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On Pseudo Tribonacci Sequence

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Abstract

We consider third order non- homogeneous recurrence relation to obtain Tribonacci like sequence called Pseudo Tribonacci Sequence. We obtain some general properties of this new sequence and extend it to obtain another generalised sequence using extended circular function.

Key Words : Tribonacci Sequence, Pseudo Fibonacci Sequence, Non homogeneous Recurrence Relation.
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1 Introduction

A well known generalisation of Fibonacci sequence is the Tribonacci sequence and many have studied these sequences. Various properties of Tribonacci sequence are found in [1], [3], [4]. Phadte and Pethe [7], introduce a new extension of Fibonacci Sequence called Pseudo Fibonacci Sequence (PFS) using a non homogeneous recurrence relation. This study of PFS was further developed in [8], [9],[10],[11]. We now extend this idea to Tribonacci Sequence. Here we consider third order non homogeneous recurrence relation to derive a new sequence called Pseudo Tribonacci Sequence (PTS).

Definition: We define the Pseudo Tribonacci Sequence $\{J_n\}$ as the sequence satisfying the following non- homogeneous recurrence relation.

$$J_n = pJ_{n-1} + qJ_{n-2} + rJ_{n-3} + At^{n-3}, \qquad (1.1)$$

for $n \geq 3$, $t \neq \alpha, \beta, \gamma$ where $A, t \in Z$, p, q, r are arbitrary integers and α, β, γ are distinct roots of auxiliary equation

$$x^3 - px^2 - qx - r = 0. \qquad (1.2)$$

Let the homogeneous relation corresponding to the equation (1.1) be given by

$$H_n = pH_{n-1} + qH_{n-2} + rH_{n-3} \qquad (1.3)$$

with the seed values

$$H_0 = 0, H_1 = 0, H_2 = p. \qquad (1.4)$$

A solution of (1.1) is given by

$$J_n = H_n + J_n^{(p)}$$

where $J_n^{(p)}$ is a particular solution.
From the characteristic equation we have

$$H_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n,$$

with

$$\begin{cases} \alpha + \beta + \gamma = p, \\ \alpha\beta + \beta\gamma + \gamma\alpha = -q, \\ \alpha\beta\gamma = r \end{cases} \qquad (1.5)$$

Using initial conditions in (1.4), we get,

$$c_1 = \frac{\alpha(\gamma - \beta)}{\Delta}, c_2 = \frac{\beta(\alpha - \gamma)}{\Delta}, c_3 = \frac{\gamma(\beta - \alpha)}{\Delta}. \qquad (1.6)$$

where $\Delta = (\gamma - \beta)(\gamma - \alpha)(\beta - \alpha)$.
Let the particular solution of equation (1.1) be $J_n^{(p)} = Dt^n$. We get,

$$D = \frac{A}{(t^3 - pt^2 - qt - r)}. \qquad (1.7)$$

Using equation (1.6) and (1.7), we get
$J_n = H_n + J_n^{(P)}$
$= \frac{1}{\Delta}\{(\gamma - \beta)\alpha^{n+1} + (\alpha - \gamma)\beta^{n+1} + (\beta - \alpha)\gamma^{n+1}\}$
$+ \frac{At^n}{(t^3 - pt^2 - qt - r)}$.
This is Binet Formula for the Pseudo Tribonacci Sequence.
Denote the sequence H_n by H_n^1, and H_n^2, when the seed values are

$$H_0 = 1,\ H_1 = 0,\ H_2 = q \qquad (1.8)$$

and

$$H_0 = 0,\ H_1 = 0,\ H_2 = r \tag{1.9}$$

respectively. Denote the sequence $\{J_n\}$ by $\{J_n^1\}$ and $\{J_n^2\}$ with the initial conditions as in (1.8) and (1.9) respectively. Then we have,

$J_n^1 = \frac{1}{\Delta}\{(\gamma^2-\beta^2)\alpha^{n+1} - (\gamma^2-\alpha^2)\beta^{n+1} + (\beta^2-\alpha^2)\gamma^{n+1}\} + \frac{At^n}{(t^3-pt^2-qt-r)}$

and

$J_n^2 = \frac{r}{\Delta}\{(\gamma-\beta)\alpha^n + (\alpha-\gamma)\beta^n + (\beta-\alpha)\gamma^n\} + \frac{At^n}{(t^3-pt^2-qt-r)}$.

2 Some identities of $\{J_n\}$

In this section we shall obtain some usual identities for Pseudo Tribonacci Sequence $\{J_n\}$.

Proposition 1. *The Generating function $G(x)$ of J_n is given by* $G(x) = \frac{x(1-tx)-Ax^3}{(1-tx)(1-px-qx^2-rx^3)}$, *provided* $|tx|<1$.

Proof.

$$Let\ \ G(x) = \sum_{n=0}^{\infty} J_n x^n. \tag{2.1}$$

Then $x^{-1}G(x) = \sum_{n=0}^{\infty} J_n x^{n-1} = \sum_{n=-1}^{\infty} J_{n+1}x^n$.

Hence

$$x^{-1}G(x) = \sum_{n=0}^{\infty} J_{n+1}x^n + J_0x^{-1}. \tag{2.2}$$

Similarly

$$x^{-2}G(x) = \sum_{n=0}^{\infty} J_{n+2}x^n + J_0x^{-2} + J_1x^{-1} \tag{2.3}$$

and

$$x^{-3}G(x) = \sum_{n=0}^{\infty} J_{n+3}x^n + J_0x^{-3} + J_1x^{-2} + J_2x^{-1}. \tag{2.4}$$

Multiply corresponding equations (2.3), (2.2), (2.1) by p, q, r respectively and subtracting them from equation (2.4), we get

$G(x)[x^{-3} - px^{-2} - qx^{-1} - r] = \sum_{n=0}^{\infty}(J_{n+3} - pJ_{n+2} - qJ_{n+1} - rJ_n)x^n + (J_0x^{-3}+J_1x^{-2}+j_2x^{-1}) - (J_0x^{-2}+J_1x^{-1}) - J_0x^{-1}$,

which yields,

$$G(x)[1-px-qx^2-rx^3] =$$

$$A\sum_{n=0}^{\infty}(t^nx^{n+3}) + J_0 + J_1x + j_2x^2 - p(J_0x+J_1x^2) - rJ_0x^2.$$

Thus

$$G(x) = \frac{x - Ax^3A\sum_{n=0}^{\infty}(tx)^n}{1-px-qx^2-rx^3}.$$

Hence

$$G(x) = \frac{x(1-tx)-Ax^3}{(1-tx)(1-px-qx^2-rx^3)},\ \text{provided } |tx|<1.$$

□

The Exponential Generating Function $E^*(x)$ of J_n is given by

$$E^*(x) = c_1e^{\alpha x} + c_2e^{\beta x} + c_3e^{\gamma x} + ze^{tx} \tag{2.5}$$

where c_1, c_2, c_3 and z are constants .

Proposition 2. *The sum of first $n+1$ terms of the sequence $\{J_n\}$ is given by*

$\sum_{k=0}^{n} J_k = \frac{J_2+(1-p)(J_0+J_1)-qJ_0-(J_{n+2}+J_{n+1})+pJ_{n+1}-rJ_n+A\sum_{k=0}^{n-1}t^k}{(1-p-q-r)}$.

Proof. From the recurrence relation

$$J_{n+3} = pJ_{n+2} + qJ_{n+1} + rJ_n + At^n,$$

we write the following.

For $n = 0$, $J_3 = pJ_2 + qJ_1 + rJ_0 + At^0$.

For $n = 1$, $J_4 = pJ_3 + qJ_2 + rJ_1 + At^1$.

For $n = 2$, $J_5 = pJ_4 + qJ_3 + rJ_2 + At^2$.

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In general,

For $n = k$, $J_k = pJ_{k-1} + qJ_{k-2} + rJ_{k-3} + At^{k-3}$.

On summing both sides, we get,

$\sum_{k=3}^{n} J_k = p\sum_{k=2}^{n-1} J_k + q\sum_{k=1}^{n-2} J_k + r\sum_{k=1}^{n-3} J_k + A\sum_{k=0}^{n-3} t^k$.

Therefore,

$\sum_{k=0}^{n} J_k - J_0 - J_1 - J_2 = p\sum_{k=0}^{n} J_k - p(J_0+J_1+J_n) + q\sum_{k=0}^{n} J_k - q(J_0+J_{n-1}+J_n) + r\sum_{k=0}^{n} J_k - r(J_{n-2}+J_{n-1}+J_n) + A\sum_{k=0}^{n-3} t^k$.

Then,

$\sum_{k=0}^{n} J_k(1-p-q-r) = (J_0+J_1+J_2) - p(J_0+J_1+J_n) - q(J_0+J_{n-1}+J_n) - r(J_{n-2}+J_{n-1}+J_n) + A\sum_{k=0}^{n-3} t^k$.

$=(J_0+J_1+J_2) - pJ_0 - pJ_1 - (pJ_n+qJ_{n-1}+rJ_{n-2}) - qJ_0 - qJ_n - rJ_{n-1} - rJ_n + A\sum_{k=0}^{n-3} t^k$.

$=(J_0+J_1+J_2)-pJ_0+pJ_1-(J_{n+1}-At^{n-2})-qJ_0+pJ_{n+1}-(pJ_{n+1}+qJ_n+rJ_{n-1})-rJ_n+A\sum_{k=0}^{n-3}t^k.$
$=(J_0+J_1+J_2)-p(J_0+J_1)-J_{n+1}-qJ_0+pJ_{n+1}-(J_{n+2}-At^{n-1})-rJ_n+A\sum_{k=0}^{n-2}t^k.$
$=J_2+(1-p)(J_0+J_1)-qJ_0-(J_{n+2}+J_{n+1})+pJ_{n+1}-rJ_n+A\sum_{k=0}^{n-1}t^k.$
Hence,
$\sum_{k=0}^{n}J_k=\frac{J_2+(1-p)(J_0+J_1)-qJ_0-(J_{n+2}+J_{n+1})+pJ_{n+1}-rJ_n+A\sum_{k=0}^{n-1}t^k}{(1-p-q-r)}.$ □

Now we state and prove some identities involving summation of products of terms of J_n.
Let,
$v_1=1+p^2-q^2-r^2,$
$v_2=1-p^2+q^2-r^2,$
$v_3=1+p^2+q^2-r^2,$ for any integers p,q and r.
$S_1=\sum_{k=0}^{n}J_kt^k$ and $S_2=\sum_{k=0}^{n}t^{2k}$.

Proposition 3. *For any integer $n>0$,*
i) $\sum_{k=0}^{n}J_kJ_{k+1}$
$=\frac{4(p+rq)[(q+rp)m_3-qm_2)]-4p(q+rp)(1-q)m_1}{\delta},$
ii) $\sum_{k=0}^{n}J_kJ_{k+2}$
$=\frac{2[(q+rp)v_1v_3-qm_2v_3+qm_1v_2-(q+rp)m_1v_3]}{\delta},$
iii) $\sum_{0}^{n}J_k^2$
$=\frac{2p(1-q)(v_1m_2-m_1v_2)+2(p+rq)(m_3v_2-m_2v_3)}{\delta},$
provided , $\delta=\begin{vmatrix} v_1 & -2(p+rq) & 0 \\ v_2 & 0 & -2(q+rp) \\ v_3 & 0 & -2q \end{vmatrix}$
i.e.
$\delta=4(p+rq)[(q+rp)v_3-qv_2)]-4p(q+rp)(1-q)v_1\neq 0.$

Proof. Using the recurrence relation (1.1), we write $(J_{k+3}-pJ_{k+2})^2=(qJ_{k+1}+rJ_k+At^k)^2.$
$J_{k+3}^2+p^2J_{k+2}-2pJ_{k+2}J_{k+3}=q^2J_{k+1}^2+r^2J_k^2+A^2t^{2k}+2(rqJ_kJ_{k+1}+rAJ_kt^k)+AqJ_{k+1}t^k$
$J_{k+3}^2+p^2J_{k+2}-q^2J_{k+1}^2-r^2J_k^2=2[pJ_{k+2}J_{k+3}+rqJ_kJ_{k+1}+rAJ_kt^k+AqJ_{k+1}t^k+A^2t^{2k}].$
Taking summation on both the sides from 1 to n, we get,

$\sum_{k=0}^{n}J_{k+3}^2+p^2\sum_{k=0}^{n}J_{k+2}-q^2\sum_{k=0}^{n}J_{k+1}^2-r^2\sum_{k=0}^{n}J_k^2$

$$=2[p\sum_{k=0}^{n}J_{k+2}J_{k+3}+rq\sum_{k=0}^{n}J_kJ_{k+1}+rA\sum_{k=0}^{n}J_kt^k+Aq\sum_{k=0}^{n}J_{k+1}t^k+A^2t^{2k}].$$

Therefore,
$(1+p^2-q^2-r^2)\sum_{k=0}^{n}J_k^2$
$=(J_0^2+J_1^2+J_2^2)-(J_{n+1}^2+J_{n+2}^2+J_{n+3}^2)+p^2(J_0^2+J_1^2)-p^2(J_{n+1}^2+J_{n+2}^2)-q^2(j_0^2-J_{n+1}^2)+2(p+rq)\sum_{k=0}^{n}J_kJ_{k+1}-2p(J_0J_1+J_1J_2)+2p(J_{n+1}J_{n+2}+J_{n+2}J_{n+3})+2A(r+qt^{-1})\sum_{k=0}^{n}J_kt^k-2Aqt^{-1}(J_0-J_{n+1}t^n)+A^2\sum_{k=0}^{n}t^{2k},$
which is written as

$$v_1X=2(p+rq)Y+m_1, \qquad (2.6)$$

where $m_1=(J_0^2+J_1^2+J_2^2)-(J_{n+1}^2+J_{n+2}^2+J_{n+3}^2)+p^2(J_0^2+J_1^2)-p^2(J_{n+1}^2+J_{n+2}^2)-q^2(j_0^2-J_{n+1}^2)-2p(J_0J_1+J_1J_2)+2p(J_{n+1}J_{n+2}+J_{n+2}J_{n+3})+2A(r+qt^{-1})S_1-2Aqt^{-1}(J_0-J_{n+1}t^{n+1})+A^2S_2,$
$X=\sum_{k=0}^{n}J_k^2$, and $Y=\sum_{k=0}^{n}J_kJ_{k+1}$.
Similarly we obtain the equations in terms of X,Y and Z.

$$v_2X=2(q+rp)Z+m_2, \qquad (2.7)$$

where $Z=\sum_{k=0}^{n}J_kJ_{k+2}$,

$m_2=(J_0^2+J_1^2+J_2^2)-(J_{n+1}^2+J_{n+2}^2+J_{n+3}^2)-p^2(J_0^2+J_1^2)+p^2(J_{n+1}^2+J_{n+2}^2)+q^2(J_0^2-J_{n+1}^2)+2[A(r+pt^{-2})S_1-q(J_0J_2-J_{n+1}J_{n+3})-Ap(J_0t^{-2}+J_1t^{-1}+J_{n+1}t^{n-1}-J_{n+2}t^n)]+S_2,$
and

$$v_3X=2p(1-q)Y+qZ+m_3, \qquad (2.8)$$

where $m_3=(J_0^2+J_1^2+J_2^2)-(J_{n+1}^2+J_{n+2}^2+J_{n+3}^2)-p^2(J_{n+1}^2+J_{n+2}^2-J_1^2)+q^2(J_0^2-J_{n+1}^2)+2[pq(J_0J_1-J_{n+1}J_{n+2})+p(J_{n+1}J_{n+2}+J_{n+2}J_{n+3}-J_0J_1-J_1J_2)+q(J_{n+1}J_{n+3}-J_0J_2)+Aqt^{-1}(J_0-J_{n+1}t^{n+1})+Apt^{-2}(J_0+J_1t-J_{n+1}t^{n+1}-J_{n+2}t^{n+2})-At^{-3}(J_0+J_1t+J_2t^2-J_{n+1}t^{n+1}-J_{n+2}t^{n+2}-J_{n+3}t^{n+3})-A(qt^{-1}+pt^{-2}-t^{-3})]S_1-A^2S_2.$

Solving the equations (2.6),(2.7) and (2.8), we get,
$X=\sum_{k=0}^{n}J_kJ_{k+1}$

$= \frac{4(p+rq)[(q+rp)m_3-qm_2)]-4p(q+rp)(1-q)m_1}{\delta}$,
$Y = \sum_{k=0}^{n} J_k J_{k+2}$
$= \frac{2[(q+rp)v_1v_3-qm_2v_3+qm_1v_2-(q+rp)m_1v_3]}{\delta}$,
$Z = \sum_{k=0}^{n} J_k^2$
$= \frac{2p(1-q)(v_1m_2-m_1v_2)+2(p+rq)(m_3v_2-m_2v_3)}{\delta}$.

□

We Illustrate the above result with an example.

Example

Let $p = 2, q = 1, r = 1, t = -1$ and $A = 1$.
Few first few terms of the sequence $\{J_n\}$ are
$J_0 = 0, J_1 = 1, J_2 = 2, J_3 = 6, J_4 = 14,$
$J_5 = 37, J_6 = 93$ etc.
Hence $v_1 = 3$, $v_2 = -3$ and $v_3 = 5$.
For, $n = 2$, $m_1 = -69, m_2 = -219, m_3 = -43$,
$s_1 = 3$ and $s_2 = 1$.
$\delta = 216$, Numerator of $X = 1080$, Numerator of $Y = 3024$, Numerator of $Z = 7344$.
Verification of Proposition 4.(i)
L.H.S.$= \sum_{k=0}^{2} J_k J_{k+1} = 5$.
R.H.S.$= \frac{4(p+rq)[(q+rp)m_3-qm_2)]-4p(q+rp)(1-q)m_1}{\delta}$
$= \frac{1080}{216} = 5$.
Result is verified. Verification of Proposition 4.(ii)
L.H.S.$= \sum_{k=0}^{2} J_k J_{k+2} = 14$.
R.H.S.$= \frac{2[(q+rp)v_1v_3-qm_2v_3+qm_1v_2-(q+rp)m_1v_3]}{\delta}$
$= \frac{3024}{216} = 14$.
Result is verified. Verification of Proposition 4.(iii)
L.H.S.$= \sum_{0}^{2} J_k^2 = 34$.
R.H.S.$= \frac{2p(1-q)(v_1m_2-m_1v_2)+2(p+rq)(m_3v_2-m_2v_3)}{\delta}$
$= \frac{7344}{216} = 34$.
Result is verified.
The following theorem gives an expression for J_n in terms of H_n.

Theorem 4. $J_n = -rJ_0^{(P)}H_{n-2} - (pJ_1^{(P)} - J_2^{(p)})H_{n-1} + (1 - J_1^{(P)})H_n + J_n^{(P)}$.

Proof. Let H_n be homogeneous relation and $J_n^{(P)}$ be a particular solution of equation (1.1). Then we have

$$J_n = H_n + J_n^{(P)} = c_1\alpha^n + c_2\beta^n + c_3\gamma^n + J_n^{(P)}, \quad (2.9)$$

with the seed values, $J_0 = 0,\ J_1 = 1$ and $J_2 = p$. Using these values and solving we get,
$c_1 = \frac{\gamma-\beta}{\Delta}[-J_0^{(P)}\beta\gamma - (1-J_1^{(P)})(\beta+\gamma) + (p-J_2^{(P)})]$,
$= \frac{\gamma-\beta}{\Delta}[\frac{-rJ_0^{(P)}}{\alpha} - (1-J_1^{(P)})(p-\alpha) + (p-J_2^{(P)})]$.
$c_2 = -\frac{\gamma-\alpha}{\Delta}[\frac{-rJ_0^{(P)}}{\beta} - (1-J_1^{(P)})(p-\beta) + (p-J_2^{(P)})]$,
and
$c_3 = \frac{\beta-\alpha}{\Delta}[\frac{-rJ_0^{(P)}}{\gamma} - (1-J_1^{(P)})(p-\gamma) + (p-J_2^{(P)})]$.
Substituting $c_1,\ c_2$ and c_3 in equation(1.3) and with some computations,
we get
$H_n = \frac{1}{\Delta} - rJ_0^{(P)}[\alpha^{n-1}(\gamma-\beta) - \beta^{n-1}(\gamma-\alpha) + \gamma^{(n-1)})(\beta-\alpha)]$
-$p(1-J_1^{(P)})[\alpha^n(\gamma-\beta) - \beta^n(\gamma-\alpha) + \gamma^{(n)})(\beta-\alpha)]$
$+(1-J_1^{(P)})[\alpha^{n+1}(\gamma-\beta) - \beta^{n+1}(\gamma-\alpha) + \gamma^{(n+1)})(\beta-\alpha)]$
$+(p-J_2^{(P)})[\alpha^n(\gamma-\beta) - \beta^n(\gamma-\alpha) + \gamma^{(n)})(\beta-\alpha)]$
$= -rJ_0^{(P)}H_{n-2} - p(1-J_1^{(P)})H_{n-1} + (2-J_1^{(P)})H_n + (p-J_2^{(P)})H_{n-1}$.
Hence, solution of (1.1) is given by
$J_n = -rJ_0^{(P)}H_{n-2} - (pJ_1^{(P)} - J_2^{(p)})H_{n-1} + (1 - J_1^{(P)})H_n + J_n^{(P)}$. □

Similarly we can obtain expressions for the n^{th} term of the other two sequences as
$J_n^1 = r(1-J_0^{(P)})H_{n-2}^1 + (pJ_1^{(P)} + q - J_2^{(P)})H_{n-1}^1 - J_1^{(P)}H_n^1 + J_n^{(P)}$
and
$J_n^2 = -rJ_0^{(P)}H_{n-2}^2 + (pJ_1^{(P)} + r - J_2^{(P)}))H_{n-1}^2 - J_1^{(P)}H_n^2 + J_n^{(P)}$.

3 E-Operator

We define operator E such that

$$EJ_n = J_{n+1}.$$

Let α, β, γ be distinct roots of auxillary equation

$$x^3 - px^2 - qx - r = 0.$$

Therefore,

$$(x-\alpha)(x-\beta)(x-\gamma) = (x^2-px+q)(x-\gamma) = 0, \quad (3.1)$$

where $\alpha+\beta = p$ and $\alpha\beta = q$.
Hence, from (1.1), the recurrence relation

$$J_{n+3} = pJ_{n+2} + qJ_{n+1} + rJ_n + At^n$$

reduce to

$$(E^2 - pE + q)(E - \gamma)J_n = At^n.$$

Therefore,

$$(E^2 - pE + q)u_n = At^n,$$

if $(E-\gamma)J_n = u_n$.
Hence $u_{n+2} - pu_{n+1} + qu_n = At^n$

$$u_{n+2} = pu_{n+1} + qu_n + At^n, n \geq 0$$

with $u_0 = 0$ and $u_1 = 1$.
The various properties of u_n can be utilised for J_n.

4 Further Generalisation of $\{J_n\}$

In this section, we state the generalized circular function,studied by Mikusinski, which are used to obtain more generalized form of the sequence $\{J_n\}$[5]. Let

$$N_{r,j}(t) = \sum_{n=0}^{\infty} \frac{t^{nr+j}}{(nr+j)!}, \quad j = 0, 1, ..., r-1; \quad r \geq 1, \tag{4.1}$$

$$M_{r,j}(t) = \sum_{n=0}^{\infty} (-1)^r \frac{t^{nr+j}}{(nr+j)!}, \quad j = 0, 1, ..., r-1; \quad r \geq 1. \tag{4.2}$$

Observe that
$N_{1,0}(t) = e^t, N_{2,0}(t) = cosht, N_{2,1}(t) = sinht$ and
$M_{1,0}(t)= e^{-t}$, $M_{2,0}(t)= cost$, $M_{2,1}(t)= sint$.
One obtains following result by differentiating (4.1) term by term with respect to t.

$$N_{r,j}^{(p)}(t) = \begin{cases} N_{r,j-p}(t), & 0 \leq p \leq j, \\ N_{r,r+j-p}(t), & 0 \leq j < j < p \leq r. \end{cases} \tag{4.3}$$

In particular, note from (4.3) that

$$N_{r,0}^{(r)}(t) = N_{r,0}(t),$$

so that in general,

$$N_{r,0}^{(nr)}(t) = N_{r,0}(t), r \geq 1. \tag{4.4}$$

Further note that

$$N_{r,0}(0) = N_{r,0}^{(nr)}(0) = 1.$$

Now we further extend Pseudo Tribonacci sequence to a new sequence $\{K_n\}$, using Pethe and Phadte techniques [6], wherein Elmor M. [2] concept of exponential generating function is used to generalize Fibonacci function.
Let,

$$J_0^*(x) = J^*(x) = c_1e^{\alpha x} + c_2e^{\beta x} + c_3e^{\gamma x} + Ae^{xt}$$

be the exponential generating function of $\{J_n\}$ where c_1, c_2 and c_3 are as in (1.6),. Further, let $J_n^*(x)$ of the sequence $\{J_n^*(x)\}$ be defined as the n^{th} derivative with respect to x of $J_0^*(x)$, then

$$J_n^*(x) = c_1\alpha^n e^{\alpha x} + c_2\beta^n e^{\beta x} + c_3e^{\gamma x} + At^n e^{xt}, \tag{4.5}$$

which is a generalization by Elmore's Method. Now we use extended circular functions to generalize $J_n^*(x)$ as follows.
Let,

$$K_0(x) = c_1N_{r,0}(\alpha^* x) + c_2N_{r,0}(\beta^* x) + c_3N_{r,0}(\gamma^*) + AN_{r,0}(t^* x) \tag{4.6}$$

where $\alpha^*=\alpha^{1/r}, \beta^* = \beta^{1/r}$ and $t^* = t^{1/r}$, r being the positive integer.
Now define the sequence $\{K_n(x)\}$ successively as follows:
$K_1(x) = K_0^{(r)}(x)$,
$K_2(x) = K_0^{(2r)}(x)$,
Hence in general
$K_n(x) = K_0^{(nr)}(x)$,
where derivatives are with respect to x.
Since,
$N_{r,j} = \sum_{n=0}^{\infty} \frac{t^{(nr+j)}}{(nr+j)!}, \quad j = 0, 1, ..., r-1; \quad r \geq 1,$
and

$$N_{1,0}(t) = e^t \tag{4.7}$$

using equation (4.6), we get,
$K_1(x) = c_1\alpha N_{r,0}(\alpha^* x) + c_2\beta N_{r,0}(\beta^* x) + c_3\gamma N_{r,0}(\gamma^* x) + AtN_{r,0}(t^* x)$
$K_2(x) = c_1\alpha^2 N_{r,0}(\alpha^* x) + c_2\beta^2 N_{r,0}(\beta^* x) + c_3\gamma^2 N_{r,0}(\gamma^* x) + At^2 N_{r,0}(t^* x)$.
In general,
$K_n(x) = c_1\alpha^n N_{r,0}(\alpha^* x) + c_2\beta^n N_{r,0}(\beta^* x) + c_3\gamma^n N_{r,0}(\gamma^* x) + At^n N_{r,0}(t^* x)$.
We have the following result.

Theorem 5. *$K_n(x)$ satisfies the non-homogeneous recurrence relation*

$$K_{n+3}(x) = pK_{n+2}(x) + qK_{n+1}(x) + rK_n(x) + At^n. \tag{4.8}$$

Proof. We prove
R.H.S. $=c_1\alpha^{n+3}N_{r,0}(\alpha^* x) + c_2\beta^{n+3}N_{r,0}(\beta^* x) + c_3\gamma^{n+3}N_{r,0}(\gamma^*) + At^{n+3}N_{r,0}(t^* x)$.
From equation(4.8)
R.H.S.$=p(c_1\alpha^{n+2}N_{r,0}(\alpha^* x) + c_2\beta^{n+2}N_{r,0}(\beta^* x) + c_3\gamma^{n+2}N_{r,0}(\gamma^*) + At^{n+2}N_{r,0}(t^* x))$
$+ q(c_1\alpha^{n+1}N_{r,0}(\alpha^* x) + c_2\beta^{n+1}N_{r,0}(\beta^* x) + c_3\gamma^{n+1}N_{r,0}(\gamma^*) + At^{n+1}N_{r,0}(t^* x))$
$+r(c_1\alpha^n N_{r,0}(\alpha^* x) + c_2\beta^n N_{r,0}(\beta^* x) + c_3\gamma^n N_{r,0}(\gamma^*) + At^n N_{r,0}(t^* x)) + At^n$.
$=c_1\alpha^n N_{r,0}(\alpha^* x)[p\alpha^2 + q\alpha + r]$
$+c_2\beta^n N_{r,0}(\beta^* x)[p\beta^2 + q\beta + r] + c_3\gamma^n N_{r,0}[p\gamma^2 + q\gamma + r]$
$+ t^n N_{r,0}[pt^2 + qt + r + A]$.
Using the fact that, α, β, γ are the roots of $x^3 - px^2 - qx - r = 0$, we get,
R.H.S.$=c_1\alpha^{n+3}N_{r,0}(\alpha^* x) + c_2\beta^{n+3}N_{r,0}(\beta^* x) + c_3\gamma^{n+3}N_{r,0}(\gamma^*) + At^{n+3}N_{r,0}(t^* x)$,
which is the required result. □

Remark

Observe that if $r = 1$, then $\alpha^* = \alpha, \beta^* = \beta, \gamma^* = \gamma$, and hence $N_{r,0}(x) = e^x$. Hence for $r = 1$, we have $K_n(x) = c_1 e^{\alpha x} + c_2 e^{\beta x} + c_3 e^{\gamma x} + Ae^{xt}$.
$=J_n^*(x)$, which is Elmore's generalisation of $\{G_n\}$ [2].
Further with $p = 1, q = 1, r = 1$, $A = 0$ and $x = 0$, $K_n(x)$ reduces to Tribonacci sequence.

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