)}}\prod_{j=1}^{D'}(d'_j)_{k_1\delta'_j}\cdots\prod_{j=1}^{D^{(r)}}(d_j^{(r)})_{k_r\delta_j^{(r)}}} \tag{1.8}$$

The M-serie is defined, see Sharma [5].

$$_{p'}M_{q'}^{\alpha}(y)=\sum_{s'=0}^{\infty}\frac{[(a_{p'})]_{s'}}{[(b_{q'})]_{s'}}\frac{y^{s'}}{\Gamma(\alpha s'+1)} \tag{1.9}$$

Here $\alpha \in \mathbb{C}, Re(\alpha) > 0$. $[(a_{p'})]_{s'} = (a_1)_{s'}\cdots(a_{p'})_{s'}$; $[(b_{q'})]_{s'} = (b_1)_{s'}\cdots(b_{q'})_{s'}$.
The serie (1.9) converge if $p' \leqslant q'$ and $|y| < 1$.

The Aleph-function of several variables generalize the multivariable h-function defined by H.M. Srivastava and R. Panda [9] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $$\aleph(z_1,\cdots,z_r) = \aleph_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix}\right|$$

$$[(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}] \quad ,[\tau_i(a_{ji};\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{\mathfrak{n}+1,p_i}]:$$
$$\cdots\cdots\cdots\cdots \quad ,[\tau_i(b_{ji};\beta_j^{(1)},\cdots,\beta_j^{(r)})_{m+1,q_i}]:$$

$$\left.\begin{matrix}[(c_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ ;[(c_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(d_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;[(d_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{matrix}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.10}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.11}$$

and $$\theta_k(s_k) = \frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.12}$$

where $j=1$ to r and $k=1$ to r

Suppose , as usual , that the parameters

$a_j, j=1,\cdots,p; b_j, j=1,\cdots,q;$

$c_j^{(k)}, j=1,\cdots,n_k; c_{ji^{(k)}}^{(k)}, j=n_k+1,\cdots,p_{i^{(k)}};$

$d_j^{(k)}, j=1,\cdots,m_k; d_{ji^{(k)}}^{(k)}, j=m_k+1,\cdots,q_{i^{(k)}};$

with $k=1\cdots,r, i=1,\cdots,R\ , i^{(k)}=1,\cdots,R^{(k)}$

are complex numbers , and the $\alpha's,\beta's,\gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} + \tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} + \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} - \sum_{j=1}^{m_k}\delta_j^{(k)}$$

$$-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.13}$$

The reals numbers τ_i are positives for $i=1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)}=1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)$ with $j=1$ to m_k are separated from those of $\Gamma(1-a_j+\sum_{i=1}^{r}\alpha_j^{(k)}s_k)$ with $j=1$ to n and $\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)$ with $j=1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} - \tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} - \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0, \text{ with } k=1\cdots,r, i=1,\cdots,R\ , i^{(k)}=1,\cdots,R^{(k)} \tag{1.14}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1}\ldots|z_r|^{\alpha_r}\,)\,,max(\,|z_1|\ldots|z_r|\,)\rightarrow 0$$

$$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\beta_1}\ldots|z_r|^{\beta_r}\,)\,,min(\,|z_1|\ldots|z_r|\,)\rightarrow \infty$$

where, with $k=1,\cdots,r:\alpha_k=min[Re(d_j^{(k)}/\delta_j^{(k)})],j=1,\cdots,m_k$ and

$$\beta_k=max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})],j=1,\cdots,n_k$$

We will use these following notations in this paper

$$U=p_i,q_i,\tau_i;R\;;\;V=m_1,n_1;\cdots;m_r,n_r \tag{1.15}$$

$$W=p_{i^{(1)}},q_{i^{(1)}},\tau_{i(1)};R^{(1)},\cdots,p_{i^{(r)}},q_{i^{(r)}},\tau_{i(r)};R^{(r)} \tag{1.16}$$

$$A=\{(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,n}\},\{\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.17}$$

$$B=\{\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.18}$$

$$C=\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\},\cdots,\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\},\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.19}$$

$$D=\{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\},\cdots,\{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\},\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.20}$$

The multivariable Aleph-function write :

$$\aleph(z_1,\cdots,z_r)=\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\left|\begin{array}{c}A:C\\ \cdot\;\cdot\;\cdot\\ B:D\end{array}\right.\right) \tag{1.21}$$

2. Formulas

The following result of Srivastava-Daoust [8 , eq.(1.2), p.15], see eq.(1.7), and Chaurasia [1, p.194, eq. (2.3)] respectively also required in our investigations :

$$F_{\sigma:N';\cdots;N^{(s)};1;1}^{\upsilon:M';\cdots;M^{(s)};0;0}\left(\begin{array}{c}z_1\\ \cdot\;\cdot\;\cdot\\ z_s\\ \text{-xt}\\ \text{(1-x)t}\end{array}\left|\begin{array}{c}[(\alpha_\upsilon);\eta',\cdots,\eta^{(s)},\gamma,\gamma];[m';\rho'];\cdots;[m^{(s)};\rho^{(s)}]\\ [(\beta_\sigma);\zeta',\cdots,\zeta^{(s)},\mu,\mu];[l';\tau'];\cdots;[l^{(s)};\tau^{(s)}];[\alpha+1;1];[\beta+1;1]\end{array}\right.\right)$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{\upsilon}(\alpha_j)_{n\gamma_j}}{(\alpha+1)_n(\beta+1)_n\prod_{j=1}^{\sigma}(\beta_j)_{n\mu_j}}P_n^{(\alpha,\beta)}(1-2x)$$

$$F_{\sigma:N';\cdots;N^{(s)}}^{\upsilon:M';\cdots;M^{(s)}}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ z_s\end{array}\left|\begin{array}{c}[\alpha_\upsilon+n\gamma_\upsilon;\eta',\cdots,\eta^{(s)},\gamma,\gamma];[m';\rho'];\cdots;[m^{(s)};\rho^{(s)}]\\ [\beta_\sigma+n\mu_\sigma;\zeta',\cdots,\zeta^{(s)},\mu,\mu];[l';\tau'];\cdots;[l^{(s)};\tau^{(s)}]\end{array}\right.\right)t^n \tag{2.1}$$

$$\int_0^1 y^k \aleph(y^{h_1}z_1,\cdots,y^{h_r}z_r)\aleph_{P_i,Q_i,c_i;r}^{M,N}\left(xy^L\left|\begin{matrix}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right){}_{p'}M_{q'}^{\alpha}(\tau y^{L'})$$

$$S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}[\tau_1 y_1^{L_1},\cdots,\tau_t y_t^{L_t}]\,\mathrm{d}y$$

$$=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{l=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_t=0}^{[N_t/M_t]}A_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r}^{M,N}(\eta_{G,g})[(a_{p'})]_l}{B_G g!}\frac{\tau^l}{[(b_{q'})]_l\,\Gamma(\alpha l+1)}x^{\eta_{G,g}}\tau_1^{K_1}\cdots\tau_t^{K_t}$$

$$\aleph_{U_{11}:W}^{0,\mathfrak{n}+1:V}\left(\begin{matrix}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{matrix}\left|\begin{matrix}(\text{-k -L}\tau_{G,g}-L'l-K_1L_1-\cdots-K_tL_t;h_1,\cdots,h_r),A:C\\ \cdots\\ \cdots\\ (\text{-1 - g -L}\tau_{G,g}-L'l-K_1L_1-\cdots-K_tL_t;h_1,\cdots,h_r),B:D\end{matrix}\right.\right) \quad (2.2)$$

Where : $U_{11}=p_i+1,q_i+1,\tau_i;R$

provided :

a) $Re(\alpha)>0,h_i>0,i=1,\cdots,r\,;p'\leqslant q'\,and\,|\tau|<1$

b) $Re[1+L\min\limits_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^{r}h_i\min\limits_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

c) $|argz|<\frac{1}{2}\pi\Omega$ Where $\Omega=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}\alpha_j-c_i(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji})>0$

d)$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.15)

Proof of (2.2)

To establish the finite integral (2.2), express the generalized class of polynomials $S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}[\tau_1 y_1^{L_1},\cdots,\tau_t y_t^{L_t}]$ occuring on the L.H.S in the series form given by (1.5), the M-function in the serie given by (1.9), the Aleph-function in serie form given by (1.3) and the multivariable Aleph-function involving there in terms of Mellin-Barnes contour integral by (1.11). We interchange the order of summation and integration (which is permissible under the conditions stated). Now evaluating the y-integral, after simplifications and on reinterpreting the Mellin-Barnes contour integral, we get the desired result.

$$\int_0^1 x^{\sigma-1}(1-x)^{\beta}\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{matrix}z_1x^{h_1}\\ \cdot\\ \cdot\\ \cdot\\ z_rx^{h_r}\end{matrix}\left|\begin{matrix}A:C\\ \cdots\\ B:D\end{matrix}\right.\right)P_n^{(\alpha,\beta)}(1-2x)\mathrm{d}x=\frac{(-1)^n\Gamma(\beta+n+1)}{n!}$$

$$\aleph_{U_{22}:W}^{0,\mathfrak{n}+2:V}\left(\begin{matrix}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{matrix}\left|\begin{matrix}(-\sigma;h_1,\cdots,h_r), & (-\sigma+\alpha;h_1,\cdots,h_r),A:C\\ \cdots & \cdots\\ \cdots & \cdots\\ (-\sigma+\alpha+n;h_1,\cdots,h_r), & (-\beta-\sigma-n;;h_1,\cdots,h_r),B:D\end{matrix}\right.\right)$$

(2.3)

Where $U_{22}=p_i+2,q_i+2,\tau_i;R$

provided

a) $Re(\sigma) > 0, Re(\beta) > 0$

b) $Re[\sigma + \sum_{i=1}^{r} h_i \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}] > 0$

c) $|argz_k| < \frac{1}{2} A_i^{(k)} \pi$, where $A_i^{(k)}$ is given in (1.15)

Proof of (2.3)

Use the formula : $P_n^{(\alpha,\beta)}(1-2x) = \binom{\alpha+n}{n} {}_2F_1(-n, \alpha+\beta+n+1; \alpha+1; x)$, we express the Gauss hypergeometric function in serie and the multivariable Aleph-function involving there in terms of Mellin-Barnes contour integral by (1.11). We interchange the order of summation and integration (which is permissible under the conditions stated). Now evaluating the x-integral, see Panda [3], after simplifications and on reinterpreting the Mellin-Barnes contour integral, we get the desired result.

$$\int_0^1 x^{\sigma-1}(1-x)^{\beta} \aleph_{P_i,Q_i,c_i;r}^{M,N}\left(\tau x^L \middle| \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{matrix}\right) {}_{p'}M_{q'}^{\alpha}(\tau' x^{L'})$$

$$S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}[\tau_1 x^{L_1}, \cdots, \tau_t x^{L_t}] \; P_n^{(\alpha,\beta)}(1-2x) \aleph_{U:W}^{0,\mathfrak{n}:V}\begin{pmatrix} \mathrm{z}_1 x^{h_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r x^{h_r} \end{pmatrix} \mathrm{d}x$$

$$= \sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{l=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_t=0}^{[N_t/M_t]} A_1 \frac{(-)^g \Omega_{P_i,Q_i,c_i,r}^{M,N}(\eta_{G,g}) \tau^{\eta_{G,g}}}{B_G g!} \frac{[(a_{p'})]_l}{[(b_{q'})]_l} \frac{\tau'^l}{\Gamma(\alpha l+1)} \tau_1^{K_1} \cdots \tau_t^{K_t}$$

$$\frac{(-t)^n \Gamma(\beta+n+1)}{(\alpha+1)_n n!} \aleph_{U_{22}:W}^{0,\mathfrak{n}+2:V}\left(\begin{matrix} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{matrix} \middle| \begin{matrix} (\text{-}\sigma - L\tau_{G,g} - L'l - K_1L_1 - \cdots - K_tL_t; h_1, \cdots, h_r), \\ \cdots \\ \cdots \\ (\text{-}\sigma + \alpha + n - L\tau_{G,g} - L'l - K_1L_1 - \cdots - K_tL_t; h_1, \cdots, h_r), \end{matrix}\right.$$

$$\left.\begin{matrix} (\text{-}\sigma + \alpha; h_1, \cdots, h_r), A : C \\ \cdots \\ \cdots \\ (\text{-}\beta - \sigma - n;; h_1, \cdots, h_r), B : D \end{matrix}\right) \tag{2.4}$$

Where $U_{22} = p_i + 2, q_i + 2, \tau_i; R$

Provided

a) $Re(\beta) > -1, h_i > 0, i = 1, \cdots, r \,; p' \leqslant q' and \, |\tau| < 1$

b) $Re[\sigma + L \min_{1 \leqslant j \leqslant M} \frac{b_j}{B_j} + \sum_{i=1}^{r} h_i \min_{1 \leqslant j \leqslant m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}] > -1$

c) $|argz| < \frac{1}{2}\pi\Omega$ Where $\Omega = \sum_{j=1}^{M}\beta_j + \sum_{j=1}^{N}\alpha_j - c_i(\sum_{j=M+1}^{Q_i}\beta_{ji} + \sum_{j=N+1}^{P_i}\alpha_{ji}) > 0$

d)$|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.15)

Proof of (2.4)

To establish the finite integral (2.6), express the generalized class of polynomials $S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}[\tau_1 x_1^{L_1},\cdots,\tau_t x_t^{L_t}]$ occuring on the L.H.S in the series form given by (1.5), the M-function in serie given by (1.9), the Aleph-function in serie form given by (1.3). Now, use the formula (2.3), after simplifications and on reinterpreting the Mellin-Barnes contour integral, we get the desired result.

3. Main Result

We establish a general finite integral transformation

$$\int_0^1 x^{\sigma-1}(1-x)^{\beta}\aleph_{P_i,Q_i,c_i;r}^{M,N}\left(\tau x^L \left| \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{matrix}\right.\right)_{p'}M_{q'}^{\alpha}(\tau' x^{L'})$$

$$F_{\sigma:N';\cdots;N^{(s)};1;1}^{\upsilon:M';\cdots;M^{(s)};0;0}\left(\begin{matrix} \mathrm{z}_1 \\ \cdots \\ \mathrm{z}_s \\ \text{-xt} \\ \text{(1-x)t} \end{matrix}\left| \begin{matrix} [(\alpha_\upsilon);\eta',\cdots,\eta^{(s)},\gamma,\gamma];[m';\rho'];\cdots;[m^{(s)};\rho^{(s)}] \\ [(\beta_\sigma);\zeta',\cdots,\zeta^{(s)},\mu,\mu];[l';\tau'];\cdots;[l^{(s)};\tau^{(s)}];[\alpha+1;1];[\beta+1;1] \end{matrix}\right.\right)$$

$$S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}[\tau_1 x_1^{L_1},\cdots,\tau_t x_t^{L_t}]\aleph_{U:W}^{0,\mathfrak{n}:V}\begin{pmatrix} \mathrm{z}_1 x^{h_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r x^{h_r} \end{pmatrix}\mathrm{d}x$$

$$= \sum_{n=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{l=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_t=0}^{[N_t/M_t]}\frac{\prod_{j=1}^{\upsilon}(\alpha_j)_{n\gamma_j}}{(\alpha+1)_n(\beta+1)_n\prod_{j=1}^{\sigma}(\beta_j)_{n\mu_j}}A_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r}^{M,N}(\eta_{G,g})\tau^{\eta_{G,g}}}{B_G g!}$$

$$\frac{[(a_{p'})]_l}{[(b_{q'})]_l}\frac{\tau'^l}{\Gamma(\alpha l+1)}\tau_1^{K_1}\cdots\tau_t^{K_t}\frac{(-t)^{\eta}\Gamma(\beta+n+1)}{n!}$$

$$F_{\sigma:N';\cdots;N^{(s)}}^{\upsilon:M';\cdots;M^{(s)}}\left(\begin{matrix} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_s \end{matrix}\left| \begin{matrix} [\alpha_\upsilon+n\gamma_\upsilon;\eta',\cdots,\eta^{(s)},\gamma,\gamma];[m';\rho'];\cdots;[m^{(s)};\rho^{(s)}] \\ [\beta_\sigma+n\mu_\sigma;\zeta',\cdots,\zeta^{(s)},\mu,\mu];[l';\tau'];\cdots;[l^{(s)};\tau^{(s)}] \end{matrix}\right.\right)$$

$$\aleph_{U_{22}:W}^{0,\mathfrak{n}+2:V}\left(\begin{matrix} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{matrix}\left| \begin{matrix} (\text{-}\sigma - L\tau_{G,g} - L'l - K_1L_1 - \cdots - K_tL_t;h_1,\cdots,h_r), \\ \cdots \\ \cdots \\ (\text{-}\sigma+\alpha+n - L\tau_{G,g} - L'l - K_1L_1 - \cdots - K_tL_t;h_1,\cdots,h_r), \end{matrix}\right.\right.$$

$$\begin{pmatrix} (\text{-}\sigma+\alpha;h_1,\cdots,h_r),A:C \\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot \\ (\text{-}\beta-\sigma-n;;h_1,\cdots,h_r),B:D \end{pmatrix} \quad (3.1)$$

Where $U_{22}=p_i+2,q_i+2,\tau_i;R$

Provided

a) $Re(\beta)>-1,h_i>0,i=1,\cdots,r\,;p'\leqslant q' and\,|\tau|<1,|t|<1$

b) $Re[\sigma+L\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^{r}h_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

c) $|argz|<\frac{1}{2}\pi\Omega$ Where $\Omega=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}\alpha_j-c_i(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji})>0$

d)$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.15)

Proof :

Multiplying both sides of (2.1) by $x^{\sigma-1}(1-x)^{\beta}\aleph_{P_i,Q_i,c_i;r}^{M,N}\left(\tau x^L\left|\begin{matrix}(\text{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\(\text{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right)$

${}_pM_q^{\alpha}(\tau'x^{L'})S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}[\tau_1x_1^{L_1},\cdots,\tau_tx_t^{L_t}]\,\aleph(x^{h_1}z_1,\cdots,x^{h_r}z_r)$ and integrating it with respect to x from 0 to 1. Evaluating the right side thus obtained by interchanging the order of integration ans summations (which is justified due to a absolute convergence of the integral involved in the process) and then integrating the inner integral with the help of the result (2.4). We get the equation (3.1).

4. Particular cases

a) If $c_i=1,, i=1,\cdots,r,$ and $r=1,$ the Aleph-function of one variable degenere to the H-function of one variable and we have

$$\int_0^1 x^{\sigma-1}(1-x)^{\beta}\,H_{P,Q}^{M,N}\left(\tau x^L\left|\begin{matrix}(\text{a}_j,A_j)\\ \\(\text{b}_j,B_j)\end{matrix}\right.\right){}_{p'}M_{q'}^{\alpha}(\tau'x^{L'})$$

$$F_{\sigma:N';\cdots;N^{(s)};1;1}^{\upsilon:M';\cdots;M^{(s)};0;0}\left(\begin{matrix}\text{z}_1\\ \cdot\ \cdot\ \cdot\\ \text{z}_s\\ \text{-xt}\\ \text{(1-x)t}\end{matrix}\left|\begin{matrix}[(\alpha_\upsilon);\eta',\cdots,\eta^{(s)},\gamma,\gamma];[m';\rho'];\cdots;[m^{(s)};\rho^{(s)}]\\ [(\beta_\sigma);\zeta',\cdots,\zeta^{(s)},\mu,\mu];[l';\tau'];\cdots;[l^{(s)};\tau^{(s)}];[\alpha+1;1];[\beta+1;1]\end{matrix}\right.\right)$$

$$S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}[\tau_1x_1^{L_1},\cdots,\tau_tx_t^{L_t}]\aleph_{U:W}^{0,\mathfrak{n}:V}\begin{pmatrix}\text{z}_1x^{h_1}\\ \cdot\\ \cdot\\ \cdot\\ \text{z}_rx^{h_r}\end{pmatrix}\text{d}x$$

$$=\sum_{n=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{l=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_t=0}^{[N_t/M_t]}\frac{\prod_{j=1}^{\upsilon}(\alpha_j)_{n\gamma_j}}{(\alpha+1)_n(\beta+1)_n\prod_{j=1}^{\sigma}(\beta_j)_{n\mu_j}}A_1\frac{(-)^g\phi_{PQ}^{M,N}(\eta_{G,g})\tau^{\eta_{G,g}}}{B_Gg!}$$

$$\frac{[(a_{p'})]_l}{[(b_{q'})]_l}\frac{\tau'^l}{\Gamma(\alpha l+1)}\tau_1^{K_1}\cdots\tau_t^{K_t}\frac{(-t)^\eta\Gamma(\beta+n+1)}{n!}$$

$$F^{\upsilon:M';\cdots;M^{(s)}}_{\sigma:N';\cdots;N^{(s)}}\left(\begin{array}{c} \mathrm{z}_1\\ \cdot\\ \cdot\\ \mathrm{z}_s\end{array}\middle| \begin{array}{c}[\alpha_\upsilon+n\gamma_\upsilon;\eta',\cdots,\eta^{(s)},\gamma,\gamma];[m';\rho'];\cdots;[m^{(s)};\rho^{(s)}]\\ [\beta_\sigma+n\mu_\sigma;\zeta',\cdots,\zeta^{(s)},\mu,\mu];[l';\tau'];\cdots;[l^{(s)};\tau^{(s)}]\end{array}\right)$$

$$\aleph^{0,\mathfrak{n}+2:V}_{U_{22}:W}\left(\begin{array}{c} \mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\middle| \begin{array}{c}(\text{-}\sigma-L\tau_{G,g}-L'l-K_1L_1-\cdots-K_tL_t;h_1,\cdots,h_r),\\ \cdots\\ \cdots\\ (\text{-}\sigma+\alpha+n-L\tau_{G,g}-L'l-K_1L_1-\cdots-K_tL_t;h_1,\cdots,h_r),\end{array}\right.$$

$$\left.\begin{array}{c}(\text{-}\sigma+\alpha;h_1,\cdots,h_r),A:C\\ \cdots\\ \cdots\\ (\text{-}\beta-\sigma-n;;h_1,\cdots,h_r),B:D\end{array}\right) \tag{4.1}$$

Where $U_{22}=p_i+2,q_i+2,\tau_i;R$

Provided

a) $Re(\beta)>-1,h_i>0,i=1,\cdots,r\,;p'\leqslant q'and\,|\tau|<1,|t|<1$

b) $Re[\sigma+L\min\limits_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum\limits_{i=1}^r h_i\min\limits_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

c) $|argz|<\frac{1}{2}\pi\Omega$ Where $\Omega=\sum\limits_{j=1}^N a_i+\sum\limits_{i=N+1}^P a_i-(\sum\limits_{i=1}^M b_i+\sum\limits_{j=m+1}^Q b_i)>0$

d)$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.15)

b) If $\iota_i=\iota_{i^{(1)}}=\cdots=\iota_{i^{(r)}}=1$ and $R=R^{(1)}=\cdots=R^{(r)}=1$, then the multivariable Aleph-function degenere to the multivariable H-function defined by Srivastava et al [9]. And we have the following results.

$$\int_0^1 x^{\sigma-1}(1-x)^\beta\,\aleph^{M,N}_{P_i,Q_i,c_i;r}\left(\tau x^L\middle|\begin{array}{c}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\ (\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right){}_{p'}M^\alpha_{q'}(\tau'x^{L'})$$

$$F^{\upsilon:M';\cdots;M^{(s)};0;0}_{\sigma:N';\cdots;N^{(s)};1;1}\left(\begin{array}{c} \mathrm{z}_1\\ \cdots\\ \mathrm{z}_s\\ \text{-xt}\\ \text{(1-x)t}\end{array}\middle| \begin{array}{c}[(\alpha_\upsilon);\eta',\cdots,\eta^{(s)},\gamma,\gamma];[m';\rho'];\cdots;[m^{(s)};\rho^{(s)}]\\ [(\beta_\sigma);\zeta',\cdots,\zeta^{(s)},\mu,\mu];[l';\tau'];\cdots;[l^{(s)};\tau^{(s)}];[\alpha+1;1];[\beta+1;1]\end{array}\right)$$

$$S^{M_1,\cdots,M_t}_{N_1,\cdots,N_t}[\tau_1x_1^{L_1},\cdots,\tau_tx_t^{L_t}]H^{0,\mathfrak{n}:V}_{p,q:W}\left(\begin{array}{c}\mathrm{z}_1x^{h_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_rx^{h_r}\end{array}\right)\mathrm{d}x$$

$$= \sum_{n=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{l=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_t=0}^{[N_t/M_t]}\frac{\prod_{j=1}^{\upsilon}(\alpha_j)_{n\gamma_j}}{(\alpha+1)_n(\beta+1)_n\prod_{j=1}^{\sigma}(\beta_j)_{n\mu_j}}A_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r}^{M,N}(\eta_{G,g})\tau^{\eta_{G,g}}}{B_G g!}$$

$$\frac{[(a_{p'})]_l}{[(b_{q'})]_l}\frac{\tau'^l}{\Gamma(\alpha l+1)}\tau_1^{K_1}\cdots\tau_t^{K_t}\frac{(-t)^\eta\Gamma(\beta+n+1)}{n!}$$

$$F_{\sigma:N';\cdots;N^{(s)}}^{\upsilon:M';\cdots;M^{(s)}}\left(\begin{array}{c}\mathrm{z}_1\\ \cdot\\ \cdot\\ \mathrm{z}_s\end{array}\left|\begin{array}{c}[\alpha_\upsilon+n\gamma_\upsilon;\eta',\cdots,\eta^{(s)},\gamma,\gamma];[m';\rho'];\cdots;[m^{(s)};\rho^{(s)}]\\ [\beta_\sigma+n\mu_\sigma;\zeta',\cdots,\zeta^{(s)},\mu,\mu];[l';\tau'];\cdots;[l^{(s)};\tau^{(s)}]\end{array}\right.\right)$$

$$H_{p+2,q+2:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\left|\begin{array}{c}(\text{-}\sigma-L\tau_{G,g}-L'l-K_1L_1-\cdots-K_tL_t;h_1,\cdots,h_r),\\ \cdots\\ \cdots\\ (\text{-}\sigma+\alpha+n-L\tau_{G,g}-L'l-K_1L_1-\cdots-K_tL_t;h_1,\cdots,h_r),\end{array}\right.\right.$$

$$\left.\begin{array}{c}(\text{ -}\sigma+\alpha;h_1,\cdots,h_r),A':C'\\ \cdots\\ \cdots\\ (\text{-}\beta-\sigma-n;;h_1,\cdots,h_r),B':D'\end{array}\right) \tag{4.2}$$

Where $U_{22}=p_i+2,q_i+2,\tau_i;R$

Provided

a) $Re(\beta)>-1,h_i>0,i=1,\cdots,r\,;p'\leqslant q' and\,|\tau|<1,|t|<1$

b) $Re[\sigma+L\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^{r}h_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

c) $|argz|<\frac{1}{2}\pi\Omega$ Where $\Omega=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}\alpha_j-c_i(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji})>0$

d) $|argz_i|<\frac{1}{2}A_i\pi,k=1\cdots r$,

where $A_i=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(i)}-\sum_{j=\mathfrak{n}+1}^{p}\alpha_j^{(i)}-\sum_{j=1}^{q}\beta_j^{(i)}+\sum_{j=1}^{n_i}\gamma_j^{(i)}-\sum_{j=n_i+1}^{p_i}\gamma_j^{(i)}+\sum_{j=1}^{m_i}\delta_j^{(i)}-\sum_{j=m_i+1}^{q_i}\delta_j^{(i)}>0$

c) if $U=n=0$, the Aleph-function of r variables degenere to product of r Aleph-functions of one variable.

$$\int_0^1 x^{\sigma-1}(1-x)^\beta\,\aleph_{P_i,Q_i,c_i;r}^{M,N}\left(\tau x^L\left|\begin{array}{c}(\text{ a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\ (\text{ b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right.\right)_{p'}M_{q'}^{\alpha}(\tau'x^{L'})$$

$$F^{\upsilon:M';\cdots;M^{(s)};0;0}_{\sigma:N';\cdots;N^{(s)};1;1}\left(\begin{array}{c}\mathrm{z}_1\\ \cdots\\ \mathrm{z}_s\\ \text{-xt}\\ \text{(1-x)t}\end{array}\middle|\begin{array}{c}[(\alpha_\upsilon);\eta',\cdots,\eta^{(s)},\gamma,\gamma];[m';\rho'];\cdots;[m^{(s)};\rho^{(s)}]\\ [(\beta_\sigma);\zeta',\cdots,\zeta^{(s)},\mu,\mu];[l';\tau'];\cdots;[l^{(s)};\tau^{(s)}];[\alpha+1;1];[\beta+1;1]\end{array}\right)$$

$$S^{M_1,\cdots,M_t}_{N_1,\cdots,N_t}[\tau_1 x_1^{L_1},\cdots,\tau_t x_t^{L_t}]\prod_{u=1}^{r}\aleph^{m_u,n_u}_{p_{i^{(u)}},q_{i^{(u)}},\tau_{i^{(u)}};r^{(u)}}(z_u x^{h_u})\,\mathrm{d}x$$

$$=\sum_{n=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{l=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_t=0}^{[N_t/M_t]}\frac{\prod_{j=1}^{\upsilon}(\alpha_j)_{n\gamma_j}}{(\alpha+1)_n(\beta+1)_n\prod_{j=1}^{\sigma}(\beta_j)_{n\mu_j}}A_1\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r}(\eta_{G,g})\tau^{\eta_{G,g}}}{B_G g!}$$

$$\frac{[(a_{p'})]_l}{[(b_{q'})]_l}\frac{\tau'^l}{\Gamma(\alpha l+1)}\tau_1^{K_1}\cdots\tau_t^{K_t}\frac{(-t)^{\eta}\Gamma(\beta+n+1)}{n!}$$

$$F^{\upsilon:M';\cdots;M^{(s)}}_{\sigma:N';\cdots;N^{(s)}}\left(\begin{array}{c}\mathrm{z}_1\\ \cdot\\ \cdot\\ \mathrm{z}_s\end{array}\middle|\begin{array}{c}[\alpha_\upsilon+n\gamma_\upsilon;\eta',\cdots,\eta^{(s)},\gamma,\gamma];[m';\rho'];\cdots;[m^{(s)};\rho^{(s)}]\\ [\beta_\sigma+n\mu_\sigma;\zeta',\cdots,\zeta^{(s)},\mu,\mu];[l';\tau'];\cdots;[l^{(s)};\tau^{(s)}]\end{array}\right)$$

$$\aleph^{0,2:V}_{2,2:W}\left(\begin{array}{c}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\middle|\begin{array}{c}(\text{-}\sigma-L\tau_{G,g}-L'l-K_1L_1-\cdots-K_tL_t;h_1,\cdots,h_r),\\ \cdots\\ \cdots\\ (\text{-}\sigma+\alpha+n-L\tau_{G,g}-L'l-K_1L_1-\cdots-K_tL_t;h_1,\cdots,h_r),\end{array}\right.$$

$$\left.\begin{array}{c}(\text{-}\sigma+\alpha;h_1,\cdots,h_r):C\\ \cdots\\ \cdots\\ (\text{-}\beta-\sigma-n;;h_1,\cdots,h_r):D\end{array}\right)\tag{4.3}$$

Provided

a) $Re(\beta)>-1,h_i>0,i=1,\cdots,r\,;p'\leqslant q' and\,|\tau|<1,|t|<1$

b) $Re[\sigma+L\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^{r}h_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

c) $|argz|<\frac{1}{2}\pi\Omega$ Where $\Omega=\sum_{j=1}^{N}a_i+\sum_{i=N+1}^{P}a_i-(\sum_{i=1}^{M}b_i+\sum_{j=m+1}^{Q}b_i)>0$

d)$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)}=\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0,$$

with $k=1\cdots,r$, $i^{(k)}=1,\cdots,R^{(k)}$

d) If $r = 2$, the Aleph-function of several variables degenere to Aleph-function of two variables defined by K.Sharma [4].

$$\int_0^1 x^{\sigma-1}(1-x)^{\beta} \aleph_{P_i,Q_i,c_i;r}^{M,N}\left(\tau x^L \middle| \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{matrix}\right) {}_{p'}M_{q'}^{\alpha}(\tau' x^{L'})$$

$$F_{\sigma:N';\cdots;N^{(s)};1;1}^{\upsilon:M';\cdots;M^{(s)};0;0}\left(\begin{matrix} \mathrm{z}_1 \\ \cdots \\ \mathrm{z}_s \\ \text{-xt} \\ \text{(1-x)t} \end{matrix} \middle| \begin{matrix} [(\alpha_\upsilon);\eta',\cdots,\eta^{(s)},\gamma,\gamma];[m';\rho'];\cdots;[m^{(s)};\rho^{(s)}] \\ [(\beta_\sigma);\zeta',\cdots,\zeta^{(s)},\mu,\mu];[l';\tau'];\cdots;[l^{(s)};\tau^{(s)}];[\alpha+1;1];[\beta+1;1] \end{matrix}\right)$$

$$S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}[\tau_1 x_1^{L_1},\cdots,\tau_t x_t^{L_t}] \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{matrix} \mathrm{z}_1 x^{h_1} \\ \cdot \\ \mathrm{z}_2 x^{h_2} \end{matrix}\right) \mathrm{d}x$$

$$= \sum_{n=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{l=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_t=0}^{[N_t/M_t]} \frac{\prod_{j=1}^{\upsilon}(\alpha_j)_{n\gamma_j}}{(\alpha+1)_n(\beta+1)_n\prod_{j=1}^{\sigma}(\beta_j)_{n\mu_j}} A_1 \frac{(-)^g \Omega_{P_i,Q_i,c_i,r}^{M,N}(\eta_{G,g})\tau^{\eta_{G,g}}}{B_G g!}$$

$$\frac{[(a_{p'})]_l}{[(b_{q'})]_l}\frac{\tau'^l}{\Gamma(\alpha l+1)}\tau_1^{K_1}\cdots\tau_t^{K_t}\frac{(-t)^\eta\Gamma(\beta+n+1)}{n!}$$

$$F_{\sigma:N';\cdots;N^{(s)}}^{\upsilon:M';\cdots;M^{(s)}}\left(\begin{matrix} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \mathrm{z}_s \end{matrix} \middle| \begin{matrix} [\alpha_\upsilon+n\gamma_\upsilon;\eta',\cdots,\eta^{(s)},\gamma,\gamma];[m';\rho'];\cdots;[m^{(s)};\rho^{(s)}] \\ [\beta_\sigma+n\mu_\sigma;\zeta',\cdots,\zeta^{(s)},\mu,\mu];[l';\tau'];\cdots;[l^{(s)};\tau^{(s)}] \end{matrix}\right)$$

$$\aleph_{U_{22}:W}^{0,\mathfrak{n}+2:V}\left(\begin{matrix} \mathrm{z}_1 \\ \cdot \\ \mathrm{z}_2 \end{matrix} \middle| \begin{matrix} (\text{-}\sigma-L\tau_{G,g}-L'l-K_1L_1-\cdots-K_tL_t;h_1,h_2), \\ \cdots \\ (\text{-}\sigma+\alpha+n-L\tau_{G,g}-L'l-K_1L_1-\cdots-K_tL_t;h_1,h_2), \end{matrix}\right.$$

$$\left.\begin{matrix} (\text{-}\sigma+\alpha;h_1,h_2),A:C \\ \cdots \\ \cdots \\ (\text{-}\beta-\sigma-n;;h_1,h_2),B:D \end{matrix}\right) \tag{4.4}$$

Where $U_{22} = p_i+2, q_i+2, \tau_i; R$

e) If $\tau_2 = \cdots = \tau_s = 0$, then the class of polynomials $S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}(\tau_1,\cdots,\tau_s)$ defined of (1.14) degenere to the class of polynomial $S_N^M(\tau_1)$ defined by Srivastava [6].

$$\int_0^1 x^{\sigma-1}(1-x)^{\beta} \aleph_{P_i,Q_i,c_i;r}^{M,N}\left(\tau x^L \middle| \begin{matrix} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{matrix}\right) {}_{p'}M_{q'}^{\alpha}(\tau' x^{L'})$$

$$F_{\sigma:N';\cdots;N^{(s)};1;1}^{\upsilon:M';\cdots;M^{(s)};0;0}\left(\begin{matrix} \mathrm{z}_1 \\ \cdots \\ \mathrm{z}_s \\ \text{-xt} \\ \text{(1-x)t} \end{matrix} \middle| \begin{matrix} [(\alpha_\upsilon);\eta',\cdots,\eta^{(s)},\gamma,\gamma];[m';\rho'];\cdots;[m^{(s)};\rho^{(s)}] \\ [(\beta_\sigma);\zeta',\cdots,\zeta^{(s)},\mu,\mu];[l';\tau'];\cdots;[l^{(s)};\tau^{(s)}];[\alpha+1;1];[\beta+1;1] \end{matrix}\right)$$

$$S_{N_1}^{M_1}[\tau_1 x_1^{L_1}]\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} z_1 x^{h_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r x^{h_r} \end{array}\right)\mathrm{d}x$$

$$= \sum_{n=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{l=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]} \frac{\prod_{j=1}^{\upsilon}(\alpha_j)_{n\gamma_j}}{(\alpha+1)_n(\beta+1)_n\prod_{j=1}^{\sigma}(\beta_j)_{n\mu_j}} A_1 \frac{(-)^g \Omega_{P_i,Q_i,c_i,r}^{M,N}(\eta_{G,g})\tau^{\eta_{G,g}}}{B_G g!}$$

$$\frac{[(a_{p'})]_l}{[(b_{q'})]_l}\frac{\tau'^l}{\Gamma(\alpha l+1)}\tau_1^{K_1}\frac{(-t)^{\eta}\Gamma(\beta+n+1)}{n!}$$

$$F_{\sigma:N';\cdots;N^{(s)}}^{\upsilon:M';\cdots;M^{(s)}}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_s \end{array}\middle| \begin{array}{c} [\alpha_\upsilon+n\gamma_\upsilon;\eta',\cdots,\eta^{(s)},\gamma,\gamma];[m';\rho'];\cdots;[m^{(s)};\rho^{(s)}] \\ [\beta_\sigma+n\mu_\sigma;\zeta',\cdots,\zeta^{(s)},\mu,\mu];[l';\tau'];\cdots;[l^{(s)};\tau^{(s)}] \end{array}\right)$$

$$\aleph_{U_{22}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\middle| \begin{array}{c} (\text{-}\sigma - L\tau_{G,g} - L'l - K_1L_1;h_1,\cdots,h_r), \\ \cdots \\ \cdots \\ (\text{-}\sigma+\alpha+n-L\tau_{G,g}-L'l-K_1L_1;h_1,\cdots,h_r), \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}\sigma+\alpha;h_1,\cdots,h_r),A:C \\ \cdots \\ \cdots \\ (\text{-}\beta-\sigma-n;;h_1,\cdots,h_r),B:D \end{array}\right) \tag{4.5}$$

Where $U_{22} = p_i+2, q_i+2, \tau_i; R$

Provided

a) $Re(\beta) > -1, h_i > 0, i = 1,\cdots,r\,; p' \leqslant q' and\,|\tau| < 1, |t| < 1$

b) $Re[\sigma + L\min\limits_{1\leqslant j\leqslant M}\frac{b_j}{B_j} + \sum_{i=1}^{r} h_i \min\limits_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}] > 0$

c) $|argz| < \frac{1}{2}\pi\Omega$ Where $\Omega = \sum_{j=1}^{M}\beta_j + \sum_{j=1}^{N}\alpha_j - c_i(\sum_{j=M+1}^{Q_i}\beta_{ji} + \sum_{j=N+1}^{P_i}\alpha_{ji}) > 0$

d)$|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.15)

5. Conclusion

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions such as , multivariable H-function , defined by Srivastava et al [9] , the Aleph-function of two variables defined by K.sharma [4].

REFERENCES

[1] Chaurasia V.B.L. A theorem concerning the multivariable H-function. Bull. Inst. Math. Acad. Sinica. Vol13(2), (1985), p. 193-196.

[2] Chaurasia V.B.L and Singh Y. New generalization of integral equations of fredholm type using the Aleph-function Int. J. of Modern Math. Sci. 9(3), 2014, p 208-220.

[3] Panda. R. Some integrals associated with the generalized Lauricella function. Publ. De l'inst. Math. Nouvelle serie tome 16(30) 1973, p.115-122.

[4] Sharma K. On the integral representation and applications of the generalized function of two variables , International Journal of Mathematical Engineering and Sciences , Vol 3 , issue1 (2014) , page 1-13.

[5] Sharma M. Fractional integration and fractional differentiation of the M-series, Fractional calculus appl. Anal. Vol11(2), 2008, p.188-191.

[6] Srivastava H.M., A contour integral involving Fox's H-function. Indian J.Math. 14(1972), page1-6.

[7] Srivastava H.M. A multilinear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial, Pacific. J. Math. 177(1985), page183-191.

[8] Srivastava H.M. and Daoust M.C. Certain generalized Neumann expansions associated with Kampé de Fériet function. Nederl. Akad. Wetensch. Proc. Ser. A72 = Indag. Math, 31, (1969), p 449-457.

[9] H.M. Srivastava And R.Panda. Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24, (1975), p.119-137.

[10] Südland N.; Baumann, B. and Nonnenmacher T.F. , Open problem : who knows about the Aleph-functions? Fract. Calc. Appl. Anal., 1(4) (1998): 401-402.

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